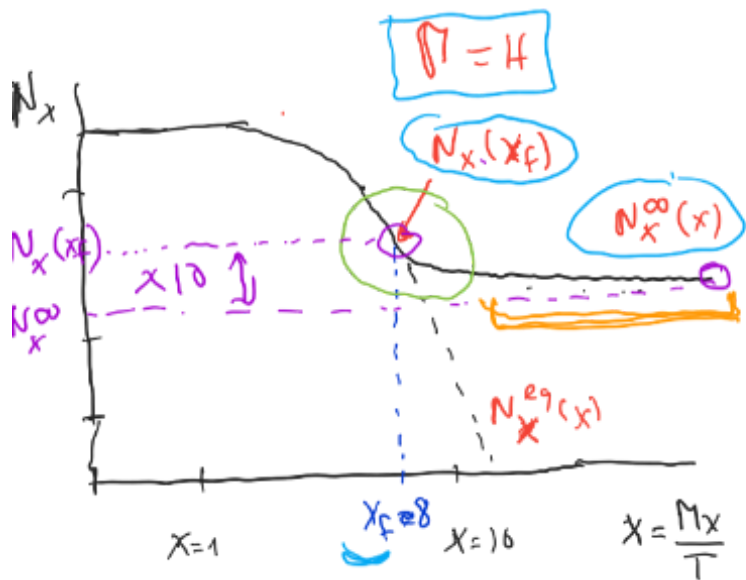


Ricatti Equation



$$\frac{dN}{dx} = -\frac{\langle \sigma v \rangle \Lambda}{x H} \left[N_x^2 - (N_x^{eq})^2 \right]$$

$$\Downarrow$$

$$\frac{dN_x}{dx} = -\frac{\lambda}{x^2} \left[N_x^2 - (N_x^{eq})^2 \right]$$

$N_x^{eq} \ll N_x$
 $x \gg x_f$

Lets assume the X, \bar{X} decouple during the epoch of radiation domination, so:

$$H = \frac{\pi}{3} \left(\frac{g_*}{10} \right)^{1/2} \frac{T^2}{M_{Pl}} = \left\{ \text{but } x = \frac{M_x}{T} \Rightarrow T = \frac{M_x}{x} \right.$$

$$= \frac{\pi}{3} \left(\frac{g_*}{10} \right)^{1/2} \left(\frac{M_x}{x} \right)^2 \frac{1}{M_{Pl}} =$$

$$= H(M_x) \frac{1}{x^2} \quad \text{where } H(M_x) = \frac{\pi}{3} \left(\frac{g_*}{10} \right)^{1/2} \frac{M_x^2}{M_{Pl}}$$

Now a similar expression can be obtain for the specific entropy:

$$\Omega = \frac{2\pi^2}{45} g_{*s} T^3 = \frac{2\pi^2}{45} g_{*s} \left(\frac{M_x}{x} \right)^3$$

So:

$$\frac{dN_x}{dx} = - \frac{\langle \sigma v \rangle}{x} \frac{\frac{2\pi}{45} g_{xs} \left(\frac{M_x}{x}\right)}{H(M_x) \frac{1}{x^2}} \left[N_x^2 - (N_x^{eq})^2 \right]$$

$$= - \frac{2\pi^2}{45} \frac{g_{xs}}{H(M_x)} \frac{\langle \sigma v \rangle M_x^3}{x^2} \left[N_x^2 - (N_x^{eq})^2 \right] =$$

$$= - \frac{\lambda}{x^2} \left[N_x^2 - (N_x^{eq})^2 \right]$$

Where

$$\lambda = \frac{2\pi^2}{45} \frac{g_{xs}}{H(M_x)} \langle \sigma v \rangle M_x^3 =$$

$$= \frac{2\pi^2}{45} \frac{g_{xs} M_x^3 \langle \sigma v \rangle}{\pi^{1/3} \left(\frac{g_x}{10}\right)^{1/2} M_x / M_{Pl}} =$$

$$= \frac{2\pi}{15} \left(\frac{10 g_{xs}^2}{g_x}\right)^{1/2} \langle \sigma v \rangle M_x M_{Pl}$$

Away from mass thresholds λ is approximately constant at high $x = M_x/T$.

At late times, i.e. $T \ll M \Rightarrow x \gg x_F$, so the Boltzmann equation simplifies as:

$$\frac{dN_x}{dx} = - \frac{\lambda}{x^2} \left[N_x^2 - \underbrace{(N_x^{eq})^2}_{\approx 0} \right] = - \frac{\lambda}{x^2} N_x^2$$

So integration with respect to x :

$$dN_x = \frac{dN_x}{dx} dx =$$

$$= -\frac{\lambda}{x^2} N_x^2 dx \Leftrightarrow$$

$$\Leftrightarrow \int_{N_x(x_f)}^{N_x(\infty)} \frac{dN_x}{N_x^2} = \int_{x_f}^{\infty} -\lambda \frac{dx}{x^2} \Rightarrow$$

assuming λ is constant

$$\Leftrightarrow \left[\frac{N_x^{-1}}{-1} \right]_{N_x(x_f)}^{\infty} = -\lambda \left[\frac{x^{-1}}{-1} \right]_{x_f}^{\infty} \Leftrightarrow$$

$$\frac{1}{N_x^{\infty}} - \frac{1}{N_x(x_f)} = -\lambda \left[\frac{1}{\infty} - \frac{1}{x_f} \right] \Leftrightarrow$$

$$\frac{1}{N_x^{\infty}} - \frac{1}{N_x(x_f)} = +\frac{\lambda}{x_f}$$

is typically 10 times smaller than $N_x(x_f)$

So one can neglect the term

$$\frac{1}{N_x(x_f)} \ll \frac{1}{N_x^{\infty}}$$

So:

$$\frac{1}{N_x^{\infty}} \approx \frac{\lambda}{x_f} \Leftrightarrow$$

$$N_x^{\infty} \approx \frac{x_f}{\lambda}$$

This result gives a good way to estimate

- to the freeze-out (relic) abundances of X, \bar{X}