# The Homogeneous Universe

## The metric and its degrees of freedom

## Expansion Dynamics

We saw that it was possible to introduce the concepts of expansion and redshift in a general way, without specifying the theory of gravitation (that drives the expansion) → they are not necessarily a consequence of general relativity.

Let us now try to derive the equations of movement of the gravitational expansion, i.e., the equations for the evolution of the scale factor a(t), or the Hubble function H(t), using Newtonian mechanics.

**Is this possible?**

*Let us consider the homogeneous and isotropic Universe as a sphere of radius r that expands radially and is filled by a homogeneous cosmological fluid with density ρ.*

i) Energy conservation (kinetic + potential)

$$E_k = v^2 / 2 \qquad E_V = - G\, M / r$$

The mass relates to the cosmological fluid density:

Mass inside of the sphere of radius $r(t)$ is $M = \frac{4}{3}\pi r^3(t)\, \rho(t)$

Energy conservation + Newtonian gravity yields:

$$\frac{\dot{r}^2}{2} - \frac{4}{3}\pi G r^2 \rho = c^{te}$$

We can introduce the scale factor by considering this equation in comoving coordinates:

$$\frac{\dot{a}^2 x^2}{2} - \frac{4}{3}\pi G a^2 x^2 \rho = K^{te}$$

$$\dot{a}^2 - \frac{8}{3}\pi G a^2 \rho = \frac{2}{x^2} c^{te}$$

$$\dot{a}^2(t) = \frac{8\pi G}{3}\rho(t)\, a^2(t) - K$$

$$\left[ H^2(t) = \frac{8\pi G}{3}\rho(t) - \frac{K}{a^2} \right]$$

Friedmann's equation

The constant is K = 2E / (x^2)

where E is the total energy of the Universe and x is the comoving coordinate of the surface of the "Newtonian Universe" - the Hubble radius.

So we get Friedmann's equation, identical to the one derived in General Relativity (although in GR the constant K has a different and well-defined meaning: it is the curvature of space.

ii) To solve Friedmann's equation for a(t) we need to know the source of gravity, i.e., the mass of the Universe, i.e., we need to know ρ(t).

The evolution of ρ(t) is constrained by the conservation of mass (the continuity equation in the Newtonian approach).

For this, let us consider the 1st law of thermodynamics for the expanding cosmologcial fluid:

$$dU = -p\, dV$$

(there is no heat dissipation to the exterior of the expanding sphere that constitutes the whole Universe)

The energy of the Universe is

$$M = \frac{4}{3}\pi r^3 \rho \quad \Rightarrow \quad U = \frac{4}{3}\pi r^3 \rho c^2$$

So, $\cancel{\frac{4}{3}\pi}\, d(r^3 \rho c^2) = -p\, \cancel{\frac{4}{3}\pi}\, d(r^3)$

Comoving coordinates $\Rightarrow$ $d(a^3 x^3 \rho c^2) = -p\, d(a^3 x^3)$

$(v = \dot{a} x,\ r = a x)$

$$\dot{\rho} a^3 + \rho\, d a^3 = -p\, d a^3$$

$\frac{d(a^3)}{dt} = 3 a^2 \dot{a}$

or $\dot{\rho} a^3 + \rho\, 3 a^2 \dot{a} = -p\, 3 a^2 \dot{a}$

=> $\dot{\rho} + \frac{3 a^2 \dot{a} (\rho + p)}{a^3} = 0$

or $\left[\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = 0\right]$ eq

This is identical to the conservation equation derived in GR.

iii) Finally, to find the equation of movement of the expanding Universe, we consider the 2nd law of Newton:

$$\ddot{r} = -\frac{GM}{r^2} = -\frac{G}{r^2}\frac{4}{3}\pi r^3 \rho$$

$$\Rightarrow \ddot{r} = -\frac{4\pi G}{3}\rho r$$

$$\Rightarrow \boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho}$$

This equation is different from its GR counterpart, which also involves pressure (in GR pressure is source of gravity, while in Newtonian gravity it is not).

However, if we combine the 1st Friedmann equation with the conservation equation that we found, we obtain the following:

(differentiate Friedmann's equation + use conservation equation → eliminate dρ/dt and get an equation for ä :

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - K$$

$$\rightarrow \quad 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left(\dot{\rho} a^2 + \rho\, 2 a \dot{a}\right)$$

$$\Leftrightarrow \quad 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left(-3\frac{\dot{a}}{a}(\rho+p)\, a^2 + \rho\, 2 a \dot{a}\right)$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\dot{a}\, a\left(-3\rho - 3p + 2\rho\right)$$

$$2\cancel{\dot{a}}\ddot{a} = -\frac{8\pi G}{3}\cancel{\dot{a}}\, a\,(\rho + 3p)$$

$$\left[\ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)\ \right]$$

This is the 2nd Friedmann equation, also called Raychadhuri equation and now it is identical to the one derived in GR.

**How was this possible? From where did the we get pressure in our Newtonian description?**

It came from using the first law of thermodynamics to get the continuity equation, i.e., we used a conservation of energy instead of conservation of mass. In other words, we wrote U from ρ, implicitly assuming mass-energy equivalence.

In conclusion: Newtonian gravity does not find the correct evolution equations. We could however find them using *relativistic Newtonian gravity*, i.e., Newtonian gravity + special relativity.

Note that relativistic Newtonian gravity is different from General Relativity. It is just Newtonian physics + the assumption that the energy is source of gravity. It does not include the concept of curvature, which also contributes to gravity.

# The Robertson-Walker metric

GR of course is not just Newtonian theory + special relativity, and so the homogeneous Universe is not just a connected set of large homogeneous Newtonian regions. GR introduces new global properties, such as curvature.

GR describes space-time as a 4D manifold whose metric tensor $g_{ab}$ is considered a dynamical field. The dynamics of this field is described by Einstein's equations, which relates the Einstein tensor to the energy-momentum tensor of the matter.

The metric keeps its traditional role of determining distances and local inertial frames, but it also plays the dynamic role of a gravitational potential, determining the geodesics (trajectories) of the space-time.

Two neighboring events in space-time with coordinate difference $dx^a$ are separated by $ds$,

$$ds^2 = g_{ab}\,dx^a dx^b = \sum_{a,b=0}^{3} g_{ab}\,dx^a dx^b$$

Let us derive the metric of a homogeneous and isotropic space-time — for observers that follow the mean motion of matter and radiation in the universe (the expansion) – the comoving observers; they have $(r, \theta, \varphi)$ = constant

The most general metric can be written as

$$ds^2 = -g_{00}\,dt^2 + 2 g_{0i}\,dx^i dt + \sigma_{ij}\,dx^i dx^j$$

The general form of the metric is fixed by the cosmological principle:

Isotropy implies $g_{0i} = 0$ $\rightarrow$ otherwise this would define a prefered direction.

(static)

The comoving observers are synchronized (comoving with the fluid)

$\Downarrow$

single time for all can be redefined and remove $g_{00}$

$$-g_{00}\,dt^2 \longrightarrow dt^2$$

Homogeneity implies there are only functions of $\underline{t}$ :

$$ds^2 = -dt^2 + a^2(t)\; dl^2(x, y, z)$$

$\downarrow$

the time-dependence may be factored out

as the scale factor

Homogeneity and isotropy leads to $\delta_{ij}$ with spherical symmetry

Does this mean that the metric is completely fixed by the cosmological principle, or are there some **degrees of freedom**?

## Scale factor

There is freedom in the time-evolution of the metric, defined by the scale factor a(t).

a(t) is not fixed by the cosmological principle (i.e., by the fact that the metric is RW). It is free to be determined by the dynamics of the Universe (the differential equations provided by GR: the Einstein equations).

**Note that in general in a physical system there are several levels of "freedom"**:

i) **Symmetries** impose general constraints (in this case determine the type of metric) but leave the physical functions f(t,x) free (in this case a(t) ).

ii) **Differential equations** provided by the theory (in this case the Einstein equations of GR) are solved to get a solution for f(t,x) (in this case the functional form a(t) ). The solution always include integration constants, which implies that the solution function can only be determined up to a constant. → This fixes the "model".

iii) **Initial conditions**, i.e., conditions at the borders of the (t,x) domain, that can be time or spatial, provide the absolute value of a physical function (or its derivatives) at a certain point of its domain (t,x) (in this case a(0), $a(t_0)$, $å(t_0)$ ). The initial conditions may be imposed (in this case a(0) = 0, $a(t_0) = 1$), or determined by observations or experiments (in this case, the value of $å(t_0)$ , i.e., $H_0$ ). → This fixes the "cosmology".

The “initial condition” absolute value of a cosmological physical function (or its derivatives) at a certain point of its domain (t,x) is what is called a **cosmological parameter**.

There are also **phenomenological models**. In these cases, the cosmological functions are proposed empirically and do not come from a solution of a differential equation of a theory. In these cases, the functional form of a cosmological function can also be parameterized → This introduces more cosmological parameters, besides the ones strictly related with initial conditions (that would determine the amplitude of the function).

## Solution for a(t) (model-dependent)

a(t) can thus be determined in a model-dependent way, i.e., by solving the relevant differential equations, and its amplitude parameterized by an initial condition.

a(t) is usually parameterized using its amplitude at z=0 (i.e. today at $t_0$).

However, differently from most functions, $a_0$ is not find by observations but it is just fixed by convention: $a_0 = 1$.

This means that all possible a(t) solutions are “distorted”, i.e., are forced to reach the value a=1 today.

This implies that the parameter $a_0$ is not useful to distinguish the various cases (the various cosmologies).

Various cosmologies are instead distinguished by looking at the slopes with which the functions a(t) reach a=1. In other words, the relevant parameter is

$\mathring{a}$ (z=0) (notation: $\mathring{a}$ is da/dt).

Since $a_0$=1, we have $\mathring{a}_0$=$H_0$ → the Hubble constant is a free parameter of the model (the parameter related to the initial condition of a(t)) and its value (determined from observations) will help in defining the cosmology.

(It is the first cosmological parameter we encounter).

## Solution for a(t) (model-independent)

Note that $\mathring{a}_0$ would also be the first parameter in the Taylor expansion of a(t) around $t_0$

In fact, if we would Taylor expand a(t) around $t_0$, introducing a potentially infinite number of parameters (the values of all-orders derivatives at $t_0$), a(t) would be fully described (in the local Universe).

If we could design a way to measure all those parameters individually, we would then reconstruct a(t) with no need for the evolution equations → in a **model-independent way**.

The set of all those parameters - called the **cosmographic parameters** - would contain the same information as the set of differential equations.

There are attempts to do this, and the lower-order parameters are defined:

$$a(t) = a(t_0) + \dot{a}|_{t_0}(t-t_0) + \frac{1}{2}\ddot{a}|_{t_0}(t-t_0)^2 + \sigma(t-t_0)^3$$

$$\Rightarrow \boxed{a(t) = \left[1 + H_0(t-t_0) + \frac{1}{2}\frac{\ddot{a}}{a}\Big|_{t_0}(t-t_0)^2 + \ldots\right] a(t_0)} \quad (a(t_0)=1)$$

The first-order term is a **velocity** term → the **Hubble constant** H_0

The second-order term is an **acceleration** term → the **deceleration parameter** q_0

(historically defined with a minus sign, hence the name deceleration instead of acceleration)

$$q_0 = -\frac{\ddot{a}}{a}\Big|_{t_0}\frac{1}{H_0^2}$$

We can continue the expansion to higher-orders, defining a series of parameters:

obtaining $$p_0^{(m)} = \frac{d^m a}{dt^m}\frac{1}{a}\Big|_{t_0}\frac{1}{H_0^m}$$

$$a(t) = a_0\left[1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \frac{1}{3!} j_0 H_0^3 (t-t_0)^3 + \frac{1}{4!} s_0 H_0^4 (t-t_0)^4 + \mathcal{O}(t-t_0)^5\right]$$

The next parameter is called the **jerk** $j_0$ (that corresponds to the change of acceleration, well-known in mechanics, felt for example when changing gears in a car)


The next orders parameters are called **snap** $s_0$, **crackle** $c_0$ and **pop** $p_0$ (taken from the names of the characters in these cereals!).

This approach to cosmology is called **cosmography**: the Universe is described in terms of **its dynamical quantities** with no need to solve the evolution equations (the Einstein equations) and no need define the sources of gravity (in the energy-momentum tensor).

## Curvature

Let us look at the spatial part of the metric that, due to the cosmological principle, must be spherically symmetric.

Usually, spherical symmetry is written like,

$$ds^2 = -dt^2 + a^2(t)\left[\underbrace{dr^2}_{\text{radial displacement}} + \underbrace{r^2}_{\text{radius of the sphere}}\overbrace{(d\theta^2 + \sin^2\theta\, d\varphi^2)}^{d\Omega^2}\right]$$

$$(d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2)$$

But as we know, gravity in GR is the space-time curvature,

and space itself (not only space-time) may also be curved.

So we need to include the additional degree of freedom of curvature in the metric.

How to do this?

To do this, we consider the following feature of a curved space:

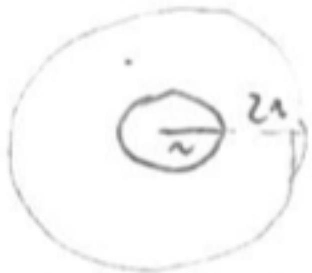
**In a curved (3D) space, the area of a spherical surface at distance r from the origin is not $4\pi r^2$ but it is smaller (positive curvature) or larger (negative curvature) than $4\pi r^2$ .**

So the spatial part of the metric may be written more generally as,

$$dl^2 = u^2(r)\, dr^2 + v^2(r)\, d\Omega^2$$

Only for flat space is $v^2(r) = r^2$ :

If the radius is the same as the radial coordinate then,



For positive curvature, we need v(r) < r or u(r) > 1

If $v(r) < r$ $\Rightarrow$ When making a displacement $ds = dr$, we move to a point belonging to a sphere with radius $< r^2$

How is this possible geometrically? Introducing curvature!

(Needs higher dimension to picture the curvature)

$u$ move $r$ but the radius does not increase proportionally to $r^2$

This is positive curvature

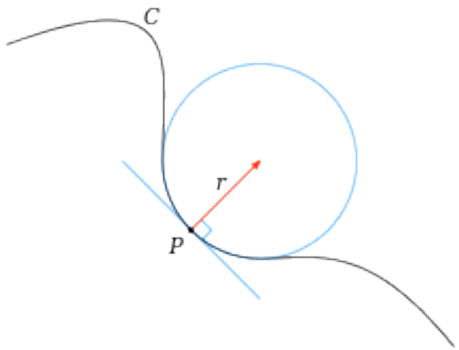
or alternatively if $u(r) > 1$ $\Rightarrow$ $dr$ means an effectively larger displacement, but the radius just grows with $r$.

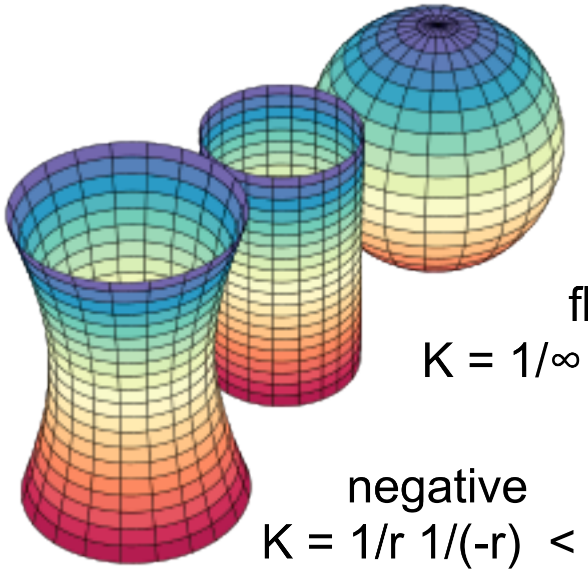
For negative curvature, we need v(r) > r or u < 1

$u>1$ { Radius increases less than the radial displacement is positive curvature

$u<1$ { Radius " more " " is negative curvature

The curvature K of a surface is the product of its **principal curvatures**.
Each principal curvature is 1/r, where r is the radius of the circle in the normal plane, that best fits the curvature.



positive
$K = 1/r \; 1/r \; > 0$

flat
$K = 1/\infty \; 1/\infty \; = 0$

negative
$K = 1/r \; 1/(-r) \; < 0$

We need now to find out from all possible functions v(r) and u(r) which are the ones that verify spherical symmetry in curved spaces. Or in other words, **what is the constraining condition of a spherical surface**: (in 3D flat space the points on a spherical surface verify the constraint $dx^2 + dy^2 + dz^2 = \text{constant} = dl^2$).

In a curved 3D space we need to consider that the space "curves into a 4D flat space", i.e., a curved 3D volume may be **embedded** as a 3D-surface in flat 4D space.

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2$$

The spherical 3D surface is then a constraint on the 4D coordinates. It is simply the surface with points at a fixed radius R from the center of the 4D space:

$$R^2 = \underbrace{x^2 + y^2 + z^2}_{r^2 \to (\text{raio } 3D)} + w^2$$

where the combination $x^2+y^2+z^2$ corresponds to the radius of the 3D volume in the 3D space.

So the points in the 4D space such that $\mathbf{w^2 + r^2 = constant}$, are the ones that define the spherical surface.

The line element of that surface is thus $dl^2 = dx^2 + dy^2 + dz^2 + (dw/dr)^2 \, dr^2$ .

Since the w coordinate of the points on the spherical surface is $w^2 = R^2 - r^2$, we get:

$$w^2 = R^2 - r^2 \Rightarrow 2w\,dw = 0 - 2r\,dr \Rightarrow dw = \frac{-r}{w}\,dr \Rightarrow \left[ dw = -r \frac{dr}{\sqrt{R^2 - r^2}} \right]$$

Then $dl^2 = dx^2 + dy^2 + dz^2 + \frac{r^2}{R^2 - r^2}\,dr^2$ is the solution

changing to spherical coordinates:

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$

$$(d\theta^2 + \sin^2\theta \, d\varphi^2 = d\Omega^2)$$

$$dl^2 = dr^2 + r^2 d\Omega^2 + \frac{r^2}{R^2 - r^2}\,dr^2$$

$$\Rightarrow dl^2 = \frac{R^2 - r^2 + r^2}{R^2 - r^2}\,dr^2 + r^2 d\Omega^2$$

$$dl^2 = \frac{1}{1 - \frac{r^2}{R^2}}\,dr^2 + r^2 d\Omega^2$$

Note that $1/R^2$ is the curvature of a spherical surface of radius R:   K = 1/R 1/R
and so the line element is,

$$dl^2 = \frac{1}{1 - Kr^2} dr^2 + r^2 d\Omega^2$$

**It corresponds then to a change in the function u(r) , which is $1/(1-Kr^2)$ instead of u(r)=1, while v(r) is kept as $r^2$**

We can also consider that the constant in the condition $w^2 + r^2$ = constant is negative, to allow for the case of negative curvature.

Note that a negative $R^2$ does not mean an imaginary radius. It simply means that the equivalent radius of one of the principal components is positive and the other is negative (meaning it curves to the opposite side).

This scenario allows for a different solution:

$$-R^2 = r^2 + w^2 \Rightarrow dw = -r\frac{dr}{w} = \frac{-r\,dr}{\sqrt{-R^2 - r^2}}$$

Then.

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2$$

$$dl^2 = dx^2 + dy^2 + dz^2 + \frac{r^2 dr^2}{-R^2 - r^2}$$

$$= dr^2 + r^2 d\Omega^2 + \frac{r^2 dr^2}{-R^2 - r^2}$$

$$= \frac{-R^2}{-R^2 - r^2} dr^2 + r^2 d\Omega^2$$

$$= \frac{1}{1 + \frac{r^2}{R^2}} dr^2 + r^2 d\Omega^2 \quad \text{or} \quad dl^2 = \frac{1}{1+Kr^2} dr^2 + r^2 d\Omega^2$$

**and now the functions are $u(r) = 1/(1+Kr^2)$ and $v(r) = r^2$**

**There is thus a degree of freedom associated with the curvature, since there are several possibilities for the curvature of a spherical symmetric space**,

Area

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right]$$

displacement
as function of area

that is encapsulated in the curvature parameter K
**(K>0: positive curvature; K=0: flat; K<0: negative curvature)**

It is also usual to write the metric using the dimensionless curvature parameter k ("small k"):

$$dl^2 = \frac{1}{1-kr^2}\,dr^2 + r^2 d\Omega^2$$

In this case, the curvature types are: **k = 1: positive curvature; k = 0: flat; k = -1: negative curvature.**
Since k is dimensionless, all values of R are equivalent. However in this case, the element $dr^2$ / ($1-kr^2$) no longer has dimensions of length, and so the scale factor a(t) needs to have units of length and is no longer dimensionless.

Now, the derivation can also be done in a way that the curvature is encapsulated in the function v(r) instead of in u(r), i.e., u(r) is kept as r, and is v(r) that changes. The result is:

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + f_K^2(r)\, d\Omega^2\right]$$

This is the most usual way to write the RW metric

with

$$f_K(\chi) = K^{-1/2}\sin(K^{1/2}\chi)\,, \quad (K > 0)$$

$$f_K(\chi) = \chi\,, \qquad (K = 0)$$

$$f_K(\chi) = (-K)^{-1/2}\sinh[(-K)^{1/2}\chi]\,, \quad (K < 0)$$

## Distances

Once we know the metric of the space-time, we can define distances.

**In general there is no unique (or correct) definition of distance**.

This happens for two reasons:

- the existence of curvature
- the existence of expansion

### Effect of curvature

As we saw, the existence of curvature introduces two different radial quantities in the spherical symmetric metric: the radial displacement u(r) and the radius of spherical surfaces v(r).

Both are legitimate ways to define a radial distance.

Looking at the metric,

$$ds^2 = -dt^2 + a^2(t)\,\underbrace{\left(d\chi^2 + f_K^2(\chi)\,d\Omega^2\right)}_{\text{Comoving element}}$$

[note that the comoving radial coordinate (i.e. in the comoving frame not affected by the expansion) is usually written with the letter χ instead of r.]

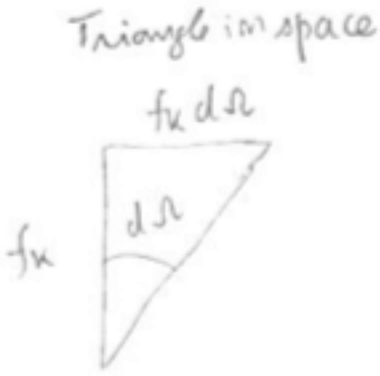
we can infer that those two distances are $\Delta\chi$ and $\Delta f_K(\chi)$ and they define the:

- comoving distance, $d_C$, also called the **line-of-sight comoving distance** [ $\Delta\chi$ ]

and

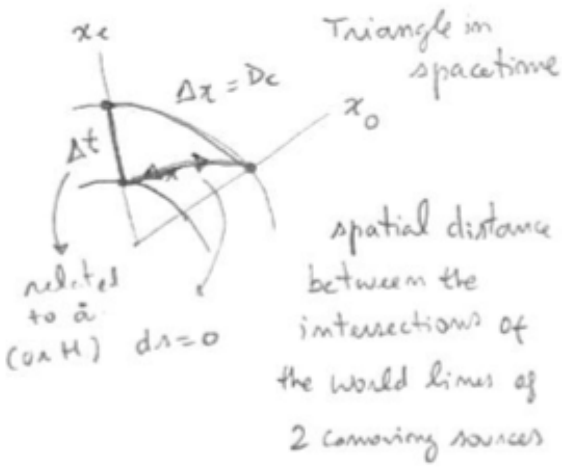
- comoving angular-diameter distance, $d_M$ , also called **transverse comoving distance** or **movement distance** [$\Delta f_K(\chi)$ ]

Since $f_K(\chi)$ is the radius of a spherical surface at comoving distance χ from the origin, it is the distance that relates an intrinsic diameter with an angular aperture, hence the name angular diameter distance.




But note that it is a radial distance and not a transverse distance.

To compute these distances from the metric elements, we can consider the **space-time triangle** defined by the line-element (not considering angular variation)




$$ds^2 = -dt^2 + a^2(t)\, dx^2 \quad ; \quad ds^2 = 0$$

(the line element is like the theorem of Pythagoras)

$$\Delta\chi = \int_{t_e}^{t_0} \frac{1}{a(t)}\, dt = \int \frac{1}{a} \frac{dt}{da}\, da = \int_{a_e}^{a_0} \frac{1}{a^2} \left( \frac{a}{\dot{a}} \right) da = \int \frac{1}{H(a)} \frac{1}{a^2}\, da$$

We can also use the redshift as a variable in the integral:

$$z = \frac{1}{a} - 1 \Rightarrow \frac{dz}{dt} = -\frac{\dot{a}}{a^2}$$

$$\Rightarrow \frac{dz}{da} = -(1+z)^2$$

The resulting expressions for the **comoving distance** are,

$$D_c = \frac{(c)}{H_0} \int_{z_0}^{z} \frac{dz}{E(z)} \qquad \text{or} \qquad D_c = \frac{(c)}{H_0} \int_{a}^{a_0} \frac{da}{E(a)\, a^2}$$

where the Hubble function is written as $H(z) = H_0\, E(z)$. This is used to separate the functional form and the $H_0$ parameter value.

The actual value of the comoving distance between two points in the Universe depends on the cosmological model of that Universe. It requires the knowledge of $E(z)$ (obtained from the Einstein equations for the particular model), and of the cosmological parameter $H_0$ (obtained from observations).

It is usual to absorb the dependence on $H_0$ into the units of distance, i.e, **cosmological distances are usually given in $M_{pc}/h$, instead of $M_{pc}$** (megaparsec), hiding the explicit dependence on the unknown value of $H_0$

The **comoving angular-diameter distance,** $\Delta f_K(\chi)$, or $d_M$ is computed from $d_c$ using $f_K(\chi)$, and is:

$$\frac{c}{H_0}\sin\left(d_C\frac{H_0}{c}\right) \quad \text{positive curvature}$$

$$d_C \quad \text{flat}$$

$$\frac{c}{H_0}\sinh\left(d_C\frac{H_0}{c}\right) \quad \text{negative curvature}$$

- volume distance, $d_V$

The so-called volume distance is not another fundamental distance defined from the metric. It is essentially a weighted geometric combination of the line-of-sight (1/3) and transverse (2/3) comoving distances.

It was introduced in the analysis of the first detection of baryon acoustic oscillations as a way to take into account the error produced by the Alcock-Paczynski effect.

It is defined as:

$$d_V(z) = \left[d_M^2(z)\frac{c\,z}{H(z)}\right]^{1/3}$$

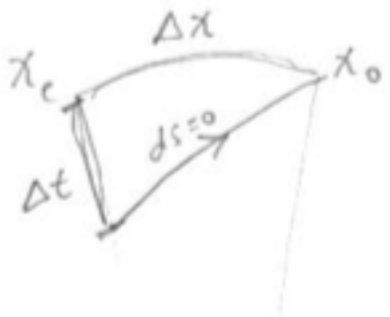
## Effect of expansion

The existence of expansion (and the fact that light propagates at finite speed) implies that the emission from a source and the detection by an observer are never at the same instant of the Universe (or at the same value of the scale factor).
Hence, they are never in the same comoving frame and **the two comoving distances are not measurable distances, and may be considered "non-physical"**.

Several "physical" non-comoving distances can be defined:

- Light-travel distance, $d_T$

This is the segment $\Delta t$ in the spacetime triangle




$$ds^2 = -dt^2 + a^2(t)\, d\chi^2$$

$$ds^2 = 0 \Rightarrow \underline{dt = a^2(t)\, d\chi^2}$$

It can be computed from,

$$\frac{\dot{a}}{a} = H_0 E(a) \Rightarrow da = a H_0 E(a)\, dt$$

$$\Rightarrow t(a) = \frac{1}{H_0} \int_{a_e}^{a_0} \frac{da}{a E(a)}$$

Note that we saw that $D_c = \Delta x = \frac{1}{H_0} \int \frac{1}{a^2 E(a)} da$

$\Rightarrow$ there is a factor $\underline{a}$ between them, i.e.

$$\underbrace{dt = a\, dx}$$ $\rightarrow$ which is exactly what we saw above in the spacetime triangle

$$D_T = c \int a(t)\, dx$$ $a(t)$ varies as the light propagates

and it is a potentially measurable distance (if the time of emission is known)

- Proper distance, $d_P$

The proper distance is similar to the light-travel distance, but for a fixed value of the scale factor (the one of the observer), and thus it is not an observable distance, since it is not connected to an actual propagation.

It is given by $d_P = a\, d_C$,

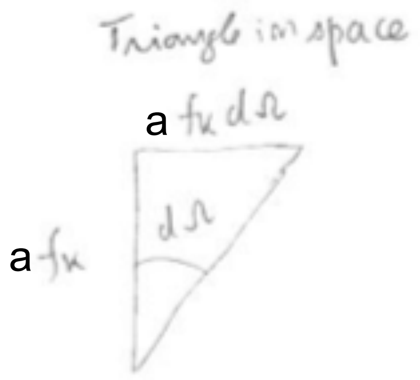
for $a = 1$, $d_P = d_C$

It is the distance that would be measured by a ruler placed between two points. Conceptually it is the simplest distance, and the one that could be thought as "the true distance". However, it is not directly measurable.

- Angular-diameter distance, $d_A$

Consider a large transversal region of the Universe (meaning with all its points at the same redshift) that occupies a solid angle $d\Omega^2$ in the sky.

Like we did for the comoving diameter-angular distance, we can draw the corresponding spatial triangle for this source-observer system:




The difference with respect to that case is that due to the expansion, the intrinsic size of the observed region is not its comoving size (fixed) but its proper size (expanding).

We can thus define the (non-comoving) **angular-diameter distance $d_A$**

which is just: $d_A = a_{source}\, d_M$ (distance from observer at $a_0$=1 to the source at the spherical surface $a_{source}$)

or $d_A = a_{source}\, d_M = a_{source}\, d_C$ for flat space.

We can also define the angular-diameter distance between two points (instead of between a point and the origin) :

$$D_A(z_1, z_2) = a(z_2)\, f_K(\chi(z_1, z_2))$$

For flat $\rightarrow$ $D_A(z_1, z_2) = D_A(z_2) - D_A(z_1)$

**The angular-diameter distance is the cosmological probe measured in observations of standard rulers.**

A **standard ruler** is an object for which the intrinsic size is known (from theory) and for which the angular size can be observed → enabling to determine its distance $d_A$, which in turn contains cosmological information because it is model-dependent (and can be computed from theory).

A useful standard ruler needs to be at a:

reasonably high redshift (to contain cosmological information),

be large (for its size to expand, so cannot be an astrophysical collapsed object; and to occupy a large enough solid angle that can be measured with good precision),

be observable;

**Do such objects exist in the Universe?**

Yes, an **horizon** is a good candidate for this! In particular, the **sound horizon** at recombination (z=1100). The sound horizon of the Universe is observed as a peak in the CMB power spectrum (the **first peak of CMB**) and also as a peak in the matter power spectrum (the baryon acoustic oscillations **BAO peak** - even though the analysis of BAO uses $d_V$ instead of $d_A$).

- Luminosity distance, $d_L$

As we already saw, the luminosity distance is not another fundamental distance from the metric. It is simply a version of the angular-diameter distance corrected for the effects of the redshift on the luminosity.

Consider the emission of light from a source of luminosity L, from which we measure its flux F.

$$F = \frac{L}{4\pi D_M^2} = \text{Flux}$$

note that the distance to be used here is the distance from the source to a spherical surface at $a_0$, i.e. the angular-diameter distance $d_A$ with a=1, which is the comoving angular-diameter distance $d_M$.

Measuring the flux and knowing the intrinsic luminosity enables us to measure the comoving angular-diameter distance.

However, the luminosity in our reference-frame at $a_0$ is not the one emitted at the **rest-frame**. The luminosity is “redshifted” in two different ways:

$L_o = \frac{E_o}{\Delta t_o}$ , but the photons are redshifted by the expansion :

$$E \propto \nu$$

$$\text{so} \quad E_o = \frac{E_e}{1+z}$$

and the unit time dilates with the expansion :

$$\frac{\Delta t_o}{a(t_o)} = \frac{\Delta t_e}{a(t_e)}$$

$$\text{so} \quad \Delta t_o = \Delta t_e \,(1+z)$$

This means that Luminosity is "redshifted" by $L_o = \frac{L}{(1+z)^2}$ or $L = L_o\, a^{-2}$

Luminosity dilutes faster than frequency
$(a^{-2})$ $(a^{-1})$

So in terms of rest-frame luminosity (intrinsic luminosity) we can write,

$$F = \frac{L_e}{4\pi a_0^2 D_M^2 (1+z)(1+z)}$$

The correction factors are absorbed in the definition of a **luminosity distance**, such that:

$$d_L = d_M\,(1+z) \qquad \text{or} \qquad d_L = d_C\,(1+z)\ \ \text{(flat)}$$

or, in function of the angular-diameter distance to the source:

$$d_L = d_A\,(1+z)^2 \quad \rightarrow d_L \text{ seen explicitly as a renormalization of } d_A$$

**The luminosity distance is the cosmological probe measured in observations of standard candles.**

A **standard candle** is an object for which the intrinsic luminosity is known (from theory) and for which the flux can be observed → enabling to determine its distance $d_L$, which in turn contains cosmological information because it is model-dependent (and can be computed from theory).

**Do such objects exist in the Universe?**

Yes, **supernovae of type Ia** are good candidates for this. Even though they do not all have the same intrinsic luminosity (absolute magnitude) as first thought, they can be "standardized" (meaning the observed light-curves can be shifted in a way to renormalize their absolute magnitudes).

# Volume

The metric also defines volumes.

The volume, like the distances, is a geometrical quantity that can be used for cosmological tests.

The comoving volume, $V_C$ , is the volume where the number density of objects that follow the cosmic flow remains constant as the Universe evolves.

Comoving element of Volume:

$$dV_c = \underbrace{dA}_{\text{transversal area}} \; \underbrace{dD_c}_{\text{longitudinal dimension}}$$

dA = (2D) Comoving area = $d\Omega \cdot D_M^2 = d\Omega \, D_A^2 (1+z)^2$

$D_M$ → Comoving transversal distance

and as we saw, $dD_c = \frac{c}{H_0} \frac{1}{E(z)} dz$

So the Volume element is:

(c=1)

$$\left[ dV_C = \frac{D_A^2}{H(z)} \underline{(1+z)^2 \, d\Omega \, dz} \right]$$

It is a ratio between $D_A^2$ and $H(z)$

The integrated volume, from z=0 to z and over the full angular sky is thus,

$$V_C(z) = \iiint dV_c = \frac{4}{3} \pi D_M^3 \quad (z)$$

We can also define the proper volume, $V_P$ , multiplying the comoving volume by $a^3$ :

$$dV_p = \frac{D_A^2}{H(z)} \frac{1}{1+z} d\Omega \, dz$$

**The volume is the cosmological probe measured in number counts observations.**

The **number of objects** (e.g. galaxy clusters) within a volume of the Universe (defined by a an angular size and a redshift size) is counted, usually in bins of a physical property, such as mass (building a **mass function**).
If the mass function is known (from theory), the comparison between predicted and observed number counts → enables to measure the volume $V_C$ which in turn contains cosmological information because it is model-dependent (and can be computed from theory).

# Einstein tensor

The "equations of GR" are the Einstein equations, which are **a set of constraint differential equations that relate gravity with the sources of gravity.**
For this, gravity is represented by the **Einstein tensor,** and the sources of gravity are encoded in the **energy-momentum tensor**.

In GR, gravity arises from the curvature of space-time.

Given that the curvature of a manifold in any number of dimensions is described by the **Riemann tensor**, the Einstein tensor has to be related to it.

In particular, the Einstein tensor is defined as: $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$

where R is the **Ricci scalar**, and the **Ricci tensor** is a contraction of the Riemann tensor:

$$R_{ij} = R^a_{iaj}$$

and is computed as,

$$R_{ab} = \Gamma^a_{bc,a} - \Gamma^a_{ba,c} + \Gamma^a_{da}\Gamma^d_{bc} - \Gamma^a_{dc}\Gamma^d_{ba}$$

The connection $\Gamma^a_{bc}$ is the quantity that enables to "connect" the local geometry around one point of the curved space (or space-time) with the local geometry around another point of the same space (or space-time). In other words, it describes how the basis vector change from point to point due to the curvature.

The connection is thus a needed quantity when computing derivatives in a curved space: the covariant derivative (;) of a function, includes the "normal" derivative (,) of the function and the derivative of the basis:

derivative of a vector: $\lambda^a{}_{;b} = \lambda^a{}_{,b} + \Gamma^a{}_{bc}\lambda^c$ or $\lambda_{a;b} = \lambda_{a,b} - \Gamma^c{}_{ba}\lambda_c$

derivative of a tensor: $T^{ab}{}_{;c} = T^{ab}{}_{,c} + \Gamma^a{}_{cd}T^{db} + \Gamma^b{}_{cd}T^{ad}$

derivative of a scalar: $\phi_{;a} = \phi_{,a}$ naturally, the connection is not needed when differentiating a scalar quantity

In GR the connection is completely determined by the metric, as,

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd}\left[g_{ad,b} + g_{bd,a} - g_{ab,d}\right]$$

but in general the connection and the metric could be two independent quantities related to curvature. Note: this is the case in the Palatini approach to gravity.

Hence, the Einstein tensor is computed from the connection and the metric.
We saw that the homogeneous Universe is described by the Robertson-Walker metric:

$$g_{ab} = \begin{bmatrix} -1 & & & 0 \\ & a^2 & & \\ & & a^2 f^2 & \\ 0 & & & a^2 f^2 \sin^2\theta \end{bmatrix}$$

(here f is $f_K$)

$$g^{ab} \text{ is the inverse } = \begin{bmatrix} -1 & & & 0 \\ & 1/a^2 & & \\ & & 1/a^2 f^2 & \\ 0 & & & 1/a^2 f^2 \sin^2\theta \end{bmatrix}$$

$$g^a{}_b = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = \mathbb{1}$$

Computing the various terms of the connection, metric and tensors
(for all combinations of indexes, running from 0 [time]  to 1,2,3 [space] ):

RW $\Rightarrow$ $\Gamma^0_{00}=0$ ; $\Gamma^0_{0i}=\Gamma^0_{i0}=0$ ; $\Gamma^0_{ij}=\delta_{ij}\dot{a}a$ ; $\Gamma^i_{0j}=\Gamma^i_{j0}=\frac{\dot{a}}{a}\delta^i_j$ ; $\Gamma^i_{jk}=0$

; $\Gamma^i_{00}=0$

2 or 3 0 given 0

1 0 and 2 space

no 0 also gives 0

($\delta^i_i=3$)

$\Rightarrow$ $R_{00}=-3\left(\frac{\ddot{a}}{a}\right)$

$R_{ij}=\delta_{ij}\left(2\dot{a}^2+\ddot{a}a\right)$

Ricci tensor in diagonal, like the metric and $T^a{}_b$

$\Rightarrow$ Ricci scalar $R=g^{ab}R_{ab} \Leftrightarrow R=R_{00}-\frac{1}{a^2}\delta^{ij}R_{ij} \Leftrightarrow R=-6\left[\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right]$

the resulting Einstein tensor is:

$\Rightarrow$ Einstein tensor for RW:

$$
\begin{cases}
G_{00} = \dfrac{f'^2 + 2ff'' - 1}{a^2 f^2} - \dfrac{3}{c^2}\left(\dfrac{\dot{a}}{a}\right)^2 \\
G_{11} = \dfrac{1 - f'^2}{f^2} + \dfrac{1}{c^2}\left(\dot{a}^2 + 2a\ddot{a}\right) \\
G_{22} = \dfrac{f^2}{c^2}\left(\dot{a}^2 + 2a\ddot{a}\right) - ff'' \\
G_{33} = G_{22}\sin^2\theta
\end{cases}
\qquad f = f_k(x)
$$

It is diagonal, like the RW metric, having only 4 non-zero elements.

This tensor is the left-side of the **Einstein equations**:

$$G_{ab} = 8\pi G\, T_{ab}$$