CFTC Centro de Física Teórica e Computacional

Ciências ULisboa

Percolation theory

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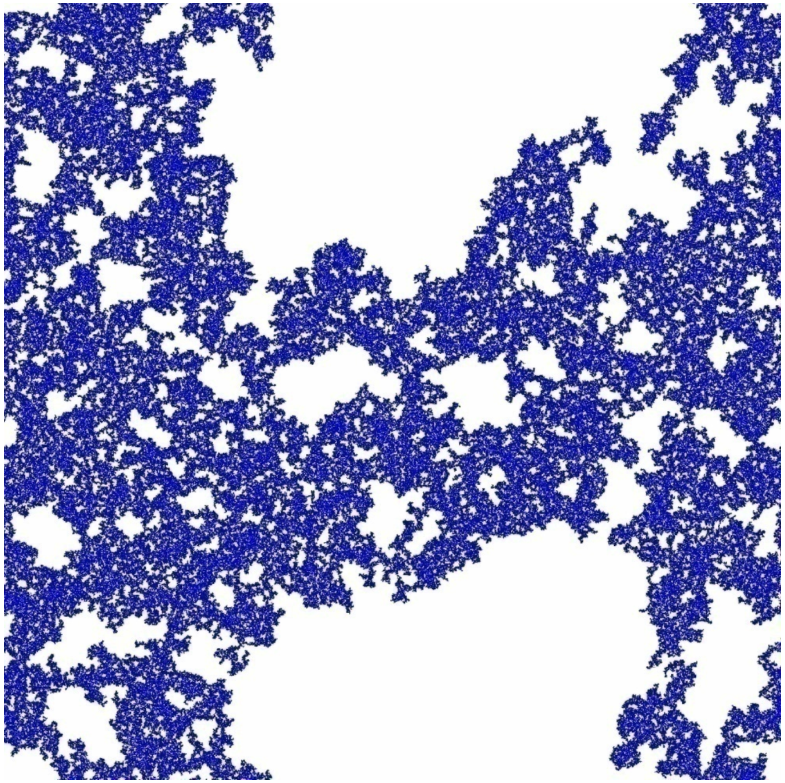
Renormalization

general concepts

$$M_\infty \sim l^{d_f}$$

rescaling transformation $\frac{\xi}{b}$



$$\xi = \frac{\xi}{b}$$

Fixed point

$$\xi = 0 \qquad \xi = \infty$$

$$p = \{0, p_c, 1\}$$

Renormalization

rescaling

rescaling transformation

$$\xi(p) = c|p - p_c|^{-\nu} \xrightarrow{\xi \mapsto \xi/b} \frac{\xi}{b} = c|T_b(p) - p_c|^{-\nu}$$

$$\frac{1}{b} = \left(\frac{|T_b(p) - p_c|}{|p - p_c|}\right)^{-\nu}$$

$$\nu = \frac{\ln b}{\ln\left(\frac{|T_b(p) - p_c|}{|p - p_c|}\right)}$$

$T_b(p_c) = p_c$

$$\nu = \frac{\ln b}{\ln\left(\left|\frac{\mathrm{d}T_b}{\mathrm{d}p}\right|_{p_c}\right)}$$

$T_b(p)$ requires *full information* about the system.

Renormaization

real-space renormalization

real-space renormalization group transformation R_b

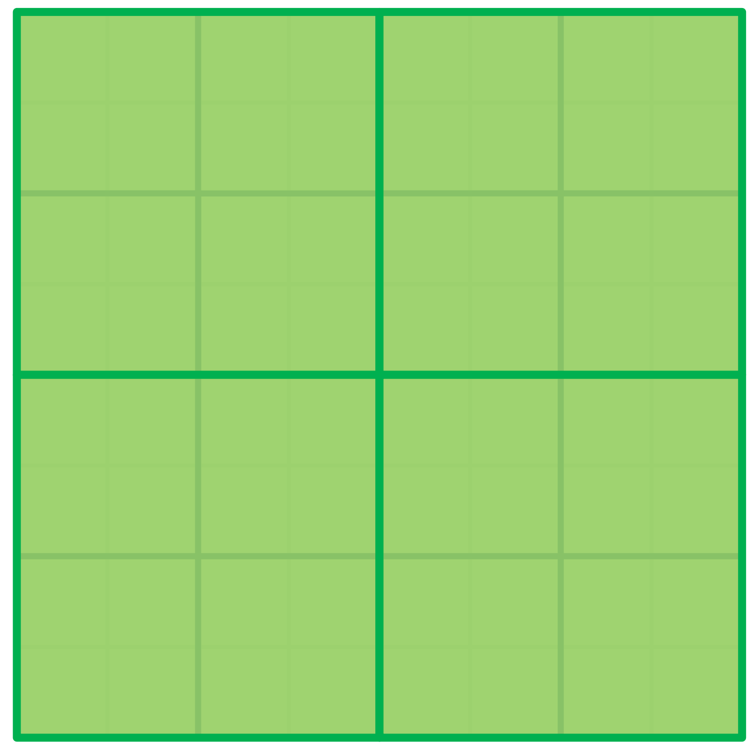
$$R_b(p^\star) = p^\star$$

fixed point

$$\nu \approx \frac{\ln b}{\ln\left(\frac{|R_b(p) - p_c|}{|p - p_c|}\right)}$$

$$\nu \approx \frac{\ln b}{\ln\left(\left|\frac{\mathrm{d}R_b}{\mathrm{d}p}\right|_{p_c}\right)}$$

$b = 2$



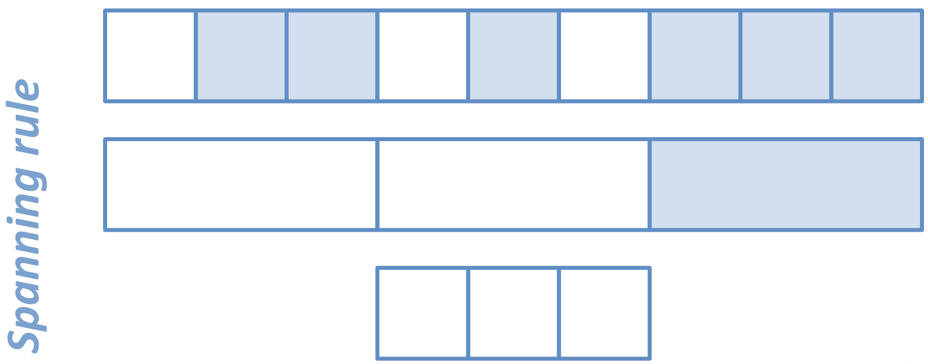
P. J. Reynolds, H. E. Stanley, and W. Klein. *J. Phys. C* **10**, L167 (1977)

Renormalization

1D

1. ***Divide*** into blocks of size b
2. ***Replace*** each block by a single one
3. ***Rescale*** all lengths by a factor of b

$$b = 3$$



$$R_b(p) = p^b$$

$$p^\star = 0 \qquad p^\star = 1$$

$$\xi(R_b(p)) = -\frac{1}{\ln R_b(p)} = -\frac{1}{\ln p^b} = \frac{\xi(p)}{b}$$

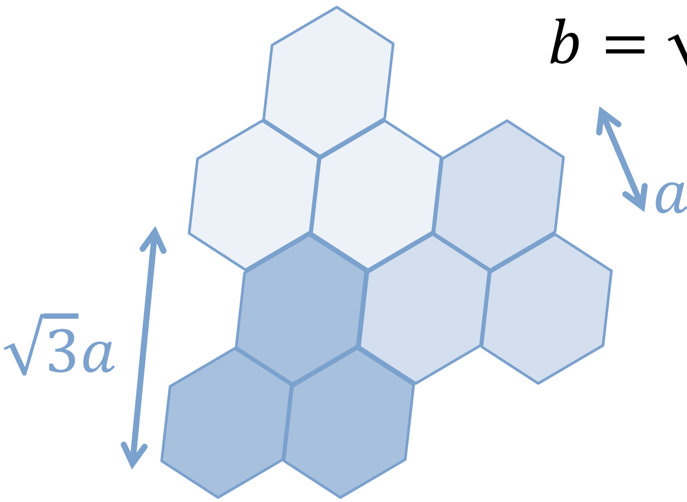
$$\left.\frac{\mathrm{d}R_b}{\mathrm{d}p}\right|_{p^\star=1} = \left. b p^{b-1}\right|_{p^\star=1} = b$$

$$\nu = \frac{\ln b}{\ln\left(\left|\frac{\mathrm{d}R_b}{\mathrm{d}p}\right|_{p_c}\right)} = 1$$

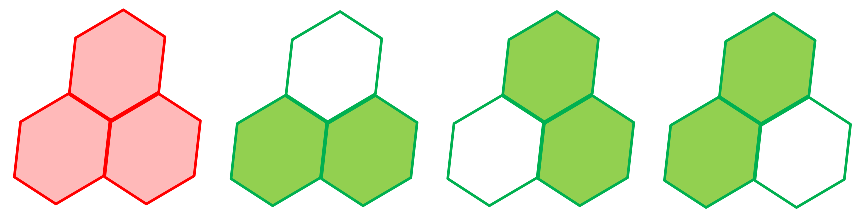
Renormalization

2D triangular lattice

Spanning rule



$$b = \sqrt{3}$$



$$R_b(p) = p^3 + 3p^2(1-p)$$

$$p^\star = 0 \qquad p^\star = 1/2 \qquad p^\star = 1$$

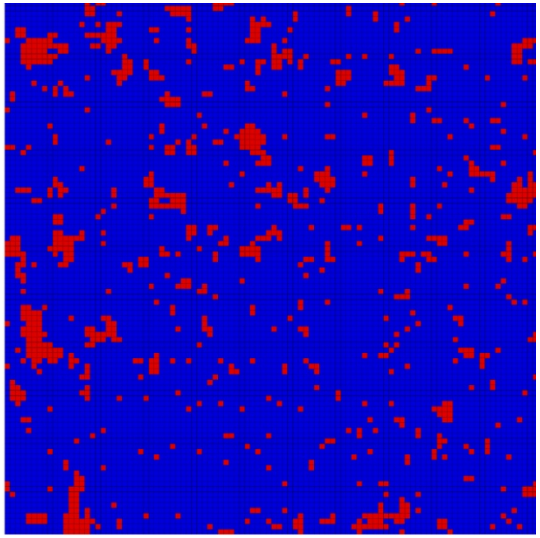
$$\nu = \frac{\ln b}{\ln\left(\left|\frac{\mathrm{d}R_b}{\mathrm{d}p}\right|_{p^\star}\right)} = \frac{\ln\sqrt{3}}{\ln(3/2)} \approx 1.355$$

$$\nu = \frac{4}{3}$$

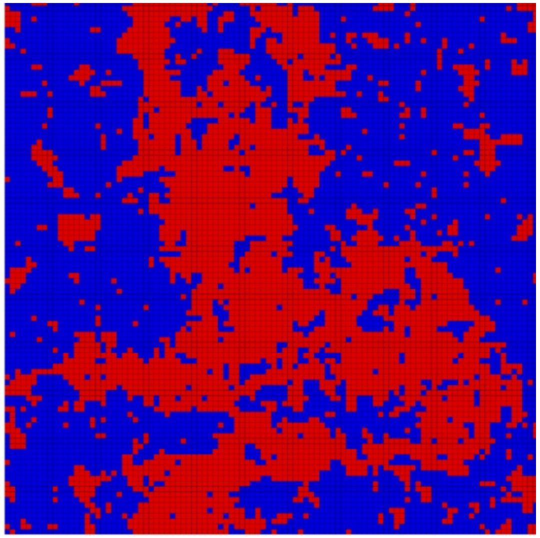
Percolation and magnetic models

Ising model (2-state Potts model)

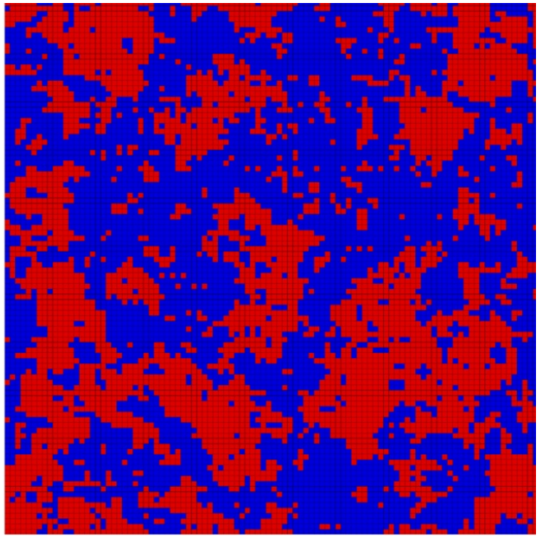
$$\mathcal{H} = J \sum_{b} \varepsilon_b \quad \text{with } \varepsilon_b = \begin{cases} 0, \text{if endpoints are in the same state} \\ 1, \text{if endpoints are in different states} \end{cases}$$



$T < T_c$



$T = T_c$

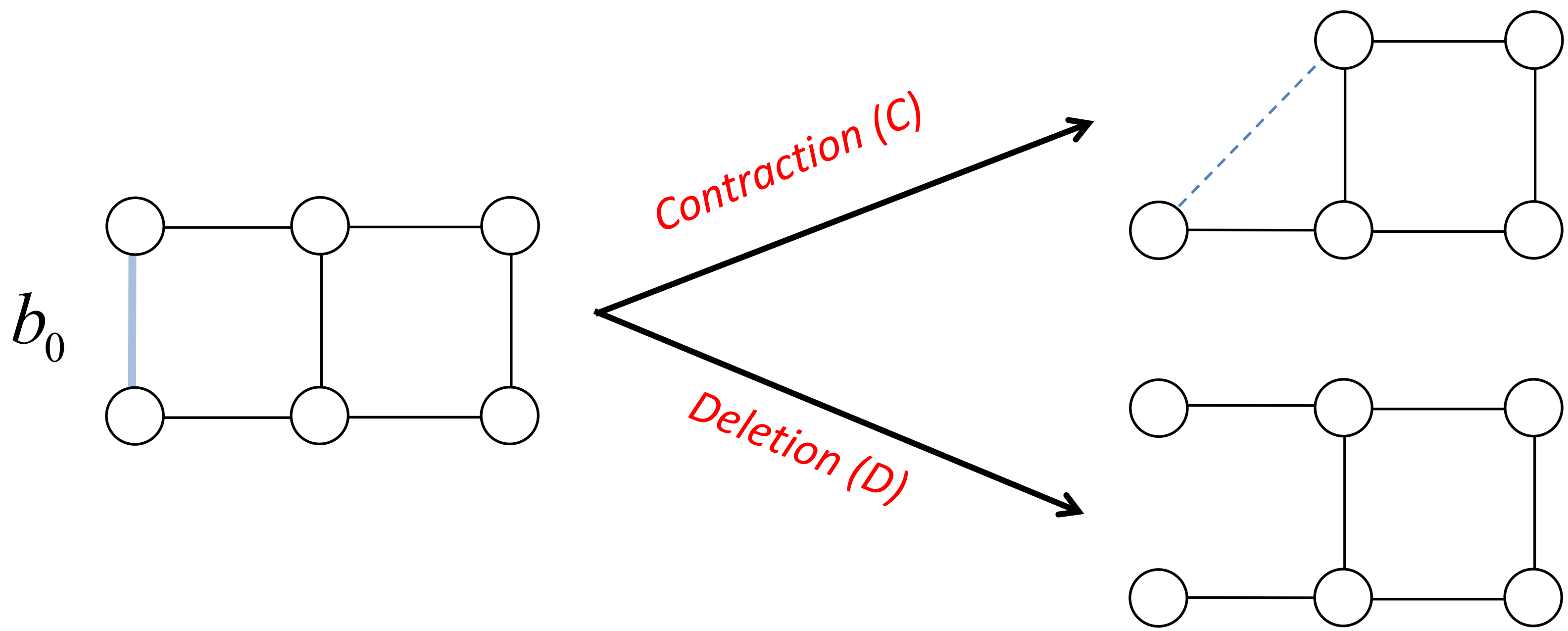


$T > T_c$

Percolation and magnetic models

theorem of Kasteleyn-Fortuin

$$\mathcal{H} = J\sum_{b} \varepsilon_b \quad \text{with } \varepsilon_b = \begin{cases} 0, \text{if endpoints are in the same state} \\ 1, \text{if endpoints are in different states} \end{cases}$$



Percolation and magnetic models

theorem of Kasteleyn-Fortuin

$$\mathcal{H} = J \sum_b \varepsilon_b$$

$$Z = \sum_X e^{-\beta \mathcal{H}(X)} = \sum_X e^{-\beta J \sum_b \varepsilon_b} = \sum_X \prod_b e^{-\beta J \varepsilon_b}$$

Partition Function

Consider bond b_0 with endpoints i and j:

$$Z = \sum_X e^{-\beta J \varepsilon_{b_0}} \prod_{b \neq b_0} e^{-\beta J \varepsilon_b} = \sum_{\substack{X: \\ \sigma_i = \sigma_j}} \prod_{b \neq b_0} e^{-\beta J \varepsilon_b} + e^{-\beta J} \sum_{\substack{X: \\ \sigma_i \neq \sigma_j}} \prod_{b \neq b_0} e^{-\beta J \varepsilon_b}$$

$$= Z_C + e^{-\beta J}\left(Z_D - Z_C\right) = \left(1 - e^{-\beta J}\right) Z_C + e^{-\beta J} Z_D = p Z_C + (1-p) Z_D$$

Z_C and Z_D are the partition functions of the graphs contracted and deleted at b_0

$$p \equiv 1 - e^{-\beta J}$$

Percolation and magnetic models
theorem of Kasteleyn-Fortuin

For bond b_1 *:* $Z = pZ_{C_{b_1}} + (1-p)Z_{D_{b_1}}$ *, and doing the same for* b_2 *:*

$$Z = p^2 Z_{C_{b_1}C_{b_2}} + p(1-p)Z_{C_{b_1}D_{b_2}} + (1-p)pZ_{D_{b_1}C_{b_2}} + (1-p)^2 Z_{D_{b_1}D_{b_2}}$$

Doing the same for all edges, one obtains a set of separated points/clusters (contracted graphs). Each can be in *q* different states.

$$Z = \sum_{\text{Configurations of bond percolation}} q^{\#\text{ of clusters}} p^c (1-p)^d$$

Configurations of *bond percolation*

$$p \equiv 1 - e^{-\beta J}$$

c and *d* are the number of contracted and deleted bonds.