Universo Primitivo

2024-2025 (1º Semestre)

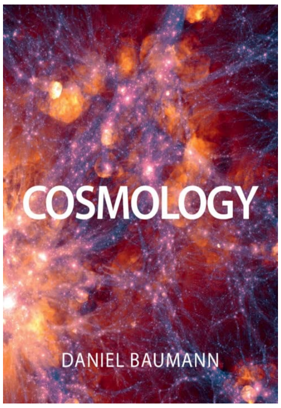
Mestrado em Física - Astronomia

Chapter 4

4 Decoupling

- Decoupling from local equilibrium;
  - Electroweak and week Interaction rates;
  - Particle distributions after decoupling;
  - Decoupling and Freeze-Out
- Neutrino decoupling;
- Electron-positron Annihilation;
- Cosmic Neutrino Background;
- Beyond thermal equilibrium: Boltzmann Equation

References



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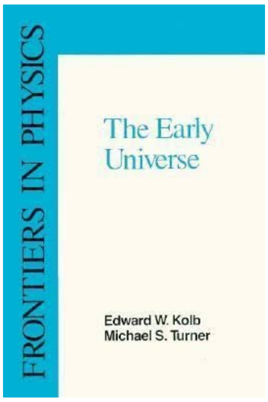

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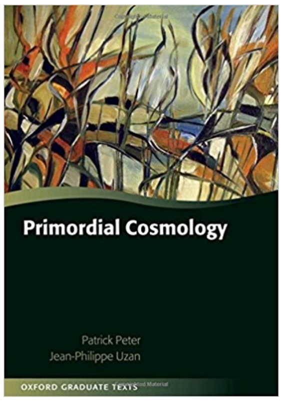

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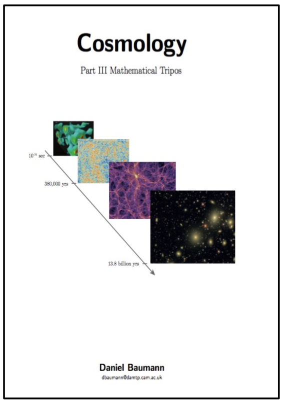

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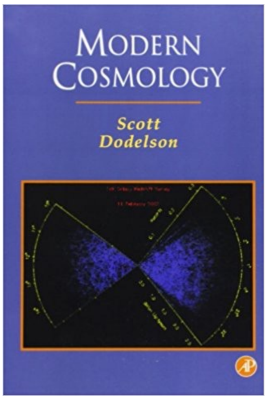

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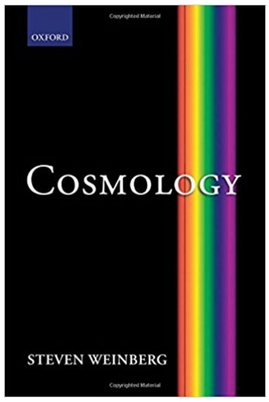


Ch. 3

Decoupling from equilibrium

Equilibrium condition, interaction timescale, and interaction rate:

Thermal equilibrium of a fluid species can be established if the interaction rate, $\Gamma(t)$, is larger than the expansion rate, $H(t) = \dot{a}/a$:

$$\Gamma(t) \gg H(t)$$

The timescale for particle interactions, $t_c = 1/\Gamma$, is therefore much shorter than the characteristic timescale of expansion, $t_H = 1/H$:

$$t_c \equiv \frac{1}{\Gamma} \ll t_H \equiv \frac{1}{H}$$

The interaction rate is the number of interaction events of the species per unit of time. It is given by:

$$\boxed{\Gamma \equiv n\sigma v}$$

where $\boldsymbol{n}$ is the **number density of target particles**, $\boldsymbol{\sigma}$, is the **interaction cross section** and, $\boldsymbol{v}$, is the **relative speed** between particles.
The SI unit of Γ ``one over second'': $[\Gamma] = [n][\sigma][v] = s^{-1}$.

Decoupling from equilibrium

Equilibrium condition, interaction timescale, and interaction rate:

For example, in the interaction process: $1 + 2 \leftrightarrow 3 + 4$ one has:

- $\Gamma_1 = n_2 \sigma v_{12}$ is the iteration rate of the particle species 1
- $\Gamma_2 = n_1 \sigma v_{21}$ is the iteration rate of the particle species 2 ($v_{21} = v_{12}$)
- $\Gamma_3 = n_4 \sigma v_{34}$ is the iteration rate of the particle species 3
- $\Gamma_4 = n_3 \sigma v_{43}$ is the iteration rate of the particle species 4 ($v_{43} = v_{34}$)

Reverting the equilibrium condition, one should expect that **a given particle specie has conditions to decouple** from the thermal bath when $\Gamma \lesssim H$, i.e.:

$$\frac{\Gamma}{H} \lesssim 1$$

For a relativistic fluid, the expansion rate of the universe reads (SI):

$$H^2 = \frac{8\pi G}{3}\rho_r = \frac{\hbar c}{3M_{\rm pl}^2}\rho_r = \frac{\hbar c}{3M_{\rm pl}^2}\frac{\pi^2}{30}g_* T^4$$

where, $M_{\rm pl}$ is the **Planck Mass**:

$$\boxed{M_{\rm pl} \equiv \sqrt{\frac{\hbar c}{8\pi G}} = 2.4 \times 10^{18}\,\mathrm{GeV}}$$

Decoupling from thermal equilibrium

Equilibrium condition, interaction timescale, and interaction rate:

Changing to natural units one has:

$$H = \sqrt{\frac{\hbar c}{3M_{\rm pl}^2}\frac{\pi^2}{30}g_* T^4} = \pi\left(\frac{g_*}{90}\right)^{1/2}\frac{T^2}{M_{\rm pl}}$$

Let us now estimate the interaction rate, $\Gamma = n\sigma v$, for the **fluid of relativistic particles**:

- Since particles are relativistic: $v \sim c = 1$
- The number density in equilibrium is: $n_i \sim \frac{\zeta(3)}{\pi^2} g_i T^3 \propto T^3$
- The interaction cross section will depend on the type of interaction and mediators. For **interactions mediated by bosons** of mass m_X:

$$\sigma_X = \begin{cases} \alpha_X^2/T^2, & T \gg m_X \quad \text{(photon, gluons, relativistic } W^{\pm}, Z^0 \text{ bosons)} \\ \alpha_X^2 T^2/m_X^4, & T \ll m_X \quad \text{(massive, non-relativistic, } W^{\pm}, Z^0 \text{ bosons)} \end{cases}$$

where α_X is the generalized structure constant with the gauge boson X

Decoupling from thermal equilibrium

Equilibrium condition, interaction timescale, and interaction rate:

So, the equilibrium condition ratio, Γ/H, becomes:

$$\frac{\Gamma}{H} \simeq \frac{n\sigma}{H} = \frac{\frac{\zeta(3)}{\pi^2} g_i T^3}{\pi\left(\frac{g_*}{90}\right)^{1/2} T^2/M_{\rm pl}} \times \begin{cases} \alpha_X^2/T^2 & T \gg m_X \text{ (photon, massless bosons)} \\ \alpha_X^2 T^2/m_X^4 & T \ll m_X \text{ (massive bosons)} \end{cases}$$

Implications:

1. **At high temperature** ($T \gtrsim 100$ GeV – the electroweak symmetry breaking $m_{W^\pm_-,Z,H} \sim 100\text{GeV}$) all interactions are mediated by relativistic massless Gauge bosons. For example, for the electroweak interaction (EW), $\alpha^2 = 0.01$, so the ratio:

$$\boxed{\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{\rm pl}}{T} \sim \frac{10^{16}\text{GeV}}{T}}$$

So, the EW interaction alone can provide equilibrium conditions for processes mediated by the EW force, within the temperature range

$$100 \text{ GeV} \lesssim T \lesssim 10^{16} \text{ GeV}$$

Above the $\sim 10^{16}$ GeV (the **Grand Unification Theory**) scale the Universe is not able to acquire an equilibrium state via the electroweak interaction. In fact, the universe may never had conditions to achieve thermal equilibrium above the 10^{16} GeV scale!

Decoupling from thermal equilibrium

Equilibrium condition, interaction timescale, and interaction rate:

So, the equilibrium condition ratio, Γ/H, becomes:

$$\frac{\Gamma}{H} \simeq \frac{n\sigma}{H} = \frac{\frac{\zeta(3)}{\pi^2} g_i T^3}{\pi\left(\frac{g_*}{90}\right)^{1/2} T^2/M_{\rm pl}} \times \begin{cases} \alpha_X^2/T^2 & T \gg m_X \text{ (photon, massless bosons)} \\ \alpha_X^2 T^2/m_X^4 & T \ll m_X \text{ (massive bosons)} \end{cases}$$

Implications:

2. **At lower temperature** ($T \lesssim 100$ GeV – below the electroweak symmetry breaking). Photons and gluons remain massless bosons and since their $\Gamma/H \sim T^{-1}$ easily allows to get $\Gamma/H \gg 1$, which provides equilibrium conditions for particles interacting via the strong and electromagnetic forces. For relativistic particles interacting via the the weak force, (which becomes mediated by massive bosons with $m_X \gg T$) one has:

$$\boxed{\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{\rm pl} T^3}{M_W^4} \sim \left(\frac{T}{1 \text{ MeV}}\right)^3}$$

which **drops below unity for $T \lesssim 1$ MeV.**

So relativistic particles interacting via weak force (e.g. neutrinos) are able to remain in equilibrium with the fluid in the temperature range:

$$1 \text{ MeV} \lesssim T \lesssim 100 \text{ GeV}$$

Below this temperature they should decouple from the fluid (Ex. Neutrinos & DM WIMPs).

Thermal history of the Universe:

Key events in the thermal history of the universe

Event	time t	redshift z	temperature T
Inflation	10^{-34} s (?)	–	–
Baryogenesis	?	?	?
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV
Reionization	100–400 Myr	11–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	13.8 Gyr	0	0.24 meV

Summary:

Electroweak interacting species may attain thermal equilibrium with the fluid up to the GUT 10^{16}GeV scale;

Pree particle species interacting via the **weak, electromagnetic and strong** forces have conditions to attain thermal equilibrium with the primordial fluid;

Particle species interacting via the **electromagnetic force** have conditions to attain thermal equilibrium with the primordial fluid;

Decoupling from thermal equilibrium

Particle distributions after decoupling

Let us now study what happens to the phase space distribution of a given particle species, $f(x, p, t > t_D)$, after that species decouples from the fluid at time t_D.

The **number of particles** within the volume element $dV d^3p$ around the point (x, p) of the phase space is:

$$dN = f(x, p, T)\; dV\; d^3p$$

If no particles are created or destroyed after decoupling, the left-hand side of this equation remains constant. On the right-hand side, we know that the volume element, dV, scales with a^3. For the momentum, from $E^2 = m^2 + p^2$ one can derive the following scaling:

$$\boxed{p \propto E = hc/\lambda \propto a^{-1}}$$

So, d^3p scales a^{-3}. The scaling of p is straightforward for massless particles (e.g. radiation) and is also valid for massive non-relativistic particles (check it – note that m is the rest mass). So, **for massless or massive relativistic particles** one can write:

$$\boxed{\frac{x_D}{x} = \frac{a_D}{a} \Leftrightarrow x_D = x\frac{a_D}{a}} \qquad \boxed{\frac{p_D}{p} = \frac{a_D^{-1}}{a^{-1}} \Leftrightarrow p_D = p\frac{a}{a_D}}$$

where the subscript D denotes quantities at the decoupling time t_D [$a_L = a(t_D)$, $p_D = p(t_D)$, $x_d = x(t_D)$].

Decoupling from thermal equilibrium

Particle distributions after decoupling

If no particles are created or destroyed as the universe ages, the number of particles in a volume element in the phase space, dN, remains constant. Since $dV \propto a^3$ and $d^3p \propto a^{-3}$,

$$dN = f(x, p, t)\, \underbrace{dV}_{a^3}\, \underbrace{d^3p}_{a^{-3}}$$

Is constant if no particles are created/destroyed after decoupling

one concludes that the **distribution function, f, keeps its functional form (i.e., it's shape) as the universe expands.** However, the arguments of f (x and p) scale with $a(t)$. So, if a **particle species decouples at t_D with a distribution function f** and one considers the scalings of its arguments one has:

$$f(x_D, p_D, t_D) = f\left(x\frac{a_D(t)}{a}, p(t)\frac{a(t)}{a_D}, t > t_D\right)$$

The **right-hand side** of this equation, is in fact, the **distribution function after decoupling**. Dropping x (because f is independent of position) this means that after decoupling f keeps its form but the momentum scales as above:

$$\boxed{f(p(t), t > t_D) \coloneqq f\left(p(t)\frac{a(t)}{a_D}, t > t_D\right) = f(p_D, t_D)}$$

Decoupling from thermal equilibrium

Particle distributions after decoupling

So, depending on the relativistic state of the particles, one has two possibilities:

1. **The species decouples while is relativistic** (here we assume $\mu = 0$, e.g. massless neutrinos)

$$f(p(t), t > t_D) \coloneqq f\left(p(t)\frac{a(t)}{a_D}, t > t_D\right) = \frac{g}{2\pi^3}\frac{1}{\exp\left(p\frac{a/a_D}{T_D}\right) \pm 1} = \frac{g}{2\pi^3}\frac{1}{\exp\left(\frac{p}{T}\right) \pm 1}$$

where $T = T(t)$ was set as:

$$\boxed{T(t) = T_D\frac{a_D}{a} = \frac{T_D}{(1+z_D)}(1+z)}$$

This $T \propto a^{-1}$ scaling arises because no particles are created or destroyed after decoupling, and the functional form of f remains unchanged. So, the **number density of a relativistic species after decoupling** scales as:

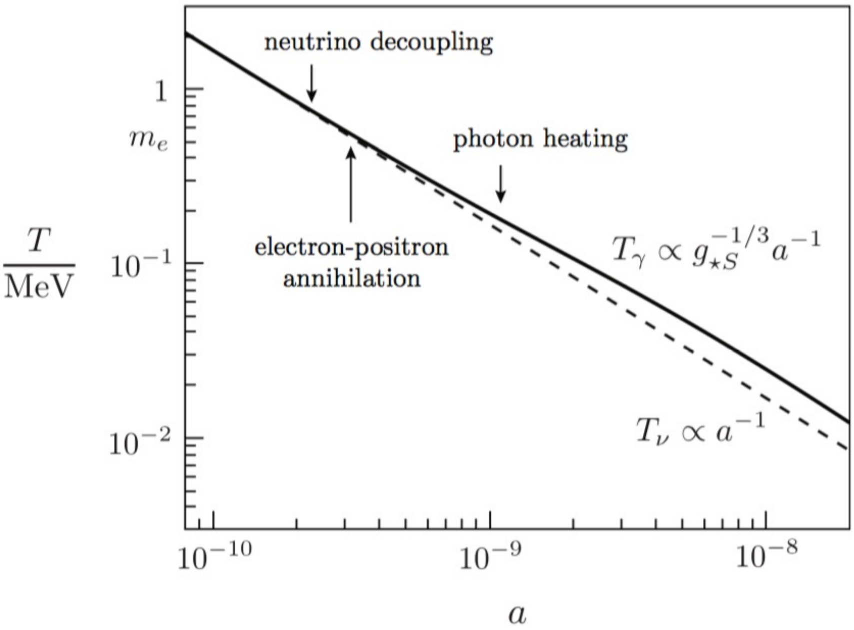
$$n_i = g_i\frac{\zeta(3)T^3}{\pi^2} = g_i\frac{\zeta(3)T_D^3}{\pi^2}\left(\frac{a_D}{a}\right)^3 \quad \Leftrightarrow \quad \boxed{n_i = n_{i,D}\left(\frac{a_D}{a}\right)^3}$$

Therefore, **species that are relativistic at decoupling their number density after decoupling scales as $n_i \propto a^{-3}$** (i.e., as before decoupling whenever g_{*S} is constant – away from mass thresholds).

Decoupling from thermal equilibrium

Particle distributions after decoupling:

We concluded that **the temperature of decoupled relativistic species** also scales with the inverse of the scale factor ($T = T_D\, a_D/a$) **as** it happens **for relativistic species in thermal equilibrium away from mass thresholds** ($T_\gamma \propto g_{*S}^{-1/3}\, a^{-1}$). The decoupling is not instantaneous (and needs to be described with the Boltzmann equation). The Figure below shows the decoupling of the neutrinos from the primordial fluid.



When neutrinos decouple, g_{*S}, decreases and therefore the temperature of the fluid, T_γ, decreases at a lower rate than the temperatures of the decoupled species.

Decoupling from thermal equilibrium

Particle distributions after decoupling

2. The species decouples while it is non-relativistic ($m \gg T$)

$$f(x,p,t \geq t_D) = f\left(p\frac{a(t)}{a_D}, t_D\right) \simeq \frac{g_i}{2\pi^3}\exp\left(-\frac{\sqrt{(pa/a_D)^2 + m^2} + \mu_D}{T_D}\right)$$

$$\simeq \frac{g_i}{2\pi^3}\exp\left(-\frac{p^2}{2mT_D}\left(\frac{a}{a_D}\right)^2 - \frac{m}{T_D} + \frac{\mu_D}{T_D}\right)$$

$$\simeq \frac{g_i}{2\pi^3}\exp\left(-\frac{p^2}{2mT} - \frac{m}{T} + \frac{\mu}{T}\right)$$

where the non-relativistic limit allows to drop the $\pm$ 1 factor in f and to expand the square root to 1st order. In the last equality

$$\boxed{T(t) = T_D\left(\frac{a_D}{a(t)}\right)^2 = \frac{T_D}{(1+z_D)^2}(1+z)^2} \qquad \boxed{\mu(t) = m + (\mu_D - m)\frac{T(t)}{T_D}}$$

These scalings render the distribution function with the same functional form of a distribution of *non-relativistic particles with temperature and chemical potential* written with the same approximations at decoupling time, $f(p_D, t_D)$.

Decoupling from thermal equilibrium

Particle distributions after decoupling

Using these scalings in the expressions for the number density of a non-relativistic species one obtains that, after decoupling the number density scales as:

$$
\begin{aligned}
n_i &= g_i \left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m-\mu)/T} \\
&= g_i \left(\frac{mT_D}{2\pi}\right)^{3/2} \left(\frac{a_D}{a}\right)^3 e^{-(m-\mu_D)/T_D}
\end{aligned}
$$

This means that if a particle species decouples when it is non-relativistic, its number density also scales as:

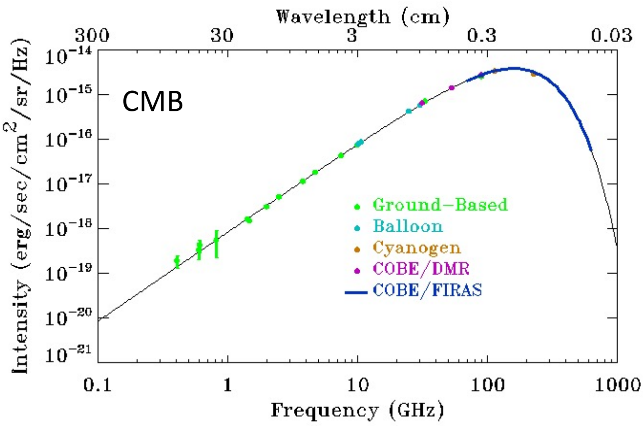
$$\boxed{n_i = n_{i,D} \left(\frac{a_D}{a}\right)^3}$$

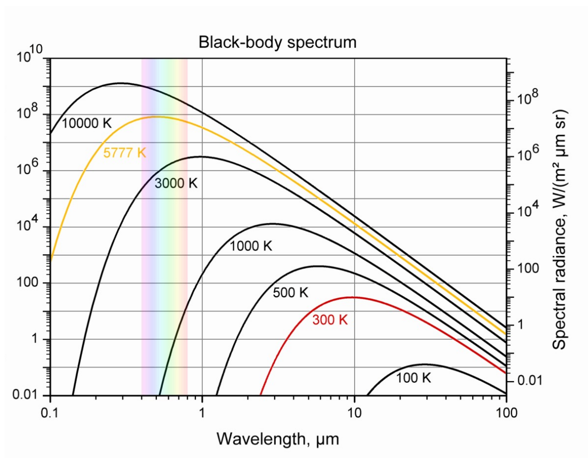
Therefore, **species that are non-relativistic at decoupling their number density after decoupling also scale as $\boldsymbol{n_i \propto a^{-3}}$**.

Decoupling from thermal equilibrium

Particle distributions after decoupling

Example: photons (relativistic)





$T_{\gamma,0} = 2.726$

$T_{\gamma,D} = 2.726\,(1+z_D) \sim 3000$ K

Relativistic:

$$\boxed{T(t) = T_D \frac{a_D}{a(t)} = \overbrace{\frac{T_D}{(1+z_D)}}^{T_0}(1+z)}$$

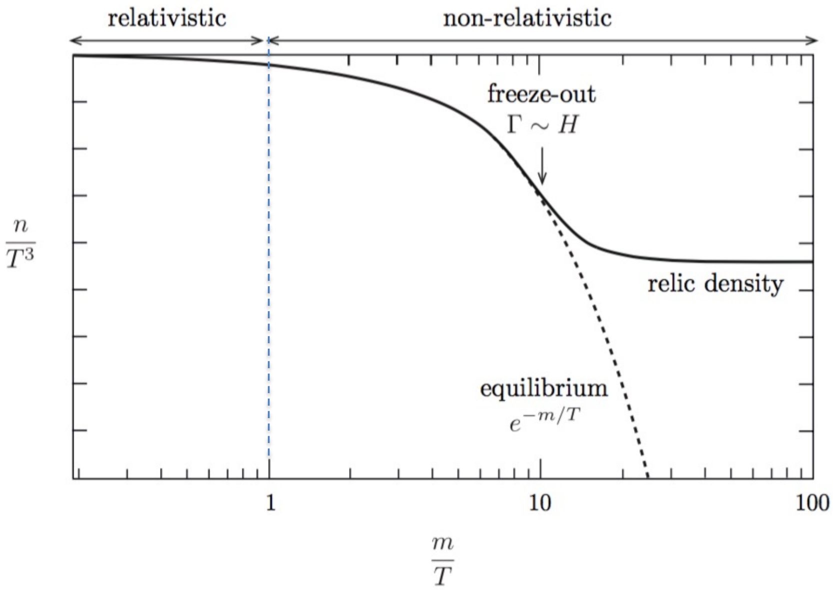
Non-Relativistic:

$$\boxed{T(t) = T_D \left(\frac{a_D}{a(t)}\right)^2 = \overbrace{\frac{T_D}{(1+z_D)^2}}^{T_0}(1+z)^2}$$

Decoupling from thermal equilibrium

Decoupling and Freeze-out

As **massive particles decouple** their abundances are **Boltzmann suppressed by $e^{-m/T}$**. While relativistic, for $T \gg m$, one should expect that n_i/T^3 is constant (because $n_i \propto \mathrm{T}^3$). However, **these predictions assume that the decoupling species is always in equilibrium** as its density is being supressed. **But this hypothesis cannot hold at very low temperatures**, $T \ll m$, because particle abundances become too small to be able to establish equilibrium.



At high enough m/T one should expect that the real number density departures from the equilibrium prediction:

$$\boxed{n_{eq} = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m/T)}}$$

In fact, beyond m/T larger than ~10 the ratio n_{non_eq}/T^3 becomes constant again. The density, n_{non_eq}, is the **non-equilibrium Freeze-Out density**.

Decoupling from thermal equilibrium

Neutrino decoupling:

Neutrinos are coupled to the thermal bath via weak interaction processes like:

$$\nu_e + \bar{\nu}_e \leftrightarrow e^+ + e^- \qquad e^+ + e^- \leftrightarrow \gamma + \gamma$$

$$e^- + \bar{\nu}_e \leftrightarrow e^- + \bar{\nu}_e$$

At 10 MeV, photons, **neutrinos**, **electrons** (and their antiparticles) are the only remaining particles of the relativistic fluid. Then, g_*,reads:

$$g_* = \underset{\gamma}{2} + \frac{7}{8}(2 \times \underset{e^\pm}{2} + 3 \times \underset{\nu}{2}) = 10.75$$

Using this in the Friedman equations

$$H = \sqrt{\frac{8\pi G}{3}}\rho^{1/2} = \sqrt{\frac{8\pi G}{3}}\left(g_*\frac{\pi^2}{30}T^4\right)^{1/2} \approx 5.44\sqrt{8\pi}\frac{T^2}{m_{\rm Pl}}$$

Combining with the expression for Γ one concludes that neutrinos decouple below at **about 1 MeV** (accurate calculation yields $T = 0.8\ \mathrm{MeV}$) .

$$\boxed{\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{\rm Pl} T^3}{M_W^4} \sim \left(\frac{T}{1\ \mathrm{MeV}}\right)^3}$$

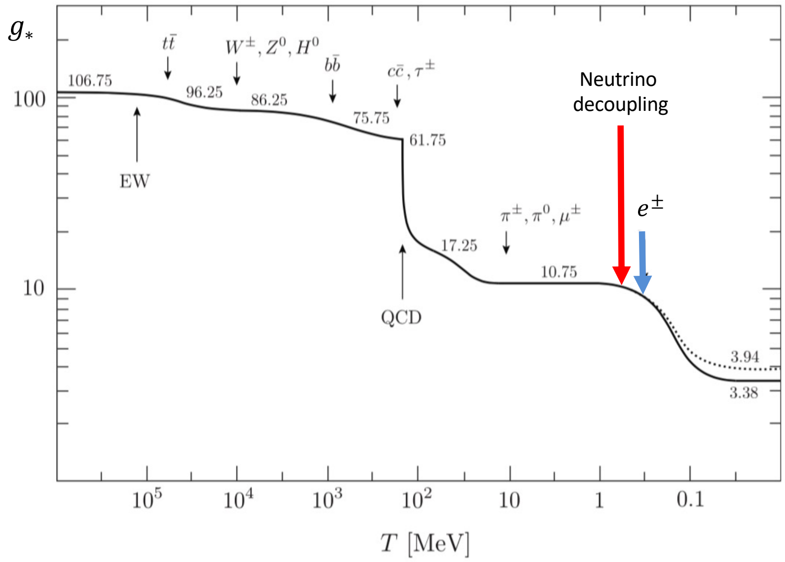
Decoupling from thermal equilibrium

Electron-positron annihilation

Electron-positron annihilation occurs soon after the neutrino decoupling. In fact, as soon as $T \lesssim 1.022$ MeV electron-positron pair creation becomes less effective, and the interaction

$$e^+ + e^- \leftrightarrow \gamma + \gamma$$

progressively moves to the right (more pairs $e^-/\,e^+$ being destroyed than created).



- Neutrino decoupling occurs around $T \sim 0.8$ MeV;
- $e^-/\,e^+$ annihilation occurs around $T \sim 0.5$ MeV, with a transition $0.1 \lesssim T/\text{MeV} \lesssim 1$
- But these processes partially overlap. **Neutrino decoupling is not over when electro-positron annihilation starts**

Decoupling from thermal equilibrium

Electron-positron annihilation

Let us now compute the effective degrees of freedom of relativistic particles before neutrino decoupling and after electron-positron annihilation.

Before neutrino decoupling ($T \gtrsim 1$ MeV):
Relativistic particles species are the γ, $e^\pm$ and ν_s, so:

$$g_{*S} = g_* = 2 + \frac{7}{8}(2 \times 2 + 3 \times 2) = 10.75$$

After electron-positron annihilation ($T \lesssim 0.5$ MeV):
Relativistic particles species are just the γ and ν_s (note that neutrinos are already decoupled but they remain relativistic and therefore contribute to the entropy):

$$g_{*S} = 2 + \frac{7}{8}(3 \times 2)\left(\frac{T_\nu}{T_\gamma}\right)^3_{\text{after}}$$

$$g_* = 2 + \frac{7}{8}(3 \times 2)\left(\frac{T_\nu}{T_\gamma}\right)^4_{\text{after}}$$

Since entropy is conserved one has:

$$S = sV = \frac{2\pi}{45} g_{*S}(T_\gamma a)^3 = const.$$

Decoupling from thermal equilibrium

Electron-positron annihilation

So, one can write:

$$g_{*S}(T_\gamma a)^3_{\text{before}} = g_{*S}(T_\gamma a)^3_{\text{after}} \Leftrightarrow$$

$$\Leftrightarrow \left(2+\frac{7}{8}(2\times 2+3\times 2)\right)(T_\gamma a)^3_{\text{before}} = \left(2+\frac{7}{8}(3\times 2)\left(\frac{T_\nu}{T_\gamma}\right)^3_{\text{after}}\right)(T_\gamma a)^3_{\text{after}} \Leftrightarrow$$

$$\Leftrightarrow \frac{43}{4}(T_\gamma)^3_{\text{before}} = \left(2+\frac{7}{8}(3\times 2)\left(\frac{T_\nu}{T_\gamma}\right)^3_{\text{after}}\right)(T_\gamma)^3_{\text{after}}\left(\frac{a_{\text{after}}}{a_{\text{before}}}\right)^3$$

But after decoupling neutrino temperature scales as: $T_{\nu,\text{before}}/T_{\nu,\text{after}} = a_{\text{after}}/a_{\text{before}}$.
Moreover, since $T_{\nu,\text{before}} = T_{\gamma,\text{ before}}$, one has:

$$\frac{43}{4}(T_\gamma)^3_{\text{before}} = \left(2+\frac{7}{8}(3\times 2)\left(\frac{T_\nu}{T_\gamma}\right)^3_{\text{after}}\right)(T_\gamma)^3_{\text{after}}\left(\frac{T_{\nu,\text{before}}}{T_{\nu,\text{after}}}\right)^3 \Leftrightarrow$$

$$\frac{43}{4}(T_{\gamma,\text{before}})^3 = \left(2+\frac{7}{8}(3\times 2)\left(\frac{T_\nu}{T_\gamma}\right)^3_{\text{after}}\right)\left(\frac{T_\gamma}{T_\nu}\right)^3_{\text{after}}(T_{\nu,\text{before}})^3 \Leftrightarrow$$

$$\Leftrightarrow \frac{43}{4}(\cancel{T_{\gamma,\text{before}}})^3 = \left(2\left(\frac{T_\gamma}{T_\nu}\right)^3_{\text{after}}+\frac{7}{8}(3\times 2)\right)(\cancel{T_{\gamma,\text{before}}})^3 \Leftrightarrow$$

Decoupling from thermal equilibrium

Electron-positron annihilation

From which one concludes that:

$$\frac{43}{4} = 2\left(\frac{T_\gamma}{T_\nu}\right)^3_{\text{after}} + \frac{7}{8}(3\times 2) \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{T_\gamma}{T_\nu}\right)^3_{\text{after}} = \frac{1}{2}\left(\frac{43}{4}-\frac{21}{4}\right) = \frac{11}{4}$$

So, after $e^-/\ e^+$ annihilation the neutrino temperature is somewhat smaller than the the photon temperature:

$$\boxed{T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma}$$

With this result one can estimate the relativistic degrees of freedom for $T \lesssim 0.5$ MeV:

$$g_{*s} = 2 + \frac{7}{8}\times 3\times 2\left(\frac{4}{11}\right) = 3.91,$$

$$g_{*} = 2 + \frac{7}{8}\times 3\times 2\left(\frac{4}{11}\right)^{4/3} = 3.36$$

Which explains the difference between g_{*S} and g_* at low temperatures.

Decoupling from thermal equilibrium

Electron-positron annihilation

However, **this result is only valid if the decoupling of neutrinos was instantaneous** (and happened before the beginning of the electron-positron annihilation).

But **these processes are not instantaneous** and in fact overlap in time. Part of the energy and entropy of the electron-positron annihilation leaks into the remaining relativistic species, increasing their temperature, via a decrease of g_{*S} and g_* (as discussed in slide 13).

A more accurate computation (considering the variation of $g_{*S}(T)$ and $g_*(T)$ and the Boltzmann formalism) gives:

$$\boxed{g_* = 3.38\,;\, g_{*S} = 3.94}$$

To keep the calculation of the effective degrees of freedom simple, it is usual to define a quantity, $\boldsymbol{N_{\mathrm{eff}}}$, known as **effective number of neutrino species**, so that:

$$g_* = 2 + \frac{7}{8} \times 3 \times 2\left(\frac{4}{11}\right)^{4/3} = 3.36$$
$$g_{*s} = 2 + \frac{7}{8} \times 3 \times 2\left(\frac{4}{11}\right) = 3.91$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} g_* &= 2 + \frac{7}{8} \times 2N_{\mathrm{eff}}\left(\frac{4}{11}\right)^{4/3} = 3.38 \\ g_{*S} &= 2 + \frac{7}{8} \times 2N_{\mathrm{eff}}\left(\frac{4}{11}\right) = 3.94\,, \end{aligned}}$$

Where $N_{\mathrm{eff}} = 3.046$ (N_{eff} is by itself a parameter that can be fit by CMB observations). If neutrino decoupling was instantaneous $N_{\mathrm{eff}} = 3$.

Decoupling from thermal equilibrium

Cosmic Neutrino Background

A **Cosmic Neutrino background** ($\mathrm{C}_\nu\mathrm{B}$) should be present in the universe since decoupling. Its temperature should scale with with the inverse of scale factor, and it is related to the CMB temperature (which also scales with the inverse of the scale factor) as:

$$T_{\nu,0} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma,0} = \left(\frac{4}{11}\right)^{1/3} 2.73 = 1.95\,\mathrm{K}$$

(which corresponds to $T_{\nu,0} = 0.17\ \mathrm{meV}$).

Plugging this result in the expression of the neutrino number and energy densities one obtains:

$$\boxed{n_\nu = \frac{3}{4} N_{\mathrm{eff}} \times \frac{4}{11} n_\gamma}$$

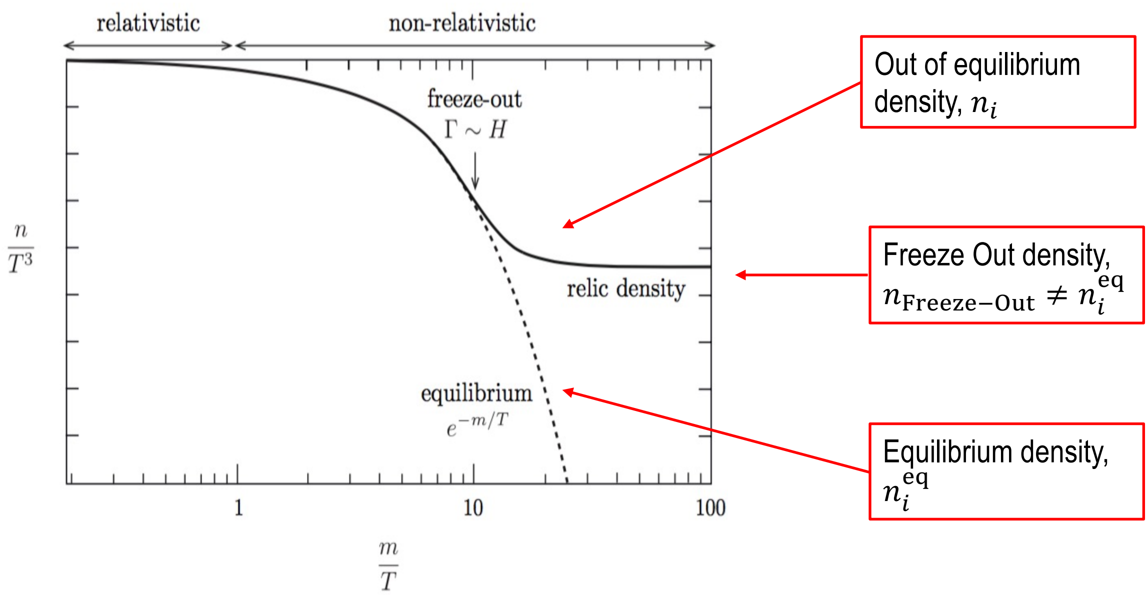
$$\boxed{\rho_\nu = \frac{7}{8} N_{\mathrm{eff}} \left(\frac{4}{11}\right)^{4/3} \rho_\gamma}$$

Assuming $N_{\mathrm{eff}} = 3.046$ and the observed values of the CMB densities, one obtains $n_{\nu,0} = 112\ \mathrm{cm}^{-3}$ and $\Omega_{\nu,0} = 0.00014$ (assuming massless neutrinos).

Decoupling from thermal equilibrium

Beyond thermal equilibrium: the Boltzmann Equation

Equilibrium quantities such as density expression derived in the previous chapter **assume that the decoupling species is always in equilibrium** as its density is being supressed. In reality, **this hypothesis cannot hold at very low temperatures**, $T \ll m$, because particle abundances become too small to be able to achieve equilibrium. The formal way of computing **out of equilibrium densities** is by using the **Boltzmann equations approach** (see next slides).



Decoupling from thermal equilibrium

Boltzmann Equation

We have established that in the **absence of interactions,** the number density of a **decoupled species, *i***, scales as $n_i = n_{0,i}\, a^{-3}$. So:

$$\frac{dn_i}{dt} = -3\, n_{0i} a^{-2} \dot{a} = -3 \frac{\dot{a}}{a} n_i \quad \Leftrightarrow \quad \frac{dn_i}{dt} + 3 \frac{\dot{a}}{a}\, n_i = 0$$

where the last equation is the continuity equation (up to a multiplicative factor, m). Multiplying and dividing by a^3 one can also express this equation as:

$$\boxed{\frac{1}{a^3} \frac{d(n_i a^3)}{dt} = 0}$$

This equation also holds when the particle species is in equilibrium way from mass thresholds (i.e. when the net number of particles remains constant).
One can **generalise** this expression to **include interactions**:

$$\boxed{\frac{1}{a^3} \frac{d(n_i a^3)}{dt} = C_i\left[\{n_j\}\right]}$$

Where the left-hand side is the same equation as above but in the right-hand side one adds a collision term, $C_i\left[\{n_j\}\right]$, that accounts for sinks / sources of the density of the species n_i due to interaction (collisions) with other species n_j.
The latter equation is known as the **collisional Boltzmann equation**.
When $C_i\left[\{n_j\}\right] = 0$ one obtains the **collisionless Boltzmann equation** (above)

Decoupling from thermal equilibrium

Boltzmann Equation

The form of the collision term $\boldsymbol{C_i[\{n_j\}]}$ **depends on the type of interaction**. For interactions of 2 particles species (3 body interactions are in principle much less likely):

$$1 + 2 \rightleftarrows 3 + 4$$

(this means that species 1 annihilates with species 2, giving rise to species 3 and 4. Conversely species 3 and 4 annihilate back to species 1 and 2).

To follow the out of equilibrium evolution of, for example, n_1 one needs to consider the balance of efficiency of the reaction between 1 and 2, that originates a sink of density, and the (reverse) reaction between 3 and 4, that originates a source of n_1. This can be translated into the collisional Boltzmann equation by replacing the collision term $C_i[\{n_j\}]$ with 2 terms:

$$\boxed{\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\alpha\, n_1 n_2 + \beta\, n_3 n_4}$$

where $\boldsymbol{\alpha n_1 n_2}$ **is a sink term describing the destruction of particles** (due to the reaction to the right) and $\boldsymbol{\beta n_3 n_4}$ **is a source term describing the creation of particles** of type 1 (due to the reaction to the left).
Naturally, each term should be proportional to the densities of each pair.

Decoupling from thermal equilibrium

Boltzmann Equation

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\alpha\, n_1 n_2 + \beta\, n_3 n_4$$

Now, the parameters α and β can be written as:

- $\alpha = \langle \sigma\, v\rangle$ is the **thermally averaged cross section** ($\alpha\, n_1 n_2 = \Gamma_1 n_1$)
- β needs to be related to α so that the right-hand side of the equation vanishes when particles are in equilibrium.

$$\beta = \left(\frac{n_1 n_2}{n_3 n_4}\right)_{\mathrm{eq}} \alpha$$

Where the densities inside the parenthesis are equilibrium densities n_i^{eq}. Thus:

$$\boxed{\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\langle\sigma v\rangle \left[n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{\mathrm{eq}} n_3 n_4\right]}$$

Decoupling from thermal equilibrium

Boltzmann Equation

It is instructive to write to write the collisional Boltzmann in terms of the number of particles in a commoving volume, defined in Chapter 3 as:

$$N_i = \frac{n_i}{s}$$

which is a conserved quantity (whenever the net number of particles is conserved) resulting from the entropy conservation equation. Setting $n_i = N_i s$ in the Boltzmann equation gives:

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\langle\sigma v\rangle \left[n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{\rm eq} n_3 n_4 \right] \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{a^3}\frac{d(N_1\, sa^3)}{dt} = -\langle\sigma v\rangle \left[N_1 N_2 s^2 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\rm eq} N_3 N_4 s^2 \right] \Leftrightarrow$$

$$\Leftrightarrow \frac{sa^3}{a^3}\frac{dN_1}{dt} = -\langle\sigma v\rangle\, s^2 \left[N_1 N_2 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\rm eq} N_3 N_4 \right]$$

where sa^3 is constant because the entropy $S = sV = V_{\rm com}\, sa^3$ is a conserved quantity. Further rearranging the previous equation, one has:

$$\frac{dN_1}{dt} = -\langle\sigma v\rangle\, sN_1 N_2 \left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\rm eq} \left(\frac{N_3 N_4}{N_1 N_2}\right) \right]$$

Decoupling from thermal equilibrium

Boltzmann Equation

Rearranging further and using the fact that $s\, N_2 = n_2$ and $\Gamma_1 = \langle\sigma v\rangle n_2$

$$\frac{1}{N_1}\frac{dN_1}{dt} = -\langle\sigma v\rangle\, s\, N_2 \left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\rm eq} \left(\frac{N_3 N_4}{N_1 N_2}\right) \right] \Leftrightarrow$$

$$\Leftrightarrow \frac{d\ln N_1}{dt} = -\langle\sigma v\rangle n_2 \left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\rm eq} \left(\frac{N_3 N_4}{N_1 N_2}\right) \right] \Leftrightarrow$$

$$\Leftrightarrow \frac{d\ln N_1}{dt} = -\Gamma_1 \left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\rm eq} \left(\frac{N_3 N_4}{N_1 N_2}\right) \right]$$

Since:

$$\frac{d\ln N_1}{dt} = \frac{d\ln N_1}{d\ln a}\frac{d\ln a}{dt} = \frac{d\ln N_1}{d\ln a}\frac{\dot a}{a} = H\,\frac{d\ln N_1}{d\ln a}$$

one can finally write the Boltzmann equation as:

$$H\,\frac{d\ln N_1}{d\ln a} = -\Gamma_1 \left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\rm eq} \left(\frac{N_3 N_4}{N_1 N_2}\right) \right] \Leftrightarrow$$

$$\Leftrightarrow \frac{d\ln N_1}{d\ln a} = -\frac{\Gamma_1}{H} \left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\rm eq} \left(\frac{N_3 N_4}{N_1 N_2}\right) \right]$$

Decoupling from thermal equilibrium

Boltzmann Equation

In summary: one can transform the collisional Boltzmann equation in terms of the equilibrium condition ratio Γ/H , and the number of particles of the intervening species, $N_i = n_i/s$:

$$\boxed{\frac{d\ln N_1}{d\ln a} = -\frac{\Gamma_1}{H}\left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\rm eq}\frac{N_3 N_4}{N_1 N_2}\right]}$$

Which is of simple interpretation :

- The factor Γ_1/H describes the **interaction efficiency**.
- The 2nd term in the parenthesis characterises **deviations from equilibrium**. When **all species are in equi**librium it gives 1 and the equation's r.h.s. is zero. So $\boldsymbol{N_1}$ **remains constant**, i.e., **freezes out**.
- **If $\boldsymbol{\Gamma_1 \ll H}$**, the r.h.s of the equation is supressed and $\boldsymbol{N_1}$**also freezes out**.
- **If $\boldsymbol{\Gamma_1 \gg H}$ equilibrium is rapidly established**. For example, if $N_1 > N_1^{\rm eq}$ the r.h.s. becomes negative (more particles will be destroyed). If $N_1 < N_1^{\rm eq}$ the r.h.s. becomes positive (more particles will be created). **Both effects push $\boldsymbol{N_1 \to N_1^{eq}}$**.

Decoupling from thermal equilibrium

Boltzmann Equation (see, Peter & Uzan Section 4.2.2.1)

Evolution of the distribution function

The evolution of the distribution function is obtained from the Boltzmann equation

$$L[f] = C[f], \tag{4.49}$$

where C describes the collisions and $L = \mathrm{d}/\mathrm{d}s$ is the Liouville operator, with s the length along a worldline. The operator L is a function of eight variables taking the explicit form

$$L[f] = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha}. \tag{4.50}$$

In a homogeneous and isotropic space-time, f is only a function of the energy and time, $f(E,t)$, so that

$$L[f] = E\frac{\partial f}{\partial t} - Hp^2\frac{\partial f}{\partial E}. \tag{4.51}$$

Using the definition (4.11) of the particle density, and integrating this equation with respect to the momentum $\boldsymbol{p}$, we obtain[7]

$$\dot{n}_i + 3Hn_i = C_i, \qquad C_i = \frac{g_i}{(2\pi)^3}\int C\left[f_i(p_i,t)\right]\frac{\mathrm{d}^3 p_i}{E_i}. \tag{4.52}$$

Decoupling from thermal equilibrium

Boltzmann Equation (see, Peter & Uzan Section 4.2.2.1)

The difficult part lies in the modelling and the evaluation of the collision term. Here, we restrain ourselves to the simple case of an interaction of the form

$$i + j \longleftrightarrow k + l, \tag{4.53}$$

for which the collision term can be decomposed as $C_i = C_{kl\to ij} - C_{ij\to kl}$ with

$$C_{ij\to kl} = (2\pi)^4 \int \frac{g_i \mathrm{d}^4 p_i}{(2\pi)^3}\delta(E_i - p_i^2 - m_i^2)\cdots\frac{g_l \mathrm{d}^4 p_l}{(2\pi)^3}\delta(E_l - p_l^2 - m_l^2)$$
$$\times \delta^{(4)}(p_i + p_j - p_k - p_l)|\mathcal{M}|^2_{ij\to kl} f_i f_j (1 \pm f_k)(1 \pm f_l), \tag{4.54}$$

with a + sign for bosons and a − sign for fermions. $|\mathcal{M}|_{ij\to kl}$ are the matrix elements describing the interaction. The Dirac delta function imposes the conservation of momentum and of energy. This form also shows that the probability for i to disappear is proportional to $f_i f_j$, i.e. roughly to the density of the interacting species.[8] The factors $(1 \pm f_k)$ arise from quantum mechanics and are related to the Pauli exclusion principle for fermions and to stimulated emission for bosons.

If $\hat{C}\hat{P}$ invariance holds, as we assume here, then $C_{kl\to ij}$ and $C_{ij\to kl}$ involve a unique matrix element, $|\mathcal{M}|^2$, determined by the physical process. Indeed, this invariance implies that the process we consider is reversible and thus that $i + j \to k + l$ and $k + l \to i + j$ have the same matrix elements. It follows that

$$C_i = (2\pi)^4 \int \delta^{(4)}(p_i + p_j - p_k - p_l)\frac{g_i \mathrm{d}^3 p_i}{2(2\pi)^3 E_i}\cdots\frac{g_l \mathrm{d}^3 p_l}{2(2\pi)^3 E_l} \times$$
$$\times |\mathcal{M}|^2 \left[f_k f_l (1 \pm f_i)(1 \pm f_j) - f_i f_j (1 \pm f_k)(1 \pm f_l)\right]. \tag{4.55}$$

Decoupling from thermal equilibrium

Boltzmann Equation (see, Peter & Uzan Section 4.2.2.1)

In cosmologically interesting situations, $E - \mu \gg T$. Quantum effects can thus be neglected and $1 \pm f \simeq 1$.

$$\boxed{\dot{n}_i + 3Hn_i = \frac{g_i \cdots g_l}{(2\pi)^8}\int \frac{\mathrm{d}^3 p_i}{2E_i}\cdots\frac{\mathrm{d}^3 p_l}{2E_l}\delta^{(4)}(p_i + p_j - p_k - p_l)|\mathcal{M}|^2 (f_k f_l - f_i f_j).} \tag{4.56}$$

In this limit, the distribution functions are of the form $f \propto \exp[(\mu - E)/T]$ so that the particle density (4.11) can be expressed as a function of that at $\mu = 0$ as

$$n_i = \mathrm{e}^{\mu_i/T}\bar{n}_i, \qquad \bar{n}_i \equiv n_i[\mu_i = 0]. \tag{4.57}$$

Furthermore, the conservation of energy implies that $E_k + E_l = E_i + E_j$ such that the term $f_k f_l - f_i f_j$ takes the form

$$\mathrm{e}^{-(E_k+E_l)/T}\left[\mathrm{e}^{(\mu_k+\mu_l)/T} - \mathrm{e}^{(\mu_i+\mu_j)/T}\right] = \mathrm{e}^{-(E_k+E_l)/T}\left(\frac{n_k n_l}{\bar{n}_k \bar{n}_l} - \frac{n_i n_j}{\bar{n}_i \bar{n}_j}\right).$$

The Boltzmann equation (4.56) can thus be written as

$$\dot{n}_i + 3Hn_i = -\langle\sigma v\rangle\left(n_i n_j - \frac{\bar{n}_i \bar{n}_j}{\bar{n}_k \bar{n}_l} n_k n_l\right), \tag{4.58}$$

where $\langle\sigma v\rangle$ is defined as

$$\bar{n}_i \bar{n}_j \langle\sigma v\rangle \equiv \int \frac{\mathrm{d}^3 p_i}{2E_i}\cdots\frac{\mathrm{d}^3 p_l}{2E_l}\delta^{(4)}(p_i + p_j - p_k - p_l)|\mathcal{M}|^2 \frac{\mathrm{e}^{-(E_i+E_j)/T}}{(2\pi)^8}. \tag{4.59}$$