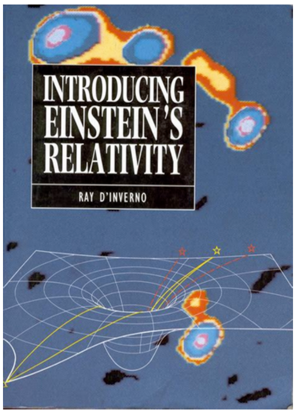
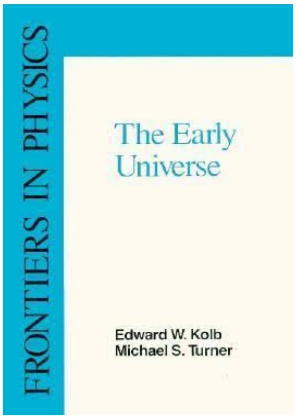
Universo Primitivo
2019-2020 (1º Semestre)

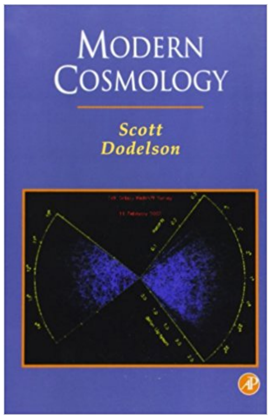
Mestrado em Física - Astronomia

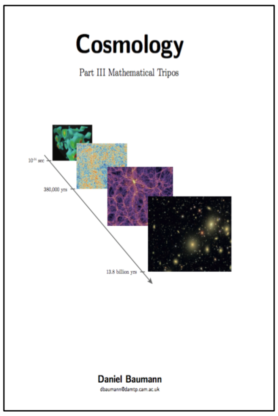
Chapter 2

2. The Standard Model of Cosmology (SMC)
   - Fundamental assumptions;
   - The GR equations and the Friedmann-Lemaitre-Robertson-Walker solution;
   - FLRW models:
     - Dynamic equations;
     - Energy-momentum conservation;
     - Fluid components and equations of state;
     - Cosmological parameters;
     - The Friedmann equation: the evolutionary phases of the Universe; exact solutions: age of the Universe;
     - Distances; horizons and volumes;
     - The accelerated expansion of the Universe;
   - Problems with the SMC: Horizon; Flatness; Relic particles; origin of perturbations; primordial Isotropy and homogeneity
   - The idea of Inflation

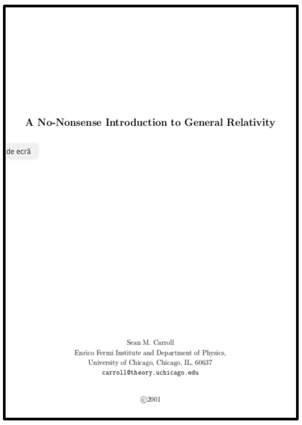


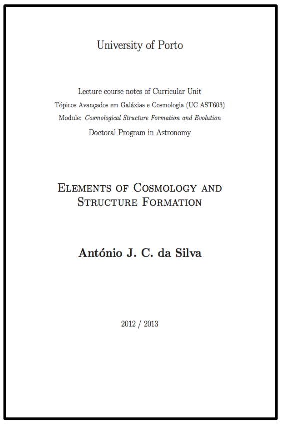






Ch. 1





Ch. 1

Standard Model of Cosmology

SMC: Mathematical framework

Fundamental assumptions:

- The Universe is homogeneous and isotropic when observed on large scales and expands uniformly with respect to any position

- The dynamics of space-time is described by Einstein's theory of general relativity (GR).

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi G}{c^4}T_{ab} + \Lambda g_{ab}$$

for the Universe to be homogeneous and isotropic the stress-energy tensor has to be that of a perfect fluid

$$T_{ab} = (\rho + \frac{p}{c^2})U_a U_b - \frac{p}{c^2}g_{ab}$$

SMC: Mathematical framework

The cosmological constant in the GR equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$ (Λ as "cosmological constant")

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu} \right) = 8\pi G \tilde{T}_{\mu\nu},$$ (Λ as "vacuum energy")

The Einstein tensor, Ricci tensor and Ricci scalar are:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu}$$

$$R = g^{\mu\nu}R_{\mu\nu}$$

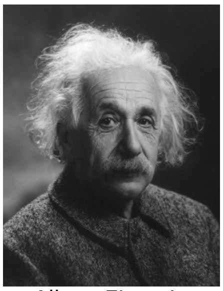
$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}g^{\mu\nu}(g_{\alpha\nu,\lambda} + g_{\alpha\lambda,\nu} - g_{\nu\lambda,\alpha}) \qquad g_{\mu\nu,\lambda} \equiv \partial g_{\alpha\nu}/\partial x^{\lambda}$$

where,

$$\mathrm{d}s^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}\mathrm{d}X^{\mu}\mathrm{d}X^{\nu} \equiv g_{\mu\nu}\mathrm{d}X^{\mu}\mathrm{d}X^{\nu}$$

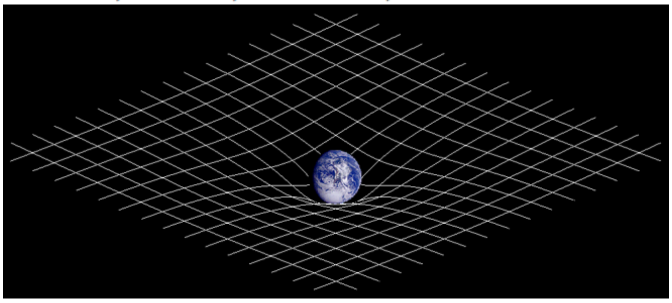
SMC: Mathematical framework

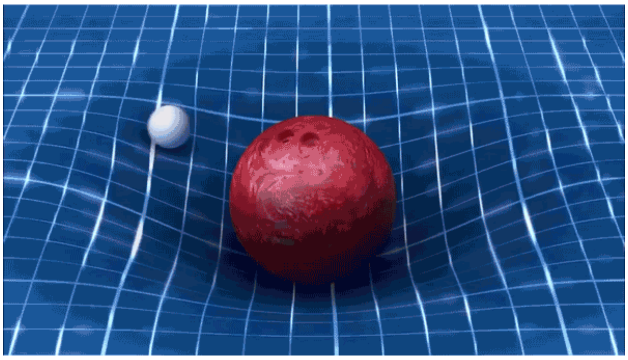
Einstein Equation:

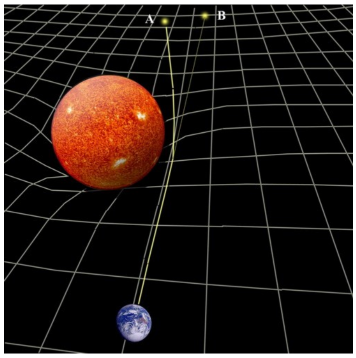


Albert Einstein
1879-1955

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$







SMC: Mathematical framework

Geodesic Equation:

In the absence of non-gravitational forces, free falling particles move along "geodesics", described by the so called Geodesic equation.

$$\boxed{\frac{dU^\mu}{ds} + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta = 0}$$

where,

$$U^\mu \equiv \frac{dX^\mu}{ds}$$

four-velocity of the particle along its free-falling path $X^\mu(s)$

$$S = -m \int_A^B \mathrm{d}s \ .$$

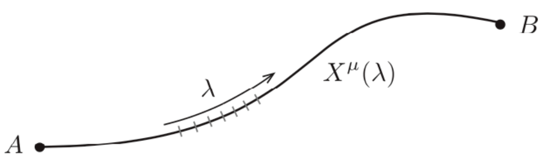


Figure 1.4: *Parameterisation of an arbitrary path in spacetime, $X^\mu(\lambda)$.*

SMC: Mathematical framework

Fundamental assumptions:

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- The dynamics of space-time is described by Einstein's theory of general relativity (GR).

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab} + \Lambda g_{ab} \qquad T_{ab} = (\rho + \frac{p}{c^2}) U_a U_b - \frac{p}{c^2} g_{ab}$$

In these conditions **the solution of the Einstein equation** is the Friedmann-Lemaitre-Robertson-Walker (**FLRW**) metric:

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right]$$

SMC: Mathematical framework

- Dynamical equations:

  (result from the Einstein equations and govern the time evolution of $a(t)$)

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}$$ Friedmann equation

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3}$$ Raychaudhuri (or acceleration) equation

- Energy momentum conservation: $\nabla_\mu T^\mu{}_\nu \equiv T^\mu{}_{\nu;\mu} = 0$

  the covariant derivative reads: $\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\lambda} T^\lambda{}_\nu - \Gamma^\lambda_{\mu\nu} T^\mu{}_\lambda = 0$

  the $\nu = 0$ (time) component of this equation gives:

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \quad \Rightarrow \quad d\left(\rho c^2 a^3\right) = -p\, d\left(a^3\right)$$ Energy conservation equation

$$p = w\rho c^2 \qquad -1 \le w \le 1$$ Equation of State (EoS)

for fluids with constant EoS parameter, w, the solution is:

$$\rho(t) = \rho_i \left(\frac{a(t)}{a_i}\right)^{-3(1+w)}$$

SMC: Mathematical framework

Covariant derivative:

Covariant derivative.—The covariant derivative is an important object in differential geometry and it is of fundamental importance in general relativity. The geometrical meaning of ∇_μ will be discussed in detail in the GR course. In this course, we will have to be satisfied with treating it as an operator that acts in a specific way on scalars, vectors and tensors:

- There is no difference between the covariant derivative and the partial derivative if it acts on a scalar
$$\nabla_\mu f = \partial_\mu f \,. \tag{1.3.83}$$
- Acting on a contravariant vector, V^ν, the covariant derivative is a partial derivative plus a correction that is linear in the vector:
$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \,. \tag{1.3.84}$$
Look carefully at the index structure of the second term. A similar definition applies to the covariant derivative of covariant vectors, ω_ν,
$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \,. \tag{1.3.85}$$
Notice the change of the sign of the second term and the placement of the dummy index.
- For tensors with many indices, you just repeat (1.3.84) and (1.3.85) for each index. For each upper index you introduce a term with a single $+\Gamma$, and for each lower index a term with a single $-\Gamma$:
$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} = {} & \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \\ & + \Gamma^{\mu_1}{}_{\sigma\lambda} T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}{}_{\sigma\lambda} T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \cdots \\ & - \Gamma^{\lambda}{}_{\sigma\nu_1} T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} - \Gamma^{\lambda}{}_{\sigma\nu_2} T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots \,.\end{aligned} \tag{1.3.86}$$
This is the general expression for the covariant derivative. Luckily, we will only be dealing with relatively simple tensors, so this monsterous expression will usually reduce to something managable.

SMC: Mathematical framework

- EoS for different energy density components:
  - w=1/3 (radiation)

$$\rho_\gamma = \rho_{\gamma 0}\left(\frac{a_0}{a}\right)^4 \xrightarrow{(1)} \left(\frac{\dot a}{a}\right)^2 \propto \frac{1}{a^4} \rightarrow a \propto t^{1/2}.$$

  - w=0 (matter)

$$\rho_{\rm m} = \rho_{\rm m0}\left(\frac{a_0}{a}\right)^3 \xrightarrow{(2)} \left(\frac{\dot a}{a}\right)^2 \propto \frac{1}{a^3} \rightarrow a \propto t^{2/3}.$$

  - w=-1 (cosmological constant)

$$\rho_\Lambda = \Lambda/8\pi G = -P_\Lambda \xrightarrow{(3)} a \propto e^{\sqrt{\Lambda}t}.$$

(1) after integration of the Friedmann equation with $k = 0,\ \Lambda = 0,\ \rho = \rho_\gamma$.
(2) after integration of the Friedmann equation with $k = 0,\ \Lambda = 0,\ \rho = \rho_m$.
(3) after integration of the Friedmann equation with $k = 0,\ \Lambda = 8\pi G\rho_\Lambda,\ \rho = 0$

SMC: FLRW models

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}$$

- Cosmological parameters:

$$\frac{8\pi G}{3H^2}\rho + \frac{\Lambda c^2}{3H^2} - \frac{kc^2}{a^2H^2} = 1 \quad \Leftrightarrow \quad \sum_i \Omega_i + \Omega_\Lambda + \Omega_k = 1$$

$$H(t) = \frac{\dot a(t)}{a(t)}$$ Hubble parameter

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\rm crit}} \qquad \Omega_\Lambda = \frac{\Lambda c^2}{3H^2}, \qquad \Omega_k = -\frac{kc^2}{a^2H^2}, \qquad \rho_c = \frac{3H^2}{8\pi G}$$ Critical energy density

Mater-energy density parameters: $(\Omega = \frac{\rho}{\rho_c} = \sum \frac{\rho_i}{\rho_c} = \sum \Omega_i)$

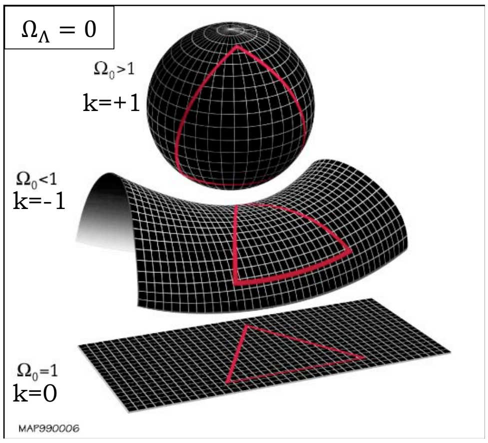
Vacuum or dark energy density parameter

Curvature density parameter

$$\rho \equiv \sum_i \rho_i$$

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\rm crit}}$$

includes all matter and radiation components (baryons, dark mater, radiation, ...)

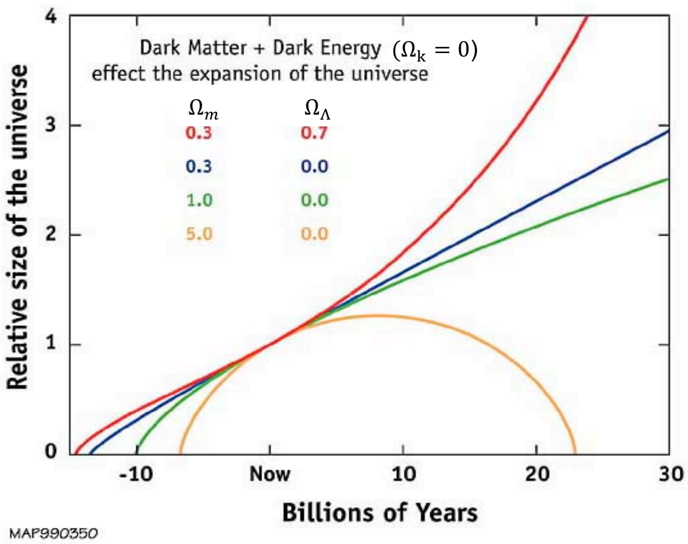


SMC: FLRW models

- Friedmann equation revisited

$$H^2(t) = \frac{8\pi G}{3}(\rho_r + \rho_m) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$
$$= H_0^2\left[\Omega_{r0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda 0}\right]$$

The evolutionary fate of the Universe is determined by cosmological parameters



SMC: Exact solutions of the Friedmann equation

- Scale factor:

$$\frac{d}{dt}\frac{a(t)}{a_0} = H_0\sqrt{1-\Omega_0+\Omega_{m0}\left(\frac{a}{a_0}\right)^{-1}+\Omega_{r0}\left(\frac{a}{a_0}\right)^{-2}-\Omega_{\Lambda 0}\left[1-\left(\frac{a}{a_0}\right)^{2}\right]}$$

for a critical density ($\Omega_k = \Omega_\Lambda = 0$) universe, gives:

$$\frac{a(t)}{a_0} = \left(\frac{3(1+w)}{2}H_0 t\right)^{2/(3(1+w))}$$

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3(w+1)t}$$

- Age of the Universe:

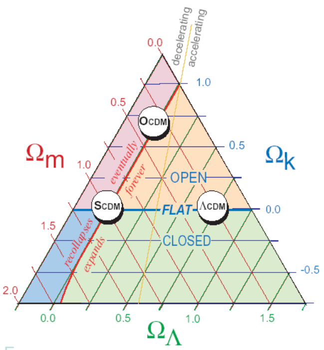
$$t = H_0^{-1}\int_0^{\frac{a(t)}{a_0}=(1+z)^{-1}} \frac{1}{\sqrt{1-\Omega_0+\Omega_{m0}x^{-1}+\Omega_{r0}x^{-2}-\Omega_\Lambda\left(1-x^2\right)}}\,dx$$

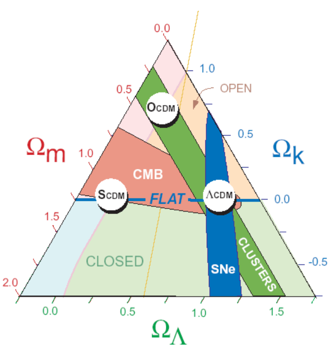
Redshift:

$$z = \frac{E-E_0}{E_0} = $$
$$= \frac{\nu}{\nu_0} - 1 =$$
$$= \frac{\lambda_0}{\lambda} - 1 =$$
$$= \frac{a_0}{a} - 1$$

SMC: Concordance Cosmology

Combination of different observational datasets…

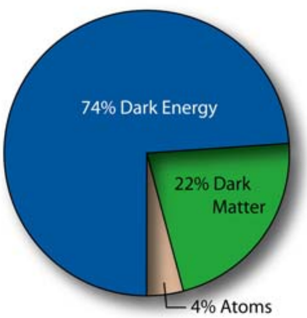




From:

… allow us to impose constraints on cosmological parameters

$$\sum_i \Omega_i + \Omega_\Lambda + \Omega_k = 1$$



WMAP3 parameters

Parameter	Value	Description
Basic parameters		
H_0	$70.9^{+2.4}_{-3.2}$ km s^{-1} Mpc^{-1}	Hubble parameter
Ω_b	$0.0444^{+0.0042}_{-0.0035}$	Baryon density
Ω_m	$0.266^{+0.025}_{-0.040}$	Total matter density (baryons + dark matter)
τ	$0.079^{+0.029}_{-0.032}$	Optical depth to reionization
A_s	$0.813^{+0.042}_{-0.052}$	Scalar fluctuation amplitude
n_s	$0.948^{+0.015}_{-0.018}$	Scalar spectral index
Derived parameters		
ρ_0	$0.94^{+0.06}_{-0.09} \times 10^{-26}$ kg/m^3	Critical density
Ω_Λ	$0.732^{+0.040}_{-0.025}$	Dark energy density
z_{ion}	$10.5^{+2.6}_{-2.9}$	Reionization red-shift
σ_8	$0.772^{+0.036}_{-0.048}$	Galaxy fluctuation amplitude
t_0	$13.73^{+0.13}_{-0.17} \times 10^9$ years	Age of the universe

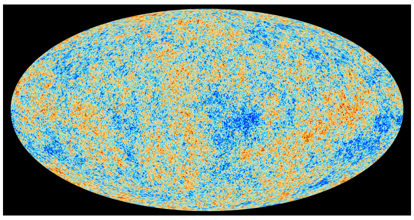
SMC: Cosmological parameters after Planck

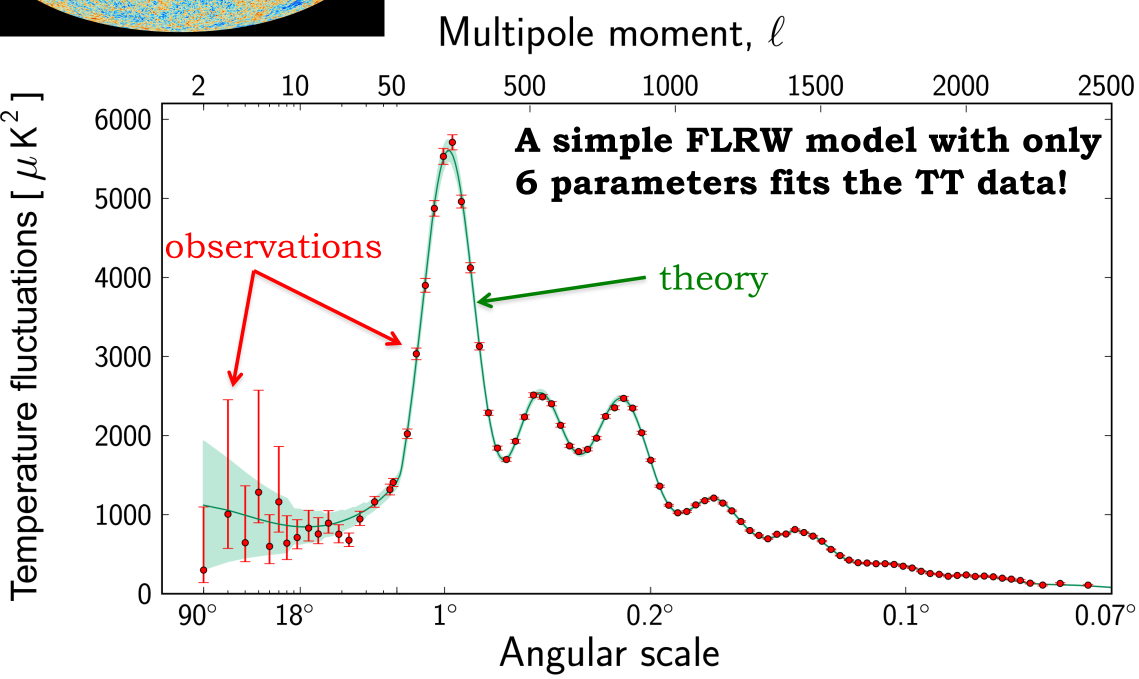
From: Planck collaboration. XVI. arXiv:1303.5076

Table 2. Cosmological parameter values for the six-parameter base ΛCDM model. Columns 2 and 3 give results for the *Planck* temperature power spectrum data alone. Columns 4 and 5 combine the *Planck* temperature data with *Planck* lensing, and columns 6 and 7 include *WMAP* polarization at low multipoles. We give best fit parameters (i.e. the parameters that maximise the overall likelihood for each data combination) as well as 68% confidence limits for constrained parameters. The first six parameters have flat priors. The remainder are derived parameters as discussed in Sect. 2. Beam, calibration parameters, and foreground parameters (see Sect. 4) are not listed for brevity. Constraints on foreground parameters for *Planck*+WP are given later in Table 5.

	Planck		*Planck*+lensing		*Planck*+WP	
Parameter	Best fit	68% limits	Best fit	68% limits	Best fit	68% limits
$\Omega_b h^2$	0.022068	0.02207 ± 0.00033	0.022242	0.02217 ± 0.00033	0.022032	0.02205 ± 0.00028
$\Omega_c h^2$	0.12029	0.1196 ± 0.0031	0.11805	0.1186 ± 0.0031	0.12038	0.1199 ± 0.0027
$100\theta_{MC}$	1.04122	1.04132 ± 0.00068	1.04150	1.04141 ± 0.00067	1.04119	1.04131 ± 0.00063
τ	0.0925	0.097 ± 0.038	0.0949	0.089 ± 0.032	0.0925	$0.089^{+0.012}_{-0.014}$
n_s	0.9624	0.9616 ± 0.0094	0.9675	0.9635 ± 0.0094	0.9619	0.9603 ± 0.0073
$\ln(10^{10} A_s)$	3.098	3.103 ± 0.072	3.098	3.085 ± 0.057	3.0980	$3.089^{+0.024}_{-0.027}$
Ω_Λ	0.6825	0.686 ± 0.020	0.6964	0.693 ± 0.019	0.6817	$0.685^{+0.018}_{-0.016}$
Ω_m	0.3175	0.314 ± 0.020	0.3036	0.307 ± 0.019	0.3183	$0.315^{+0.016}_{-0.018}$
σ_8	0.8344	0.834 ± 0.027	0.8285	0.823 ± 0.018	0.8347	0.829 ± 0.012
z_{re}	11.35	$11.4^{+4.0}_{-2.8}$	11.45	$10.8^{+3.1}_{-2.5}$	11.37	11.1 ± 1.1
H_0	67.11	67.4 ± 1.4	68.14	67.9 ± 1.5	67.04	67.3 ± 1.2
$10^9 A_s$	2.215	2.23 ± 0.16	2.215	$2.19^{+0.12}_{-0.14}$	2.215	$2.196^{+0.051}_{-0.060}$
$\Omega_m h^2$	0.14300	0.1423 ± 0.0029	0.14094	0.1414 ± 0.0029	0.14305	0.1426 ± 0.0025
$\Omega_m h^3$	0.09597	0.09590 ± 0.00059	0.09603	0.09593 ± 0.00058	0.09591	0.09589 ± 0.00057
Y_P	0.247710	0.24771 ± 0.00014	0.247785	0.24775 ± 0.00014	0.247695	0.24770 ± 0.00012
Age/Gyr	13.819	13.813 ± 0.058	13.784	13.796 ± 0.058	13.8242	13.817 ± 0.048
z_*	1090.43	1090.37 ± 0.65	1090.01	1090.16 ± 0.65	1090.48	1090.43 ± 0.54
r_*	144.58	144.75 ± 0.66	145.02	144.96 ± 0.66	144.58	144.71 ± 0.60
$100\theta_*$	1.04139	1.04148 ± 0.00066	1.04164	1.04156 ± 0.00066	1.04136	1.04147 ± 0.00062
z_{drag}	1059.32	1059.29 ± 0.65	1059.59	1059.43 ± 0.64	1059.25	1059.25 ± 0.58
r_{drag}	147.34	147.53 ± 0.64	147.74	147.70 ± 0.63	147.36	147.49 ± 0.59
k_D	0.14026	0.14007 ± 0.00064	0.13998	0.13996 ± 0.00062	0.14022	0.14009 ± 0.00063
$100\theta_D$	0.161332	0.16137 ± 0.00037	0.161196	0.16129 ± 0.00036	0.161375	0.16140 ± 0.00034
z_{eq}	3402	3386 ± 69	3352	3362 ± 69	3403	3391 ± 60
$100\theta_{eq}$	0.8128	0.816 ± 0.013	0.8224	0.821 ± 0.013	0.8125	0.815 ± 0.011
$r_{drag}/D_V(0.57)$	0.07130	0.0716 ± 0.0011	0.07207	0.0719 ± 0.0011	0.07126	0.07147 ± 0.00091

SMC: Cosmological parameters after Planck





SMC: Cosmological parameters after Planck

$$\sum_i \Omega_i + \Omega_\Lambda + \Omega_k = 1$$

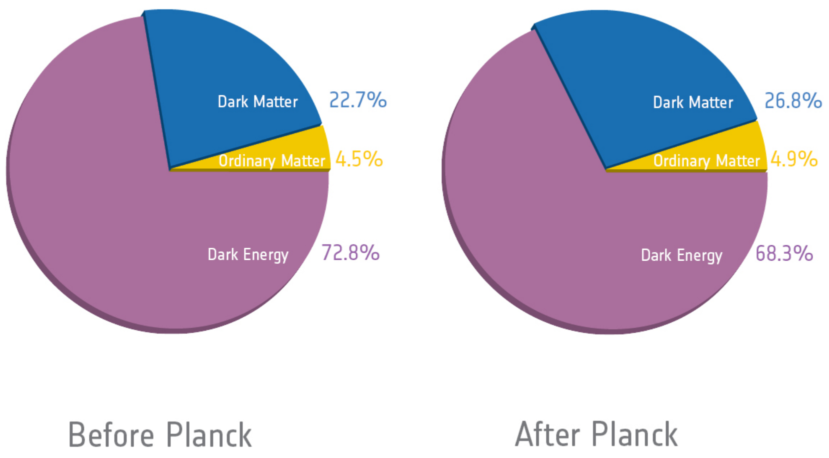
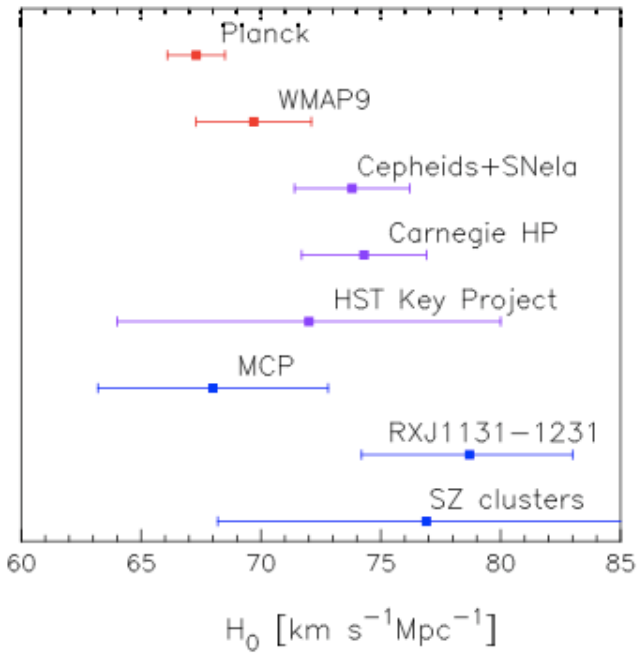
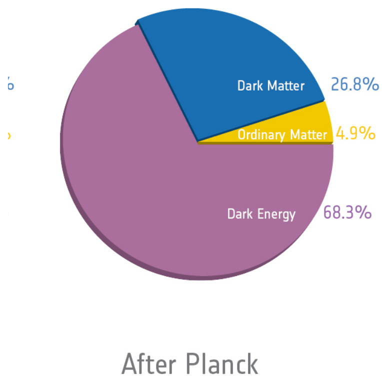


Fig. credits: ESA / PLANCK collaboration

SMC: Cosmological parameters after Planck

From: Planck collaboration. XVI. arXiv:1303.5076

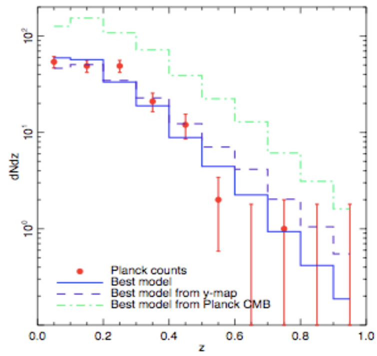


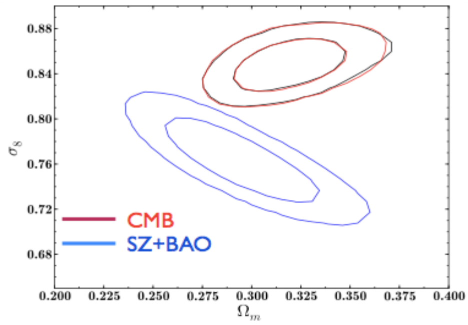


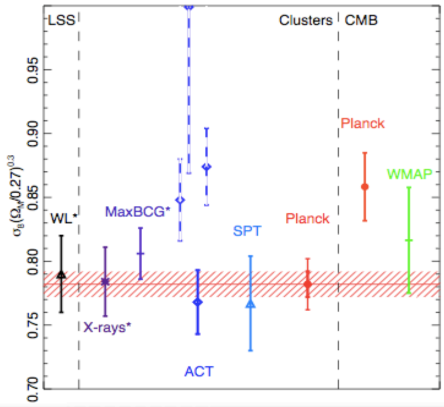
SMC: Limitations of a 6 parameter model...

Comparing primary CMB with other datasets

- Higher values of Ω_m, σ_8 in *Planck* CMB analysis
- 3σ tension
- More general tension between clusters and CMB ?



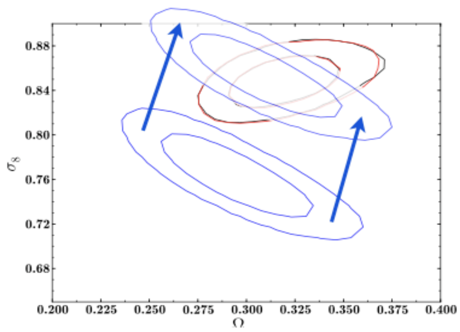


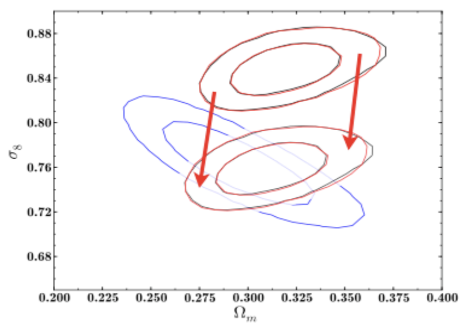


SMC: Limitations of a 6 parameter model...

Comparing primary CMB with other datasets

- Getting higher σ_8 from clusters
  - Change scaling
  - Change bias
  - Account for missing clusters
- Getting lower σ_8 from CMB
  - Change initial power spectrum
  - Change transfert function





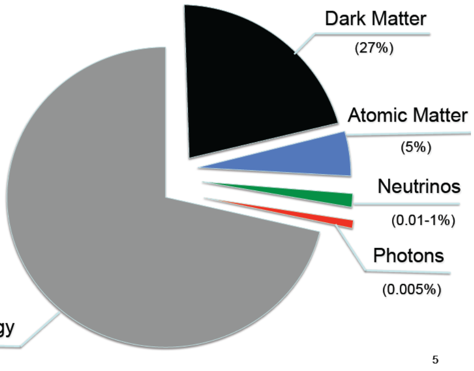
Planck Legacy: A new baseline cosmological model?

The (new) concordance model: ΛCDM + massive neutrinos

From: Planck collaboration. XIII (2015)

Parameter	TT	TT+lensing	TT+lensing+ext	TT, TE, EE	TT, TE, EE+lensing	TT, TE, EE+lensing+ext
Ω_K	$-0.052^{+0.049}_{-0.055}$	$-0.005^{+0.016}_{-0.017}$	$-0.0001^{+0.0054}_{-0.0052}$	$-0.040^{+0.038}_{-0.041}$	$-0.004^{+0.015}_{-0.015}$	$0.0008^{+0.0040}_{-0.0039}$
Σm_ν [eV]	< 0.715	< 0.675	< 0.234	< 0.492	< 0.589	< 0.194
$N_{\rm eff}$	$3.13^{+0.64}_{-0.63}$	$3.13^{+0.62}_{-0.61}$	$3.15^{+0.41}_{-0.40}$	$2.99^{+0.41}_{-0.39}$	$2.94^{+0.38}_{-0.38}$	$3.04^{+0.33}_{-0.33}$
$Y_{\rm P}$	$0.252^{+0.041}_{-0.042}$	$0.251^{+0.040}_{-0.039}$	$0.251^{+0.035}_{-0.036}$	$0.250^{+0.026}_{-0.027}$	$0.247^{+0.026}_{-0.027}$	$0.249^{+0.025}_{-0.026}$
$dn_s/d\ln k$	$-0.008^{+0.016}_{-0.016}$	$-0.003^{+0.015}_{-0.015}$	$-0.003^{+0.015}_{-0.014}$	$-0.006^{+0.014}_{-0.014}$	$-0.002^{+0.013}_{-0.013}$	$-0.002^{+0.013}_{-0.013}$
$r_{0.002}$	< 0.103	< 0.114	< 0.114	< 0.0987	< 0.112	< 0.113
w	$-1.54^{+0.62}_{-0.50}$	$-1.41^{+0.64}_{-0.56}$	$-1.006^{+0.085}_{-0.091}$	$-1.55^{+0.58}_{-0.48}$	$-1.42^{+0.62}_{-0.56}$	$-1.019^{+0.075}_{-0.080}$

$$\sum m_\nu = 0.16^{+0.08}_{-0.11}\,\text{eV} \quad (\textit{Planck}\ \text{TT+lowP+aggressive lensing + BAO; 68\%})$$



SMC: Particle and Event horizons

Consider light travelling along radial ($d\theta = d\phi = 0$) geodesics in a FLRW metric (c=1):

$$\begin{aligned} ds^2 &= dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \\ &= dt^2 - a^2(t)\left[d\chi^2 + f_k(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right], \end{aligned}$$

written in a **conformal** way with the introduction of the **conformal time** $d\tau = dt/a$

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2\right]$$

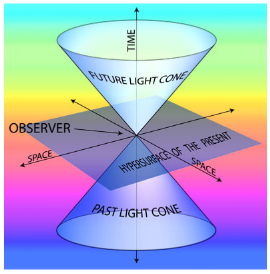
(with $d\chi = dr$ for flat geometries). **Light rays travel along null** ($ds^2 = 0$) **geodesics,** so:

$$d\chi = \pm d\tau$$

From integrating this we can define the notions of:

- Particle horizon: $\chi_{\rm ph}(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{\mathrm{d}t}{a(t)}$ with $t_i = 0$
- Event horizon: $\chi_{\rm eh}(\tau) = \tau_f - \tau = \int_{t}^{t_f} \frac{\mathrm{d}t}{a(t)}$ with $t_f = \infty$

SMC: Particle and Event horizons



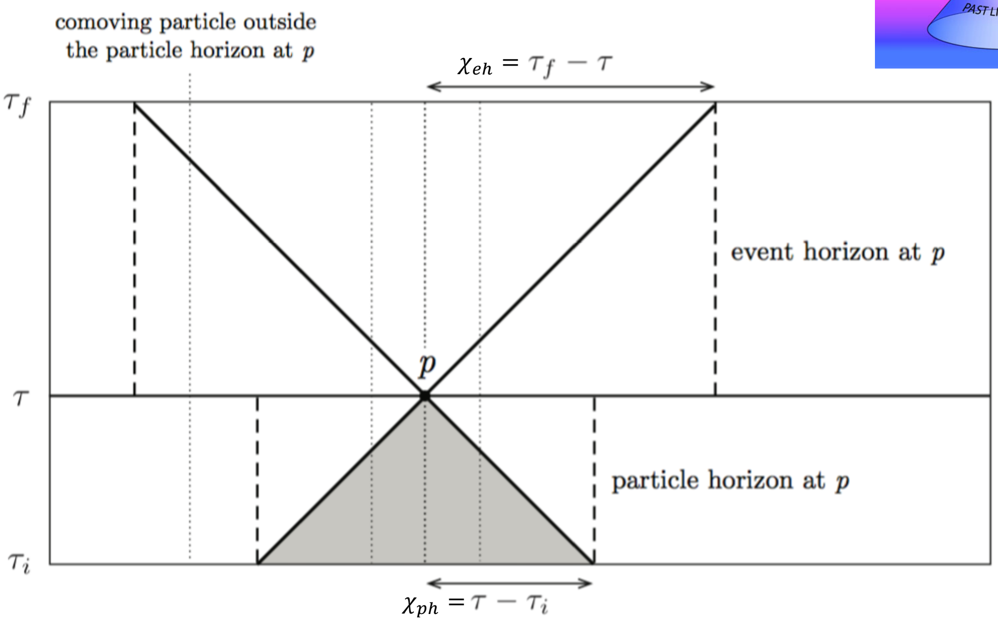


Figure 2.1: *Spacetime diagram illustrating the concept of horizons. Dotted lines show the worldlines of comoving objects. The event horizon is the maximal distance to which we can send signal. The particle horizon is the maximal distance from which we can receive signals.*

SMC: distances, angular sizes and volumes

- Comoving coordinate distance:

(*also computed using photons that travel along null geodesics*, $ds^2 = 0$, *with* $d\theta = d\phi = 0$)

$$ds^2 = c^2\,dt^2 - a(t)^2 \frac{dr^2}{1 - kr^2} = 0 \quad \longrightarrow \quad \boxed{\int_{r_0}^{r} \frac{dr}{\sqrt{1 - kr^2}} = c \int_{t_0}^{t} \frac{dt'}{a(t')}}$$

- Proper (physical) distance:

$$\boxed{d(t) = a(t) \int_{r_0}^{r} \frac{dr}{\sqrt{1 - kr^2}} \equiv \int_{r_0}^{r} \sqrt{|g_{rr}|} = a(t) c \int_{t_0}^{t} \frac{dt'}{a(t')}}$$

From:

for a $\Omega_\Lambda = 0$ universe this gives:

$$d\ (t) \simeq \frac{2}{3w+1} \frac{c}{H_0} \Omega_{w0}^{1/2} \left(\frac{a}{a_0}\right)^{3(1+w)/2} = 3\frac{1+w}{1+3w} ct$$

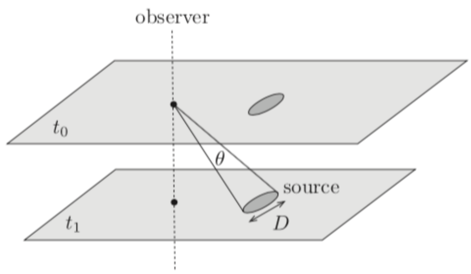
SMC: distances angular sizes and volumes

- Angular size of a region at a given time:

$$\theta = \frac{D}{d_A(t)}$$

where

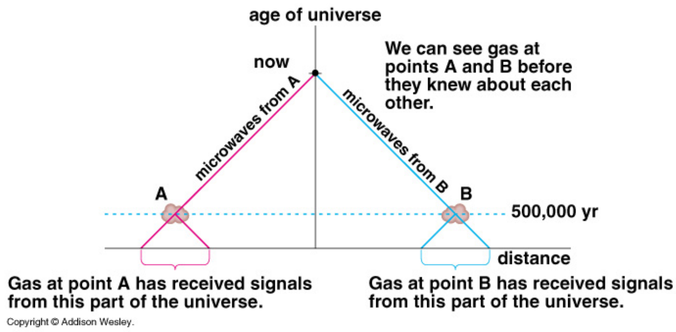
$$d_A(t) = a(t)\int_{r_0}^{r}\frac{dr}{\sqrt{1-kr^2}} \equiv \int_{r_0}^{r}\sqrt{|g_{rr}|} = a(t)c\int_{t_0}^{t}\frac{dt'}{a(t')}$$



Angular size of the **particle horizon** at a given time for a critical density universe $(\Omega_\Lambda = 0)$

From:

$$\theta_H \simeq 2\tan\frac{\theta_H}{2} = \frac{\Omega_0^{3/2}\sqrt{1+z}}{\Omega_0 z + (\Omega_0 - 2)\left(\sqrt{1+\Omega_0 z} - 1\right)}$$



SMC: distances, angular sizes and volumes

- Hubble length:

$$R_H(t) = \frac{c}{H(t)} = \frac{3(w+1)}{2}ct$$

where the last equality holds for a critical density universe $\Omega=1$

- Physical volume element:

$$dV = \sqrt{|g|}\,dr\,d\theta\,d\phi = (ar)^2\frac{a\,dr}{\sqrt{1-kr^2}}\,d\Omega$$

$$\frac{dV}{d\Omega\,dz} = \frac{c}{H(z)}\frac{(a_0 r)^2}{(1+z)^3} = \frac{c}{H_0}\frac{d_A^2}{\mathscr{H}(z)(1+z)}$$

where:

$$\mathscr{H}(z) = H(z)/H_0$$

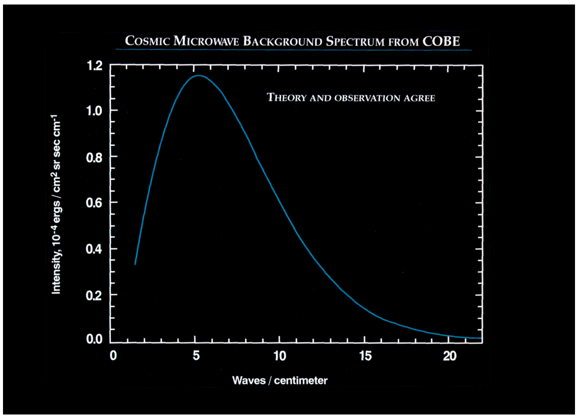
Problems of the FLRW models as a sole ingredient of the SMC

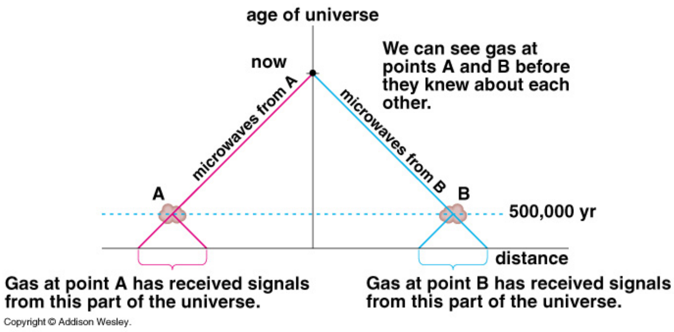
The Horizon Problem

At high redshift ($z \gg 1$):

$$\theta_H \simeq \frac{180}{\pi}\sqrt{\frac{\Omega_0}{z}}\,\mathrm{deg}$$

there are ~54000 causal disconnected angular areas in the CMB sky. So, why the CMB has a thermal spectrum with a so uniform temperature in all directions (2.725 °K)?





The Flatness Problem

From the Friedmann Equation, written at early times:

$$|\Omega(t) - 1| = \frac{|k|}{a^2(t)H^2(t)} = \frac{|k|}{\dot{a}^2(t)}$$

is a decreasing function of time:
So as $t \rightarrow 0$, $\Omega \rightarrow 1$

decreases as time approaches the big bang instant.

This means that as we go back in time the energy density of universe has to be extremely close to critical density. For t=1e-43 s (Planck time) Ω should deviate no more than 1e-60 from the unity.

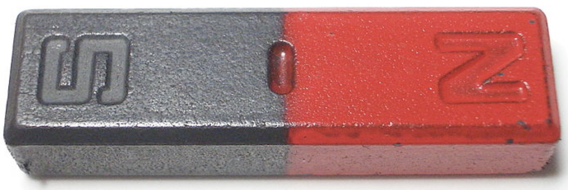
Why has the universe to start with Ω(t) so close to 1?

The Monopoles & other relics Problem

Particle physics predicts that a variety of **"exotic" stable particles**, such as the magnetic monopoles, should be produced in the early phase of the Universe and remain in measurable amounts until the present.

No such particles have yet been observed. Why?

This either implies that the predictions from particle physics are wrong, or their densities are very small and therefore there's something missing from this evolutionary picture of the Big Bang.

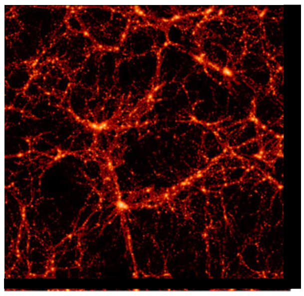


The Origin of Perturbations Problem

Locally the universe is not homogeneous. It displays a complex hierarchical pattern of galaxies, clusters and super clusters.

What's the origin of cosmological structure?
Does it grew from gravitational instability?
What is the origin of the initial perturbations?

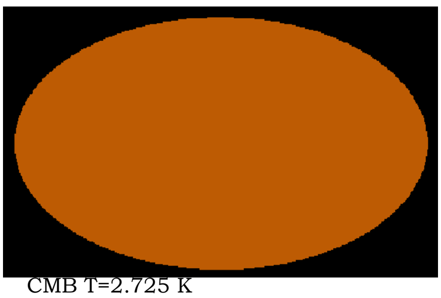
Without a mechanism to explain their existence one has to assume that they ``were born'' with the universe already showing the correct amplitudes on all scales, so that gravity can correctly reproduce the present-day structures?



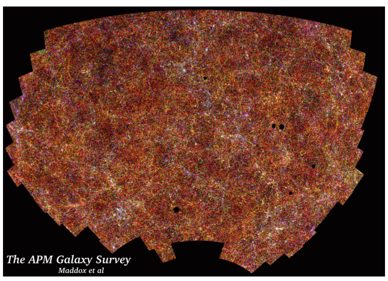
The homogeneity and isotropy Problem

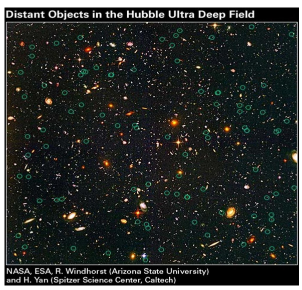
Why is the universe homogeneous on large scales? At early times homogeneity had to be even more "perfect".

The FLRW universes form a very special subset of solutions of the GR equations. So *why nature "prefers" homogeneity and isotropy from the beginning as opposed to having evolved into that stage?*



CMB T=2.725 K





The Theory of Inflation…

Inflation can be defined as

$$\text{Inflation} \Leftrightarrow \ddot{a} > 0 \Leftrightarrow \frac{d}{dt}\left(cH^{-1}/a\right) < 0.$$

This happens when

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) \quad \Rightarrow \quad p < -\rho c/3$$

… this continues in Chapter 9