WINPs: Weakly interactive Massive Particles

Let's assume decoupling of the Form (anahilation):

$$X + \bar{X} \rightleftarrows l + \bar{l}$$

l and $\bar{l}$ are massless an tightly coupled particles to the Fluid. Let's assume the Following conditions

- $n_l = n_l^{eq}$
- $n_{\bar{l}} = n_{\bar{l}}^{eq}$
- there is no initial assymetry of X and $\bar{X}$:
  $n_X = n_{\bar{X}}$, $n_X^{eq} = n_{\bar{X}}^{eq}$

The Boltzmann equation reads:

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\langle\sigma v\rangle\left[n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} n_3 n_4\right] \rightarrow$$

$$\rightarrow \frac{1}{a^3}\frac{d}{dt}(n_X a^3) = -\langle\sigma v\rangle\left[n_X n_{\bar{X}} - \left(\frac{n_X n_{\bar{X}}}{\cancel{n_l}\,\cancel{n_{\bar{l}}}}\right)_{eq} \overset{n_l^{eq}}{\cancel{n_l}}\;\overset{n_{\bar{l}}^{eq}}{\cancel{n_{\bar{l}}}}\right]$$

$$= -\langle\sigma v\rangle\left[n_X n_{\bar{X}} - n_X^{eq} n_{\bar{X}}^{eq}\right] =$$

$$= -\langle\sigma v\rangle\left[n_X^2 - n_X^{eq\,2}\right]$$

We can convert this equation in a Number of particles equation using $N_X = n_X / s$, $N_X^{eq} = n_X^{eq}/s$

$$\frac{1}{a^3}\frac{d}{dt}\Big[N_X \underbrace{s a^3}_{S'}\Big] = -\langle\sigma v\rangle\left[N_X^2 s^2 - {N_X^{eq}}^2 s^2\right] \iff$$

$$\frac{\cancel{s}\cancel{a^3}}{\cancel{a^3}}\frac{dN_X}{dt} = -\langle\sigma v\rangle s^{\cancel{2}}\left[N_X^2 - {N_X^{eq}}^2\right] \iff$$

$$\frac{dN_X}{dt} = -\langle\sigma v\rangle s\left[N_X^2 - \left(N_X^{eq}\right)^2\right]$$

Lets make a change of varible $\underbrace{x = \frac{M_X}{T}}$. So the left-hand-side gives:

$$\underbrace{\frac{dN_X}{dt}} = \frac{dN_X}{dx}\frac{dx}{dt} = \frac{dN_X}{dx}\underbrace{\frac{d}{dt}\left(\frac{M_X}{T}\right)} =$$

$$= \frac{dN_X}{dx}\frac{d}{dt}\left(M_X T^{-1}\right) = \frac{dN_X}{dx}\left(M_X (-1) T^{-2}\frac{dT}{dt}\right)$$

(1) $$= -\frac{dN}{dx}\frac{1}{T}\overset{x}{\left(\frac{M_X}{T}\right)}\left(\frac{dT}{dt}\right) = -x\frac{dN}{dx}\frac{1}{T}\frac{dT}{dt}$$

But using entropy conservation we know that

$$T = A\, g_{*s}^{-1/3}\, a^{-1}$$

lets start with the case when $g_{*s} \simeq$ constant (away from mass thresholds)

$$\frac{dT}{dt} = A\, g_{xs}^{-1/3} \frac{d a^{-1}}{dt} = A\, g_{xs}^{-1/3} (-1)\, a^{-2} \dot{a}$$

$$= -A g_{xs}^{-1/3} \frac{\dot{a}}{a}\, a^{-1} = -A g_{xs}^{-1/3} H a^{-1}$$

So going back to (1):

$$\frac{dN_x}{dt} = -x \frac{dN_x}{dx} \frac{1}{T} \left(-A g_{xs}^{-1/3}\right) H\, a^{-1} =$$

$$= -x \frac{dN_x}{dx} \frac{1}{\left(\cancel{A g_{xs}^{-1/3} a^{-1}}\right)} \left(-\cancel{A g_{xs}^{-1/3} a^{-1}}\right) H =$$

$$= x\, H \frac{dN_x}{dx}$$

This allows us to go back to Boltzmann equation:

$$\frac{dN_x}{dt} = x\, H \frac{dN_x}{dx} = -\langle \sigma v \rangle s \left[N_x^2 - \left(N_x^{eq}\right)^2\right]$$

$$\boxed{\frac{dN_x}{dx} \simeq -\frac{\langle \sigma v \rangle s}{x\, H} \left[N_x^2 - N_x^{eq\,2}\right]}$$

Ricatti equation