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The Lagrangian - From fields to events

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Dark Matter, Phase transitions and Gravitational Waves

The Lagrangian, the Action and Natural units

The classical Lagrangian

In the old days of Analytical Mechanics the Action (S) was given by

$$S = \int_{t_1}^{t_2} dt\, L(q_i(t), \dot{q}_i(t)) \qquad with \qquad L = \int d^3x \mathscr{L}(x)$$

where $\mathscr{L}$ is the Lagrangian density and $q_i(t), \dot{q}_i(t)$ are the generalised position and velocity, respectively. The minimisation of the action leads to Euler-Lagrange equations of motion. To go from just classical to a classical field theory we have to replace

$$q_i \to \phi(x^\mu) \equiv \phi(x); \qquad \dot{q}_i \to \partial_\mu \phi(x) \equiv \frac{\partial}{\partial x^\mu}\phi(x)$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$x_\mu \equiv (t, -\vec{x}) = (t, -x, -y, -z)$$

Our starting point is a Lagrangian density with an Action written as

$$S = \int d^4x\, \mathscr{L}(\phi(x), \partial_\mu \phi(x))$$

The Lagrangian density (only Lagrangian for us) will be our best friend during these lectures.

The fields

The fields are the basic entities in QFT. Particles are excitations of fields. Fields exist of course at the classical level. Once they are operators and quantised, and incorporating relativity in our description we will have a full QFT. Fields corresponding to already observed particles

ϕ Scalar or pseudoscalar - spin zero field (Higgs)

A_μ Vector or Axial vector - spin one field (gluons, weak gauge bosons, photon)

ψ Spinor - spin 1/2 field (Fermions - quarks and leptons)

Is dark matter one of these (kind of) particles? We don't know. Is one favoured over the others? Not that we know. Is any of them excluded by experiment. No.

Can we build models with any of them? Yes we can!

Building the Lagrangian - dimensions

$\hbar = c = 1$ - In Natural Units all quantities are measured in units of mass/energy to some power.

$$E = m;\ E = \nu$$

$$[p_\mu] = [\partial_\mu] = m \qquad [x_\mu] = m^{-1}$$

m stands for mass/energy

The action is now dimensionless (because Planck constant is 1)

$$S = \int \mathscr{L}\ d^4x \ \Rightarrow\ [\mathscr{L}] = m^4$$

The canonical dimension of the field is obtained from the free Lagrangian

$$\mathscr{L}^{KG}_{free} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) \ \Rightarrow\ [\phi] = m$$

$$\mathscr{L}^{EM}_{free} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ \Rightarrow\ [F_{\mu\nu}] = m^2 \ \Rightarrow\ [A_\mu] = [F/\partial] = m$$

$$\mathscr{L}^{Dirac}_{free} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi \ \Rightarrow\ [\psi] = m^{3/2}$$

$$\begin{aligned} 1\,\mathrm{GeV}^{-2} &= 0.389\,\mathrm{mb} \\ 1\,\mathrm{GeV}^{-1} &= 6.582 \cdot 10^{-25}\,\mathrm{s} \\ 1\,\mathrm{kg} &= 5.61 \cdot 10^{26}\,\mathrm{GeV} \\ 1\,\mathrm{m} &= 5.07 \cdot 10^{15}\,\mathrm{GeV}^{-1} \\ 1\,\mathrm{s} &= 1.52 \cdot 10^{24}\,\mathrm{GeV}^{-1} \end{aligned}$$

Building the Lagrangian - dimensions

Know your couplings; regarding interactions there are mainly three types (in what concerns dimensions)

- $\lambda_3 \phi^3 \;\Rightarrow\; [\lambda_3] = m$
- $\lambda_4 \phi^4 \;\Rightarrow\; [\lambda_4] = m^0 = 1$
- $\lambda_5 \phi^5 \;\Rightarrow\; [\lambda_5] = m^{-1}$

$$S = \int \mathscr{L} \, d^4x \;\Rightarrow\; [\mathscr{L}] = m^4$$

When is a coupling large? The theory is perturbative if:

If the coupling has mass dimensions, the mass has to be well below the energy scale probed.

If the coupling is dimensionless, the coupling has to be below 1 or below 4π or other possibilities (more later).

If the coupling has inverse mass dimensions, the mass has to be well above the energy scale probed (Fermi theory).

Building the Lagrangian - symmetries

Consider the function

$$f(x) = x^2 + x^3 + \sin x + \cos x$$

and now suppose you want this function to be invariant under $x \to -x$. The function becomes

$$f(x) = x^2 + \cos x$$

The world after Emmy Noether - invariance

Now you do this for a Lagrangian. The Lagrangian describes the world. The world has this symmetry.

Invariance under a given transformation leads to conserved quantities.

Colloquium: A Century of Noether's Theorem https://arxiv.org/abs/1902.01989

The world after Emmy Noether - conservation theorems

Invariant Variational Problems

I. If the integral $\mathcal{I}$ is invariant under a finite continuous group $\mathcal{G}_\rho$ with ρ parameters, then there are ρ linearly independent combinations among the Lagrangian expressions that become divergences—and conversely, that implies the invariance of $\mathcal{I}$ under a group $\mathcal{G}_\rho$.

I includes all the known theorems in mechanics, etc., concerning first integrals.

II. If the integral $\mathcal{I}$ is invariant under an infinite continuous group $\mathcal{G}^\infty{}_\rho$ depending on ρ arbitrary functions and their derivatives up to order σ, then there are ρ identities among the Lagrangian expressions and their derivatives up to order σ. Here as well the converse is valid.

II can be described as the maximum generalization in group theory of "general relativity."

Noether's Theorem (1915):

For every continuous symmetry in nature, there is a corresponding conservation law.

Every conservation law has a corresponding symmetry.

Elementary Consequences of Theorem I

Translation in space *No preferred location*	Momentum Conservation
Translation in time *No preferred time*	Energy Conservation
Rotational invariance *No preferred direction*	Angular Momentum Conservation

https://arxiv.org/abs/physics/0503066v3

Noether and Klein and Hilbert

1915: Habilitation lecture to become *Privatdozent* in Göttingen, with unanimous support of the Math / Science Department of the Philosophical Faculty

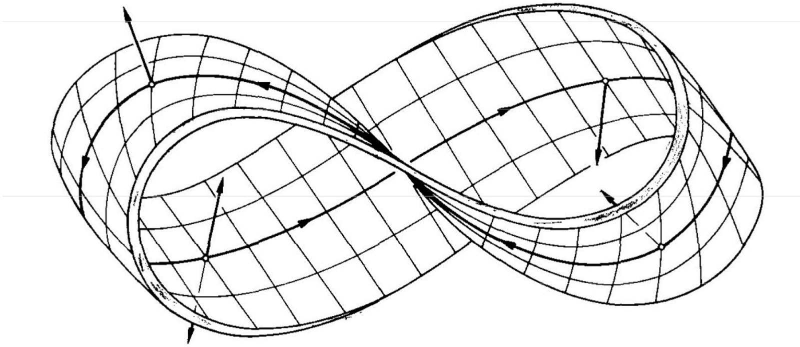
Historical-Philological Department *opposed*

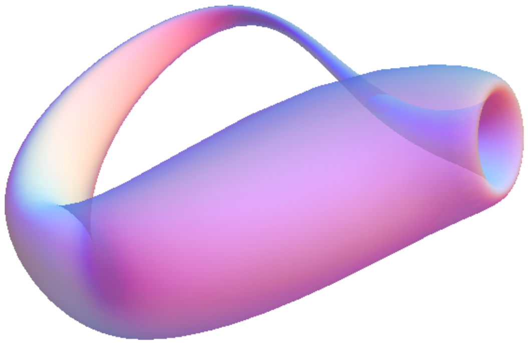
Concern that seeing a female organism might be distracting to the students.

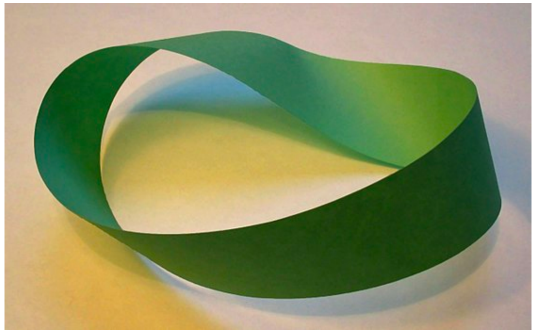
Special vote against the Habilitation of Emmy Noether, 19 November 1915

Not approved; EN permitted to lecture under Hilbert's name

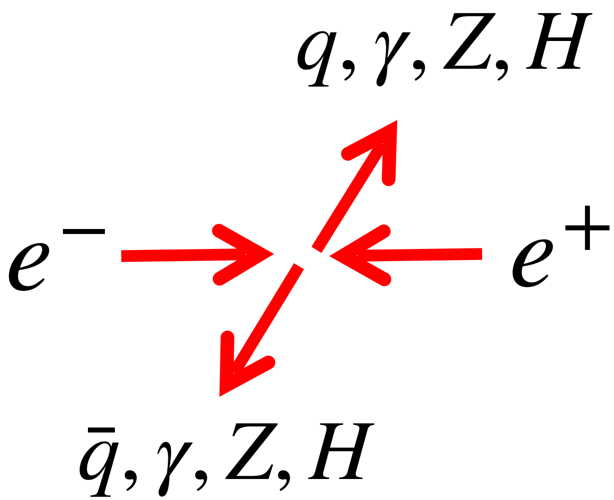
C. Quigg - A century of Noether's theorem arXiv:1902.01989







The world after Emmy Noether - internal symmetries



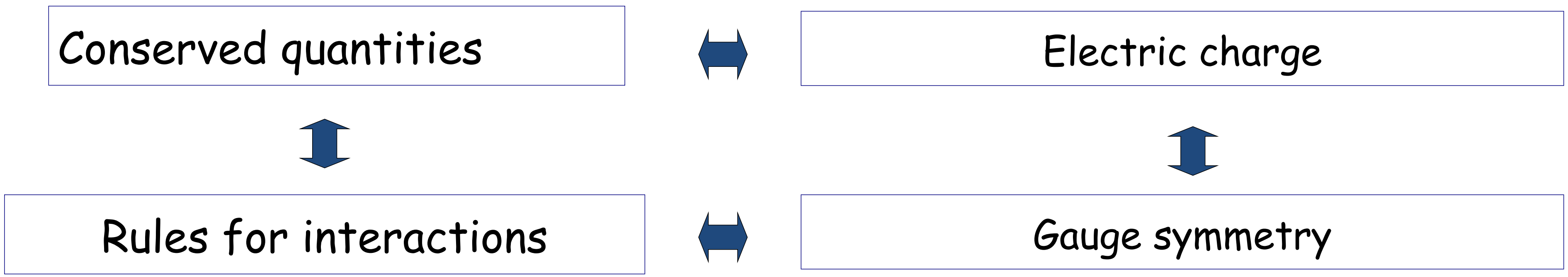
Why is charged conserved?

T1 - Global phase invariance implies a conserved quantity - we call it electric charge

T2 - Local phase (local gauge) invariance generate QED (more later).

Symmetries of the Lagrangian imply conservation laws.

$$\mathscr{L} = e \; \bar{\psi}_e \gamma_\mu \psi_e A^\mu$$



Examples of symmetries

Lagrangian built with a complex scalar field ϕ invariant under a U(1) transformation

$$\phi \to e^{i\theta}\phi$$

Terms in the Lagrangian that are invariant under this transformation

$$\phi^*\phi, \qquad (\phi^*\phi)^n, \ldots$$

Terms in the Lagrangian that break the symmetry

$\phi^2, \qquad \phi^3$ Soft breaking terms

ϕ^4 Hard breaking term

Examples of symmetries

Lagrangian built with two real scalar fields ϕ_1 and ϕ_2 invariant under a $\mathbb{Z}_2$ transformation

$$\phi_1 \to \phi_1, \qquad \phi_2 \to -\phi_2$$

Terms in the Lagrangian that are invariant under this transformation

$$\phi_1^2, \quad \phi_2^2, \quad \phi_1^3, \quad \phi_1^4, \quad \phi_1^2\phi_2^2, \ldots$$

Terms in the Lagrangian that would break the symmetry

$\phi_1^2\phi_2, \quad \phi_2^3$ — Soft breaking terms

$\phi_1^3\phi_2$ — Hard breaking term

P: Build the most general Lagrangian (up to dimension 4) with three fields ϕ_1, ϕ_2, ϕ_3 symmetries, the first two real and the third complex, with a symmetry
$\phi_1 \to \phi_1, \quad \phi_2 \to -\phi_2, \quad \phi_3 \to e^{i\theta}\phi_3.$

The charge perspective

We can also look at the problem in terms of the charge of the fields. For the $\mathbb{Z}_2$ transformation we can think of one field with charge 1 (or even under the transformation) and the other with charge -1 (or odd under the transformation).

The electric charge is related to the U(1) invariance. An operator that commutes with the Hamiltonian can be defined in terms of creation and annihilation operators

$$Q = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} (a_p^\dagger a_p - b_p^\dagger b_p)$$

In this framework, charge (in a very general sense) is the rule to build the Lagrangian.

$$\mathscr{L} = a\,b\,c \Longleftarrow \begin{cases} Q(a) = -1 \\ Q(b) = 1 \\ Q(c) = 0 \end{cases}$$

Symmetries

In Quantum field theory symmetries can be:

Local - independent transformations at every point $\phi \to e^{i\theta(x)}\phi$

Global - transformation same at every point $\phi \to e^{i\theta}\phi$

Spacetime - symmetries acting on spacetime coordinates - Parity

Internal - intrinsic symmetries (independent of coordinates) - Charge Conjugation

Internal symmetries can be represented by a set (group) of unitary matrices denoted by U(n) or by a set (group) of special unitary matrices - with unit determinant denoted by SU(n). Here, n is the size/ dimension of the matrices.
The simplest internal symmetry group is U(1). Geometrically, it is the rotational symmetry of a circle. It is the one in QED.
SU(2) is the rotational symmetry of a sphere - a set of 2 x 2 matrices with unit determinant. They model the weak nuclear interactions - between pair of fermions (e.g. electron and neutrino) and a set of three bosons: Z and W (+,-)

The free scalar field

We will come back to this point later. For now you just have to know that a renormalisable theory (predictability) can only have terms in the Lagrangian up to dimension four.

For a field not to be static we need a kinetic term in the Lagrangian. It also has to be Lorentz invariant. So the kinetic term for a scalar field is

$$\mathscr{L} = \partial_\mu \phi \partial^\mu \phi$$

Mass dimension 4

If we have a plane wave this lead to a momentum squared term. Remember that in normal quantum mechanics one starts with

$$E = \frac{(\vec{p})^2}{2m} \qquad \vec{p} \to i\overrightarrow{\nabla}; \quad E \to i\frac{\partial}{\partial t}$$

to get Schrödinger's equation. Klein-Gordon equation is obtained with the same replacement but now on the relativistic version

$$E^2 = \vec{p}^2 + m^2 \quad \to \quad (\Box + m^2)\phi = 0 \qquad \Box \equiv \partial_\mu \partial^\mu$$

The free scalar field with mass

What is the relation between the Lagrangian and the Klein-Gordon equation? Starting with a Lagrangian

$$\mathscr{L} = \frac{1}{2}[\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2]$$

Mass dimension 4

and using the equations of motion

$$\frac{\partial \mathscr{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) \quad \Rightarrow \quad -m^2 \phi = \partial_\mu \partial^\mu \phi$$

So we can write the Lagrangian that fits a given equation of motion. The solution of the Klein-Gordon equation, a free scalar field propagating in space is a plane wave

$$(\Box + m^2)\phi = 0 \quad \Rightarrow \quad \phi = N e^{-ip.x} \qquad E^2 = \vec{p}^2 + m^2$$

The scalar field with interactions

What if we include interactions? If there is a potential, the Lagrangian is L=T-V. So we have

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$$

Mass dimension 4

and if we expand the potential in powers of ϕ

$$V(\phi) = a\phi + b\phi^2 + c\phi^3 + d\phi^4 + e\phi^5 + \ldots$$

It the potential has a minimum at ϕ_0

$$\frac{\partial V}{\partial \phi}\Big|_{\phi=\phi_0} = 0 \qquad and \qquad V(\phi_0) = V_0$$

And so

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2 + M\phi^3 + \lambda\phi^4 + e\phi^5 + \ldots$$

(Auto) - Interaction terms

Mass term

Vacuum energy

P: What is the field is complex? Remember the Lagrangian is a real number.

The free vector field

To have a QFT for the photon, we start by writing everything in covariant form (in terms of the potential V and the potential vector $\overrightarrow{A}$. As a 4-vector

$$A_\mu \equiv \left(A_0, -\overrightarrow{A}\right) = \left(V, -\overrightarrow{A}\right)$$

And define the electromagnetic tensor as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{bmatrix} 0 & E_x & E_y & E_z. \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \qquad F^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu}, \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

And two more relations

$$F_{\mu\nu} F^{\mu\nu} \equiv F^2 = -2(\overrightarrow{E}^2 - \overrightarrow{B}^2)$$

$$F_{\mu\nu} \tilde{F}^{\mu\nu} \equiv F^2 = -4\overrightarrow{E} \cdot \overrightarrow{B}$$

The free vector field

The equation of motion are Maxwell equations or the Lagrangian

$$\mathscr{L}_{Free} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F^2 \qquad \frac{\partial \mathscr{L}}{\partial A_\alpha} - \partial_\beta\left(\frac{\partial \mathscr{L}}{\partial(\partial_\beta A_\alpha)}\right) = 0$$

$$\boxed{\partial_\mu F^{\mu\nu} = 0}$$

Maxwell equations

If we take $\alpha = 0$ we get

$$\partial_\mu F^{\mu 0} = \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \vec{\nabla}\cdot\vec{E} = 0$$

which is Poisson equation with no sources. With sources we would have $\vec{\nabla}\cdot\vec{E} = \rho$. How are the sources introduced in the Lagrangian?

$$\mathscr{L} = \mathscr{L}_{Free} + \mathscr{L}_{Int} = -\frac{1}{4}F^2 - A_\alpha J^\alpha$$ with $$J^\alpha = (\rho, \vec{j})$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0}$$

Maxwell equations with sources

P: Do $\partial_\mu F^{\mu 1}$. Show equations with sources.

The massive vector field

The vector field with mass is described by the Proca lagrangian

$$\mathscr{L} = -\frac{1}{4}F^2 + \frac{1}{2}m^2A^2$$

$$\frac{\partial \mathscr{L}}{\partial A_\alpha} - \partial_\beta \left(\frac{\partial \mathscr{L}}{\partial(\partial_\beta A_\alpha)} \right) = 0$$

The equations of motion for the Proca Lagrangian are

$$\boxed{\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0}$$

We can show that in this case, the <u>Lorenz</u> gauge always holds which is a consequence of loosing gauge invariance

$$\partial_\mu A^\mu = 0$$

Remember that we want to describe a particle with two degrees of freedom using a vector with four components. We will come back to this point shortly.

The free spinor field - Dirac equation

The spinor field has four components. The Lagrangian that leads to Dirac equation

$$\mathscr{L} = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi$$

Mass dimension 4

$$\frac{\partial \mathscr{L}}{\partial \bar{\psi}} = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \bar{\psi})} \right) \quad \Rightarrow \quad (i\gamma_\mu \partial^\mu - m)\psi = 0$$

Dirac equation

Important definitions (in the Dirac basis)

$$\gamma^0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}; \; \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \qquad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \; \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \; \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$\bar{\psi} = \psi^\dagger \gamma^0 \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Know your spaces

Remember a field can live in multiple spaces.

$$\Psi_{i,j,k} \Longleftarrow \begin{cases} i \Rightarrow isospin \Rightarrow (2\times 2) \\ j \Rightarrow flavour \Rightarrow (3\times 3) \\ \quad k\ldots \end{cases}$$

Take care of one space at a time. Suppose you have a field that lives both in the isospin two-dimensional space and in the flavour three-dimensional space. Suppose that you expand the SU(2) fields and they mix. You take care of the SU(2) mixing first, and leave the SU(3) vector for later

$$\begin{bmatrix} a_G & b_G \end{bmatrix} \begin{bmatrix} m_{11}^2 & m_{12}^2 \\ m_{21}^2 & m_{22}^2 \end{bmatrix} \begin{bmatrix} a_G \\ b_G \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_M & b_M \end{bmatrix} \begin{bmatrix} m_a^2 & 0 \\ 0 & m_b^2 \end{bmatrix} \begin{bmatrix} a_M \\ b_M \end{bmatrix}$$

And now expand the field in its SU(3) components

$$a_G = [a_1 \quad a_2 \quad a_3]$$

Gauge invariance

Is this Lagrangian invariant under some transformation?

$$\mathscr{L} = -\frac{1}{4}F^2 + \frac{1}{2}m^2A^2$$

If we try

$$A^\mu \to A^\mu + \partial^\mu \phi$$

It is easy to show that

$$F^{\mu\nu} \to F^{\mu\nu} \qquad\qquad A^2 \nrightarrow A^2$$

And therefore the spin 1 Lagrangian is invariant under this transformation but a mass term is not. What about the Maxwell Lagrangian with sources?

$$\mathscr{L} = \mathscr{L}_{Free} + \mathscr{L}_{Int} = -\frac{1}{4}F^2 - A_\alpha J^\alpha$$

P: Find this out. Write $A_\alpha J^\alpha \to (A_\alpha + \partial_\alpha \phi)$ and use $\partial_\alpha (J^\alpha \phi)$.

Gauge invariance

What is meant by gauge transformation?

The term gauge refers to any specific mathematical formalism to regulate redundant degrees of freedom in the Lagrangian of a physical system. The transformations between possible gauges, called gauge transformations, form a Lie group—referred to as the symmetry group or the gauge group of the theory.

What is the gauge principle?

A gauge principle specifies a procedure for obtaining an interaction term from a free Lagrangian which is symmetric with respect to a continuous symmetry—the results of localising (or gauging) the global symmetry group must be accompanied by the inclusion of additional fields (such as the electromagnetic field), with appropriate kinetic and interaction terms in the action, in such a way that the extended Lagrangian is covariant with respect to a new extended group of local transformations.

What is the U(1) symmetry?

The group U(1) corresponds to the circle group, consisting of all complex numbers with absolute value 1, under multiplication.

Local Gauge invariance - minimal coupling

Let us consider the free Klein-Gordon Lagrangian, but for a complex scalar field

$$\mathscr{L}_0 = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$ Invariant under $\phi \to \phi e^{-i\epsilon}$

Let us also suppose that the symmetry is local

$$\partial_\mu [\phi e^{-i\epsilon}] = (\partial^\mu \phi) e^{-i\epsilon} - i\phi((\partial^\mu \epsilon) e^{-i\epsilon}$$ not covariant

The fields and the derivatives transform as (infinitesimally)

$$\begin{cases} \delta\phi = -i\epsilon\phi \\ \delta(\partial_\mu \phi) = -i\epsilon(\partial_\mu \phi) - i(\partial_\mu \epsilon)\phi \end{cases}$$

P: Show these relations

The variation of the Lagrangian is

$$\delta\mathscr{L}_0 = \frac{\partial \mathscr{L}_0}{\partial \phi}\delta\phi + \frac{\partial \mathscr{L}_0}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + (\phi \leftrightarrow \phi^*)$$

Local Gauge invariance - minimal coupling

Leading to

$$\delta\mathscr{L}_0 = (\partial_\mu \epsilon) i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) = (\partial_\mu \epsilon) J^\mu$$

Noether current related to electric charge

Let us add the term

$$\mathscr{L}_1 = -eJ^\mu A_\mu \quad \text{with} \quad A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \epsilon$$

The transformation for $\mathscr{L}_1$ reads

$$\delta\mathscr{L}_1 = -e(\delta J^\mu)A_\mu - eJ^\mu \delta A_\mu = -2e\phi^*\phi\partial^\mu \epsilon A_\mu - J_\mu \partial^\mu \epsilon$$

And one more term with the corresponding variation

$$\mathscr{L}_2 = e^2 A^2 \phi^* \phi \quad \Rightarrow \quad \delta\mathscr{L}_2 = 2eA_\mu(\partial^\mu \epsilon)\phi^*\phi$$

P: Show

And finally we add

$$\mathscr{L}_3 = -\frac{1}{4}F^2 \quad \Rightarrow \quad \delta\mathscr{L}_3 = 0$$

P: Show

Local Gauge invariance - minimal coupling

We can add all pieces to get

$$\mathscr{L} = (\partial_\mu \phi + ieA_\mu \phi)(\partial^\mu \phi^* - ieA^\mu \phi^*) - m^2 \phi^* \phi - \frac{1}{4}F^2$$

and defining

$$D_\mu \phi \equiv (\partial_\mu + ieA_\mu)\phi \qquad \text{with} \qquad \delta(D_\mu \phi) = -i\epsilon(D_\mu \phi) \qquad \text{covariant derivative}$$

The Lagrangian is written as

$$\mathscr{L} = (D_\mu \phi)(D^\mu \phi)^* - m^2 \phi^* \phi - \frac{1}{4}F^2$$

Lesson: if you know the symmetry, you write the corresponding covariant derivative and you get the interaction term. A term $m^2 A^2$ would not be invariant. Enter the Higgs mechanism.
For the SM the covariant derivative is

$$D_\mu \equiv \partial_\mu - i\frac{g_1}{2}YB_\mu - i\frac{g_2}{2}\sigma_j W_\mu^j - i\frac{g_3}{2}\lambda_j G_\mu^j$$

U(1) hypercharge ($\frac{g_1}{2}YB_\mu$) — SU(2) Isospin ($\sigma_j W_\mu^j$) — SU(3) Colour ($\lambda_j G_\mu^j$)

Covariant derivative and interactions

Let us suppose that the SM has a real scalar field and a photon. The free piece of the Lagrangian is

$$\mathscr{L}_{free} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) - \frac{1}{4}F^2$$

The covariant derivative is defined (if we consider the symmetries of the SM)

$$D_\mu \equiv \partial_\mu - i\frac{g_1}{2}YB_\mu - i\frac{g_2}{2}\sigma_j W_\mu^j - i\frac{g_3}{2}\lambda_j G_\mu^j$$

The field ϕ has hypercharge 0, isospin 0 and colour 0. So

$$D_\mu = \partial_\mu$$

Lesson: <u>if such a field would exist in the SM it would have no interactions with the photon.</u>

P: What kind of interaction could this field have?

Covariant derivative and interactions

Let us now suppose that the SM has a complex scalar field and a photon. The free piece of the Lagrangian is

$$\mathscr{L}_{free} = (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) - \frac{1}{4} F^2$$

Again the covariant derivate is

$$D_\mu \equiv \partial_\mu - i\frac{g_1}{2} Y B_\mu - i\frac{g_2}{2}\sigma_j W_\mu^j - i\frac{g_3}{2}\lambda_j G_\mu^j$$

The field ϕ has now hypercharge 1, isospin 0 and colour 0. So

$$D_\mu \equiv \partial_\mu - i\frac{g_1}{2} B_\mu$$

And we recover the Lagrangian

$$\mathscr{L} = (\partial_\mu \phi - i\frac{g}{2} B_\mu \phi)(\partial^\mu \phi^* + i\frac{g}{2} B^\mu \phi^*) - m^2 \phi^* \phi - \frac{1}{4} F^2$$

P: Write the most general Lagrangian up to dimension 4 with one real field ϕ_1 and one complex field ϕ_2.

Spontaneous symmetry breaking - The Higgs mechanism

Suppose we now start with the same Lagrangian

$$\mathscr{L} = (D_\mu \phi)(D^\mu \phi)^* - m^2 \phi^* \phi - \frac{1}{4} F^2$$

but redefine the scalar field such that (this is SSB)

$$\phi \to \phi + v$$ where v is a real constant

The two last terms are not relevant, but the kinetic term has now the extra terms

$$e^2[B^2 \phi^* \phi + v^2 B^2 + v B^2 \phi + \ldots]$$

P: Show this is true.

A mass term for the vector was born

$$m_B^2 = e^2 v^2$$ this is SSB - giving a non-zero VEV to the field generates mass

The U(1) symmetry, present in the original Lagrangian was broken by the vacuum expectation value of the scalar field. We lost the U(1) symmetry and so the photon gains a mass. And electric charge is no longer conserved.

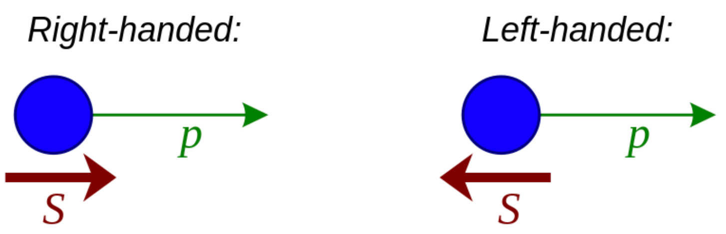
Gauge invariance

What is chirality?

A chiral phenomenon is one that is not identical to its mirror image. The spin of a particle may be used to define a handedness, or helicity, for that particle, which, in the case of a massless particle, is the same as chirality. A symmetry transformation between the two is called parity transformation. Invariance under parity transformation by a Dirac fermion is called chiral symmetry.

What is helicity?

The helicity of a particle is positive ("right-handed") if the direction of its spin is the same as the direction of its motion. It is negative ("left-handed") if the directions of spin and motion are opposite. Mathematically, helicity is the sign of the projection of the spin vector onto the momentum vector: "left" is negative, "right" is positive.



For massless particles – photons, gluons – chirality is the same as helicity.

SSB - Minimal requirements for electroweak symmetry breaking

Gauge structure

- $SU(2)_L \times U(1)_Y$ describes the electroweak interactions.
- $SU(2)_L$ factor is a left-handed isospin – an isospin charge carried by left-chirality fermions (eigenstate of left projector $\gamma_L = (1 - \gamma_5)/2$) .
- $U(1)_Y$ factor is related to the weak hypercharge.
- For each fermion type, left-handed and right-handed states are independent. Mass terms appear due to bilinear interactions which link different fields – they break electroweak symmetry.
- Electroweak symmetry also requires the gauge bosons to be massless but they do have a mass.

Interlude - Mass and gauge eigenstates

The simplest Lagrangian with two mixed fields we can write is

$$\mathscr{L} = a(\phi_1^g)^2 + 2b\phi_1^g\phi_2^g + c(\phi_2^g)^2 = \begin{bmatrix}\phi_1^g & \phi_2^g\end{bmatrix}\begin{bmatrix}a & b\\ b & c\end{bmatrix}\begin{bmatrix}\phi_1^g\\ \phi_2^g\end{bmatrix}$$

Rotate from the gauge eigenstates to the mass eigenstates

$$\Phi^m = R_\alpha \Phi^g \qquad \begin{bmatrix}\phi_1^m\\ \phi_2^m\end{bmatrix} = \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\phi_1^g\\ \phi_2^g\end{bmatrix} \qquad \tan 2\alpha = \frac{2b}{a-c}$$

$$\mathscr{L} = (\Phi^g)^T M_g \Phi^g = (\Phi^m)^T R_\alpha M_g R_\alpha^T \Phi^m = (\Phi^m)^T M \Phi^m$$

Show that

$$\begin{cases} m_1^2 = \dfrac{a+c+\sqrt{(a-c)^2-4b^2}}{2} \\[2ex] m_2^2 = \dfrac{a+c-\sqrt{(a-c)^2-4b^2}}{2} \end{cases} \quad \Rightarrow \quad b = 0 \quad \begin{cases} m_1^2 = a \\ m_2^2 = c \end{cases}$$

Interlude - Mass and gauge eigenstates

Consider the Lagrangian

$$\mathscr{L} = \partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2 - m_1^2 \phi_1^2 - m_2^2 \phi_2^2 - m_{12}^2 \phi_1 \phi_2 + \lambda \phi_1^2 \phi_2^2$$

which is not complete.

P: Find the mass eigenstates. Write the Lagrangian in terms of the mass eigenstates.

SSB in the SM

The difference to the SM is that the Higgs field in an SU(2) doublet. The electroweak covariant derivative is a 2 by 2 matrix and therefore we need a doublet field

$$D_\mu \equiv \partial_\mu - i\frac{g_1}{2}YB_\mu - i\frac{g_2}{2}\sigma_j W_\mu^j \qquad \Phi_{SM} = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \qquad <\Phi_{SM}> = \begin{pmatrix} v_1 + iv_2 \\ v_3 + iv_4 \end{pmatrix}$$

Vacuum expectation values of the Higgs doublet

If there is SSB, the gauge boson mass terms always come from the covariant derivative term.

The Higgs doublet has four components but the fact that the field is invariant under SU(2) allows to rotate fields away (it is like choosing a reference frame). The Higgs field has isospin 1/2 and hypercharge 1 (we choose). So we choose

$$<\Phi_{SM}> = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \qquad Q_{SM} <\Phi_{SM}> = \left(I_3 + \frac{Y}{2}\right) <\Phi_{SM}> = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = 0$$

P: Expand $(D_\mu \Phi)^\dagger (D^\mu \Phi)$

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \ \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \ \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Mass eigenstates - gauge bosons

- Gauge bosons

$$D_\mu \Phi = \partial_\mu \Phi - \frac{i}{2}\begin{bmatrix} g W_\mu^3 + g' B_\mu & \sqrt{2} g W_\mu^+ \\ \sqrt{2} g W_\mu^- & -g W_\mu^3 + g' B_\mu \end{bmatrix}\Phi$$

$$< \Phi_{SM} > = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2)$$

$$D_\mu \equiv \partial_\mu - i\frac{g'}{2} Y B_\mu - i\frac{g}{2}\sigma_j W_\mu^j - i\frac{g_3}{2}\lambda_j G_\mu^j$$

$$\left\langle (D_\mu \Phi)^\dagger (D^\mu \Phi) \right\rangle_{\text{mass}} = \frac{g^2 v^2}{2} W_\mu^+ W^{\mu -} + \frac{v^2}{8}\begin{bmatrix} W_\mu^3 & B_\mu \end{bmatrix}\begin{bmatrix} g & -g'g \\ -g'g & g'^2 \end{bmatrix}\begin{bmatrix} W^{3\mu} \\ B^\mu \end{bmatrix}$$

$$= \frac{g^2 v^2}{2} W_\mu^+ W^{\mu -} + \frac{(g^2 + g'^2) v^2}{4} Z_\mu Z^\mu$$

$$\begin{cases} Z_\mu = c_W W_\mu^3 - s_W B_\mu \\ A_\mu = s_W W_\mu^3 + c_W B_\mu \end{cases}$$

$$c_W = \frac{g}{\sqrt{g^2 + g'^2}} = \frac{M_W}{M_Z}$$

Charge Breaking (CB) in the SM

What would happen if we started with

$$< \Phi_{SM} > = \begin{pmatrix} v_1 + i v_2 \\ v_3 + i v_4 \end{pmatrix}$$

We would find the same mass spectrum (for the gauge bosons)

$$m_1^2 = m_2^2 = \frac{g^2 v^2}{4} \qquad v^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2$$

$$m_3^2 = \; = \frac{v^2}{4}(g^2 + g'^2 Y^2)$$

$$m_4^2 = 0$$

It's the photon!

So U(1) survives and charge is always conserved independently of the pattern of symmetry breaking - a consequence of gauge invariance.

Parity (P), Charge conjugation (C), and CP

Parity is an operation that works like a reflection relative to an axis. If $\vec{r}$ is the particle's position $P(\vec{r}) = -\vec{r}$ and $P[F(\vec{r})] = \pm F(-\vec{r})$.

Observables are even (if invariant) and odd (if not invariant) under P. A space derivative is odd under P. Angular momentum is even.

https://radiolab.org/podcast/122382-desperately-seeking-symmetry

Parity is violated by the weak interactions

C.S. Wu proved in 1956 that Parity was not conserved in weak interactions following a proposal by T.D. Lee and C.N. Yang.

In QED and QCD parity is conserved.

^{60}Co atoms aligned by a uniform magnetic field and cooled to near T=0 K so that thermal motions did not ruin the alignment. Cobalt-60 is unstable and decays beta to ^{60}Ni. During this decay, one of the neutrons in the cobalt-60 nucleus decays to a proton emitting an e^- and an ν_e (weak). The resulting NI nucleus is an excited state and decays to its ground state by emitting two gammas (electromagnetic).

QED conserves parity - the distribution of the emitted electrons could be compared to the distribution of the emitted photons in order to compare whether they too were being emitted isotropically. Distribution of gammas acted as a control for the distribution of the electrons.

Results: photons anisotropy was approximately 0.6. Wu observed that electrons were emitted in a direction preferentially opposite to that of photons with an asymmetry significantly greater than the photons. That is, most of the electrons favoured a very specific direction of decay, specifically opposite to that of the nuclear spin.

C is also violated in the weak interactions. A P-violating decay distribution is also a C-violating distribution. If charge conjugation symmetry held, polarized anti-cobalt would have produced a decay asymmetry with the opposite sign.

C is also violated in the weak interactions. A P-violating decay distribution is also a C-violating distribution. If charge conjugation symmetry held, polarised anti-cobalt would have produced a decay asymmetry with the opposite sign (but the experiment was not done).

Observations of the Failure of Conservation of Parity and Charge Conjugation in Meson Decays: the Magnetic Moment of the Free Muon*

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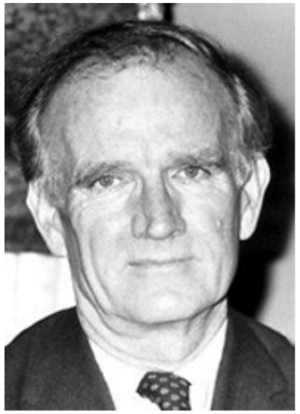
(Received January 15, 1957)

At the same time a new experiment proved both C and P violation. They measured opposite polarisations for $\mu+$ and $\mu-$ produced from unpolarised pions. (Pions are spinless, and therefore unpolarisable.) The existence of the polarisation is a P-violation; the correlation of polarisation with charge is a C-violation; the pion-producing decays separately conserve CP.

CP connects matter and anti-matter and is also violated

Landau proposed CP as the true matter antimatter symmetry.

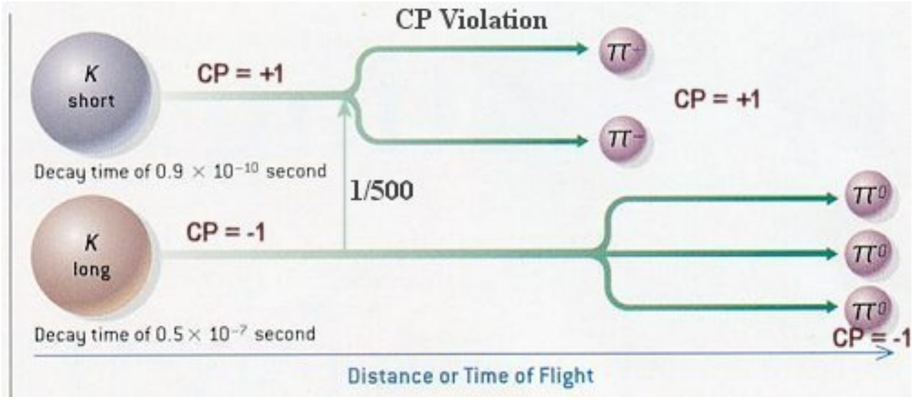


J.W.Cronin, V.L.Fitch - Discovered CP violation in 1964.

CP conservation would imply that a state K_2 (CP-odd) should decay into three particles, never two.

They detected a small but significant number (45 out of 22,700) of K_2 decays into two pions (a CP-even state).

So K_2 had to be a mixed CP state.



The need for more quarks - In 1973, Kobayashi & Maskawa showed that CP violation could not be accommodated in a theory with only four quark fields.

At that time only up, down & strange were known

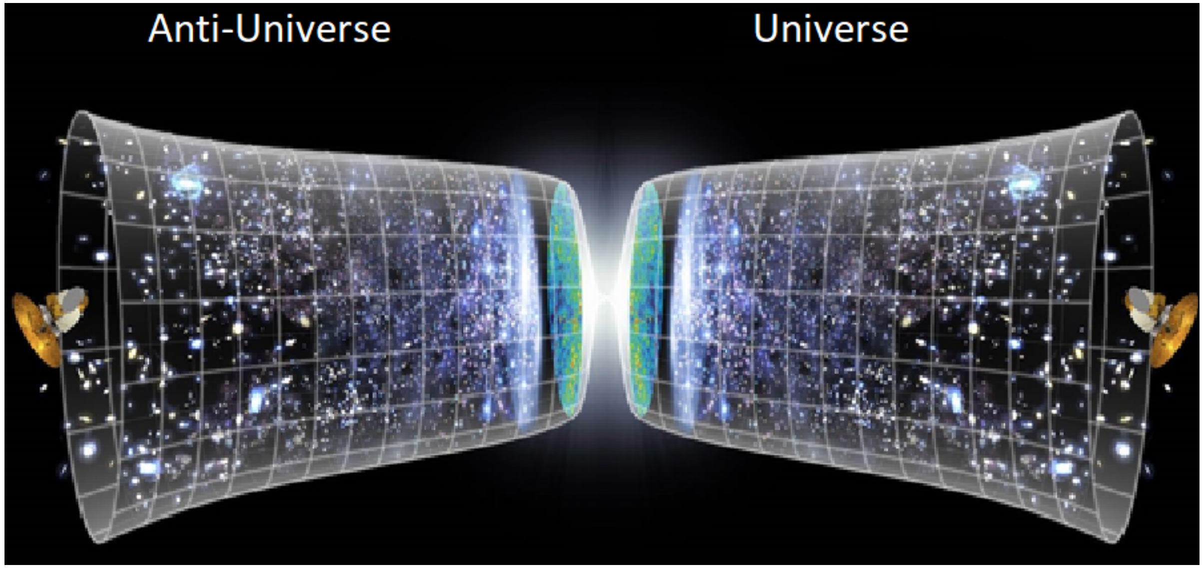
- Quarks largely considered as a mathematical model, not as real physical entities;
- Existence of charm hypothesised (GIM mechanism) ... but discovery not until the next year.

Among possible extensions, KM considered

- Introduction of a third family (bottom and top) of quarks;
- Quark mixing following the scheme introduced by Cabibbo.

Still not enough!

But where is anti matter? Is this it?

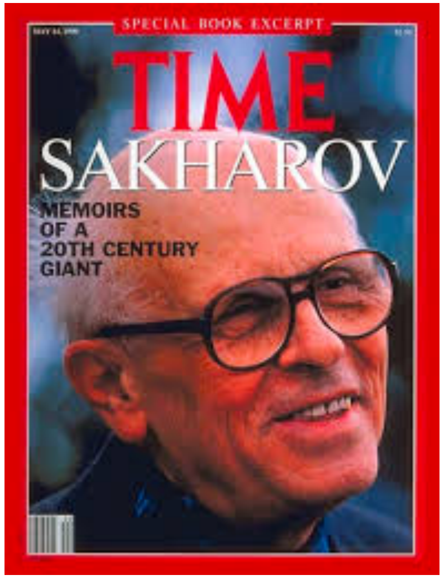


A. D. Sakharov, JETP Lett. 5, 24 (1967)

VIOLATION OF CP INVARIANCE, C ASYMMETRY, AND BARYON ASYMMETRY OF THE UNIVERSE

A. D. Sakharov
Submitted 23 September 1966
ZhETF Pis'ma 5, No. 1, 32-35, 1 January 1967

The theory of the expanding Universe, which presupposes a superdense initial state of matter, apparently excludes the possibility of macroscopic separation of matter from antimatter; it must therefore be assumed that there are no antimatter bodies in nature, i.e., the Universe is asymmetrical with respect to the number of particles and antiparticles (C asymmetry). In particular, the absence of antibaryons and the proposed absence of baryonic neutrinos implies a non-zero baryon charge (baryonic asymmetry). We wish to point out a possible explanation of C asymmetry in the hot model of the expanding Universe (see [1]) by making use of effects of CP invariance violation (see [2]). To explain baryon asymmetry, we propose in addition an approximate character for the baryon conservation law.



Sakharov's conditions necessary to obtain baryon asymmetry: **baryon number violation**, **loss of thermal equilibrium**, C and CP violation.

One of the conditions is C and CP-violation

The most subtle of Sakharov's requirements is C and CP-violation. Consider some process $X \to Y + B$ and suppose that C is a symmetry. Then the rate of the C-conjugate process is the same. That is

Cline, hep-ph 0609145

$$\Gamma(X \to Y + B) = \Gamma(\bar{X} \to \bar{Y} + \bar{B})$$

The net rate of baryon production goes like the difference of these rates

$$\frac{dB}{dt} \propto \Gamma(X \to Y + B) - \Gamma(\bar{X} \to \bar{Y} + \bar{B}) = 0$$

in a C-symmetric theory. However, even if C is violated this is not enough, we also need CP-violation. Let us consider an example where X decays into two left-handed or two right-haded quarks

$$X \to q_L q_L; \qquad X \to q_R q_R$$

Under CP

$$q_L \to \bar{q}_R$$

C and CP in Sakharov's conditions

While under C

Cline, hep-ph 0609145

$$q_L \to \bar{q}_L$$

Therefore, even though C violation implies that

$$\Gamma(X \to q_L q_L) \neq \Gamma(\bar{X} \to \bar{q}_L \bar{q}_L)$$

CP conservation would imply

$$\Gamma(X \to q_L q_L) = \Gamma(\bar{X} \to \bar{q}_R \bar{q}_R) \qquad \Gamma(X \to q_R q_R) = \Gamma(\bar{X} \to \bar{q}_L \bar{q}_L)$$

And therefore

$$\Gamma(X \to q_L q_L) + \Gamma(X \to q_R q_R) = \Gamma(\bar{X} \to \bar{q}_R \bar{q}_R) + \Gamma(\bar{X} \to \bar{q}_L \bar{q}_L)$$

And again we would have no asymmetry between matter and anti-matter.

CP not enough in the SM

Baryon asymmetry is characterised by the η parameter defined as

Cline, hep-ph 0609145

$$\frac{1}{7.04}\eta = \frac{n_B - n_{\bar{B}}}{s}$$

where n_B, $n_{\bar{B}}$ and s are the densities of baryons, density of anti baryons and entropy respectively.

This value has been fixed by the Wilkinson Microwave Anisotropy Probe experiment as

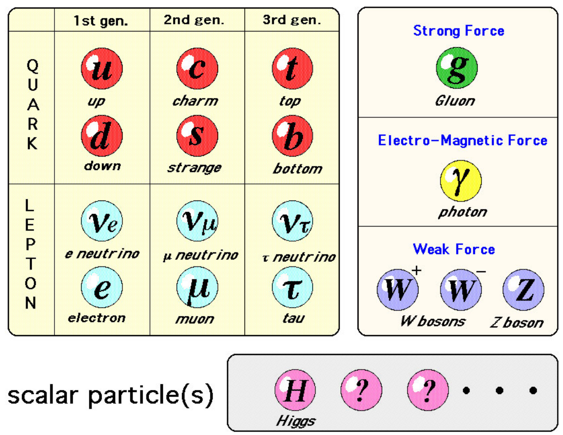
$$\eta = (6.14 \pm 0.25) \times 10^{-10}$$

From studying Jarlskog invariant we can conclude that Cp-violation asymmetry in the SM correspond to

$$\eta = 10^{-20}$$

showing the need for extra sources of CP-violation.

We extend the models by including new extra scalars and therefore a new source of CP-violation.



Elements of the Standard Model

Minimal extension with more CP-violation - The 2-Higgs doublet model

Potentials are usually used in minimal versions using ad-hoc symmetries. We just want them to suit our goals. The Z_2 symmetric version is

Complex - CP-violation

$$V_{2HDM} = m_{11}^2 |\Phi_1|^2 + m_{22}^2 |\Phi_2|^2 - (m_{12}^2 \Phi_1^\dagger \Phi_2 + h.c.)$$
$$\frac{\lambda_1}{2}(\Phi_1^\dagger\Phi_1)^2 + \frac{\lambda_2}{2}(\Phi_2^\dagger\Phi_2)^2 + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) \left\{ \frac{\lambda_5}{2}(\Phi_1^\dagger\Phi_2)^2 + h.c. \right\}$$

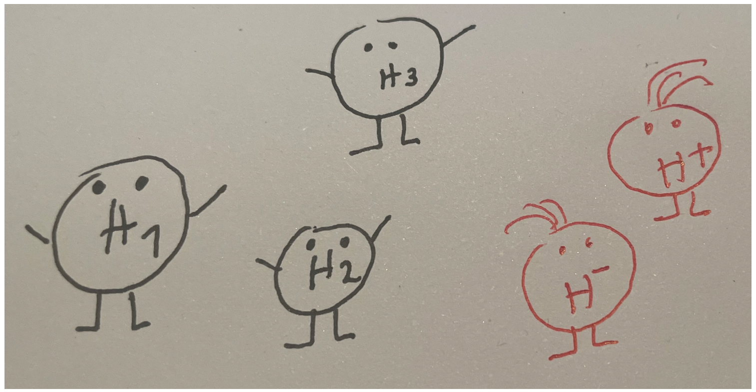
With the fields defined as (VEVs may be complex)

$v_2 = 0$, dark matter, IDM

$$\Phi_1 = \begin{pmatrix} \phi_1^+ \\ \frac{1}{\sqrt{2}}(v_1 + \rho_1 + i\eta_1) \end{pmatrix} \qquad \Phi_2 = \begin{pmatrix} \phi_2^+ \\ \frac{1}{\sqrt{2}}(v_2 + \rho_2 + i\eta_2) \end{pmatrix}$$

The particle content is the following

3 neutral Higgs with the same quantum numbers and no definite CP



2 charged Higgs bosons

CB (and CP) in the SM

You can use the SU(2) freedom to perform the rotation

$$<\Phi_{SM}> = \begin{pmatrix} v_1 + i v_2 \\ v_3 + i v_4 \end{pmatrix} \quad \rightarrow \quad <\Phi_{SM}> = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

C - Charge Conjugation

P - Parity

CP - C+P

Using a more general vacuum would just mean to redefine the charge operator.

For the same reason, any phase in the vacuum can be rotated away. This means that no spontaneous CP can occur. And the potential is also explicitly CP conserving.

Explicit breaking - if the Lagrangian is not invariant under a given symmetry

Spontaneous breaking - if the Lagrangian is invariant under a given symmetry but the vacuum is not

The SM has no CB and no CP violation in the potential.

C and P number without fermions

Suppose we have some extension of the SM but with no fermions. Also let us assume for the moment that the theory conserves C and P separately. The C and P quantum numbers of the photon and Z boson are

$$C(A_\mu) = P(A_\mu) = -1 \qquad C(Z_\mu) = P(Z_\mu) = -1$$

Not being sloppy

$$CZ_\mu C^{-1} = -Z_\mu; \quad PZ_\mu P^{-1} = Z^\mu \qquad P\partial_\mu P^{-1} = \partial^\mu$$

If the theory has the vertices hhh and HHH,

$$P(h) = P(H) = 1;\ C(h) = C(H) = 1 \qquad P(A) = 1;\ C(A) = -1$$

What about the terms below? Are they C invariant, P invariant, CP-invariant?

$$Z_\mu Z^\mu h; \qquad \partial \cdot Zh^2; \qquad \partial \cdot Zh; \qquad \partial \cdot ZAh$$