Saffman-Taylor instability

The Saffman–Taylor instability arises, or may arise, when two fluids of different viscosity are pushed by a pressure gradient through a Hele Shaw cell [§6.8] or allowed to drain through such a cell under their own weight. It would be of little practical importance were it not for the fact that creeping flow in a Hele Shaw cell is the two-dimensional analogue of creeping flow through a porous medium [§6.13]. Something very like the Saffman–Taylor instability frustrates attempts to extract, by pushing it out with pressurised water, the last traces of oil from oil wells. Theoretically, the instability has features in common with the Rayleigh–Taylor instability discussed in §8.2; it differs in that the equibrium state is a dynamic one, in which the interface between the two fluids is moving rather than stationary, but the analysis required is nevertheless distinctly similar.

Suppose the cell to be horizontal, in which case the effects of gravity may be ignored. Suppose it to be bounded by straight edges at $y = \pm\frac{1}{2}L$, and suppose there to be pressure gradients which are driving the fluid contents in the $+x$ direction with some uniform velocity U. In the equilibrium state whose stability we are to investigate, the interface between the two fluids is the straight line $x = Ut$. Where $x < Ut$, the viscosity is η'; where $x > Ut$, the viscosity is η. According to (6.47), the pressure gradients needed to maintain this motion are given in the two regions by

$$\frac{\partial p'}{\partial x} = -\frac{12\eta' U}{d^2}, \quad \frac{\partial p}{\partial x} = -\frac{12\eta U}{d^2},$$

where d is the thickness of the cell. The pressures p' and p are not necessarily equal at the interface, because the interface is liable to be curved in the vertical (z) direction. Provided that this curvature is constant, however, it does not affect the results of the analysis, so we may as well ignore it and write

$$p' = -\frac{12\eta' U}{d^2}(x - Ut) + p_o, \quad p = -\frac{12\eta U}{d^2}(x - Ut) + p_o,$$

for the equilibrium state, where p_o does not depend upon x.

Now suppose that the interface is perturbed, in such a way that at time t it lies at $x = X$, where

$$X = Ut + \zeta_k \mathrm{e}^{iky}.$$

There must be some corresponding perturbation in p' and p, and it must have the same periodicity in the y direction. However, p' and p obey Laplace's equation in two dimensions [§6.8], so any perturbing term which varies like $\exp(iky)$ must vary like $\exp(\pm kx)$ [(5.12)]. Since the perturbation cannot affect the pressure at large distances from the interface, the perturbed pressures presumably have the form

$$p' = -\frac{12\eta' U}{d^2}(x - Ut) + p_o + A' \mathrm{e}^{k(x-Ut)}\,\mathrm{e}^{iky}$$

$$p = -\frac{12\eta U}{d^2}(x - Ut) + p_o + A \mathrm{e}^{-k(x-Ut)}\,\mathrm{e}^{iky}$$

when k is positive, where the coefficients A' and A are to be determined by reference to the boundary conditions at the interface.

These boundary conditions, applicable in each case at $x = X$, and linearised by omission of terms which are of higher than first order in A or ζ_k are as follows.

(i)
$$\langle u'\rangle_x = \langle u\rangle_x = \frac{\partial X}{\partial t},$$

where $\langle \boldsymbol{u}\rangle$ is the mean velocity described by (6.47), or

$$-\frac{d^2}{12\eta'}\frac{\partial p'}{\partial x} = -\frac{d^2}{12\eta}\frac{\partial p}{\partial x} = U + \frac{\partial \zeta_k}{\partial t}\,\mathrm{e}^{\mathrm{i}ky}.$$

To first order this corresponds to

$$-\frac{d^2k}{12\eta'}A' = -\frac{d^2k}{12\eta}A = \frac{\partial \zeta_k}{\partial t}\,\mathrm{e}^{\mathrm{i}ky}. \tag{8.7}$$

(ii)
$$p' - p = -\sigma\frac{\partial^2 X}{\partial y^2} = \sigma k^2 \zeta_k\,\mathrm{e}^{\mathrm{i}ky},$$

where σ is the interfacial surface tension. To first order this corresponds to

$$A' - A = \left\{\frac{12U}{d^2}(\eta' - \eta) + \sigma k^2\right\}\zeta_k \exp(\mathrm{i}ky). \tag{8.8}$$

It is a trivial exercise to eliminate A' and A from (8.7) and (8.8), and so to obtain the result

$$s_k = \frac{1}{\zeta_k}\frac{\partial \zeta_k}{\partial t} = \frac{1}{\eta' + \eta}\left\{-U(\eta' - \eta)k - \frac{\sigma d^2 k^3}{12}\right\}. \tag{8.9}$$

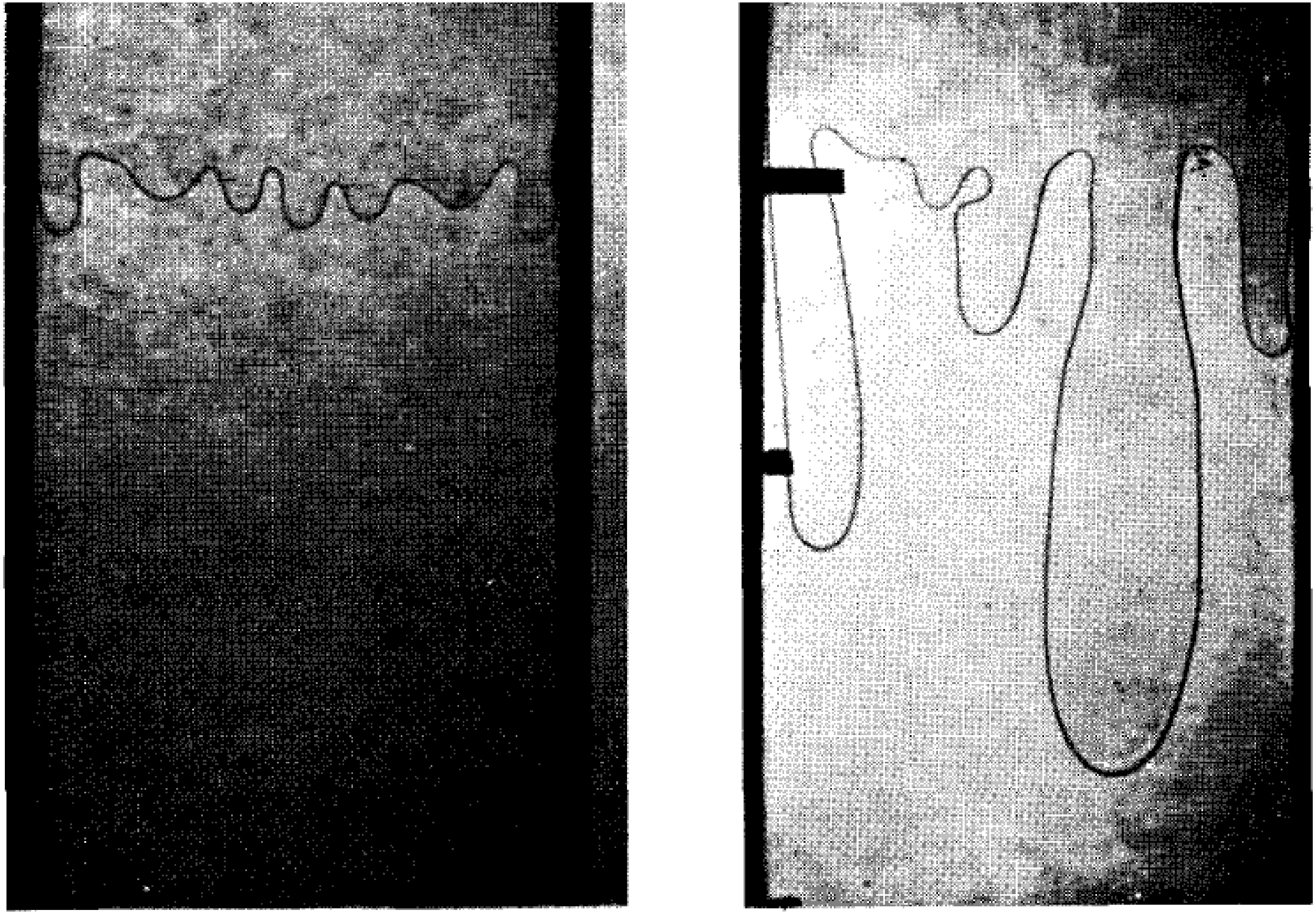
Thus if $\eta < \eta'$ the interface is stable for all k. When $\eta > \eta'$, however, i.e. when a viscous fluid is being displaced by a less viscous one, it is marginally stable with respect to a perturbation for which $k = k_c$, where

$$k_c^2 = \frac{12U(\eta - \eta')}{\sigma d^2},$$

and it is unstable with respect to perturbations for which $0 < k < k_c$. The perturbations which grow fastest (i.e. for which s_k is a maximum) have $k = k_c/\sqrt{3}$, i.e. a wavelength

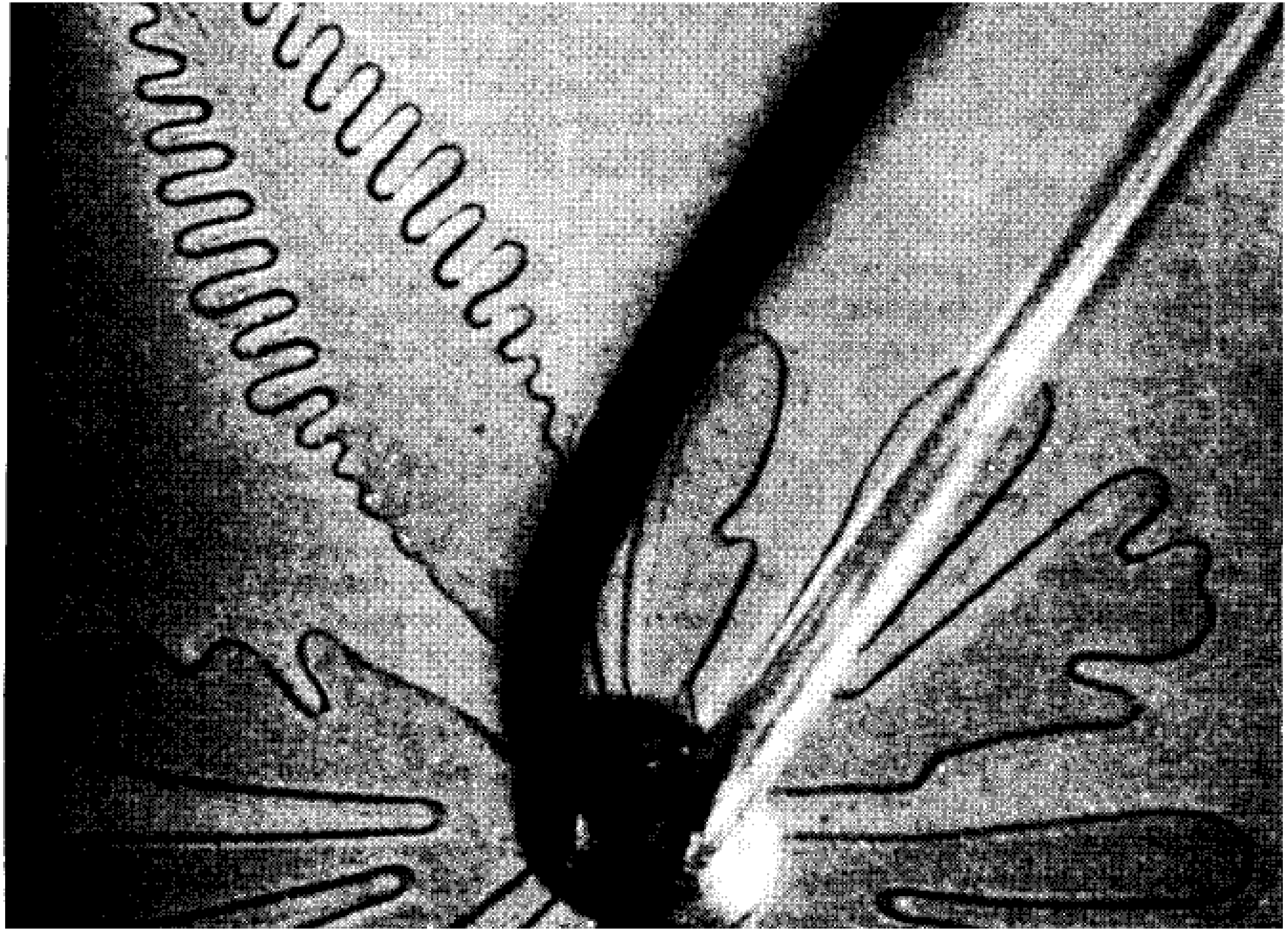
$$\lambda = \pi d \sqrt{\frac{\sigma}{U(\eta - \eta')}}. \tag{8.10}$$

The smallest value of k which is consistent with the boundary conditions at the sides of the cell, where $y = \pm\frac{1}{2}L$, is π/L, and if the cell is so narrow, or if U is so small, that this exceeds k_c then no instabilities can be observed. In the experiments conducted by Saffman and Taylor, however, in which air was used to displace glycerine through a cell whose thickness was about 1 mm, L was 12 cm and the wavelength λ predicted by (8.10) was normally a bit less than 2 cm. Thus they expected to see, when the pressure gradient was first applied, six or seven corrugations develop in the interface over the full width of the cell, and so they did; one of their photographs is reproduced as fig. 8.4(a).



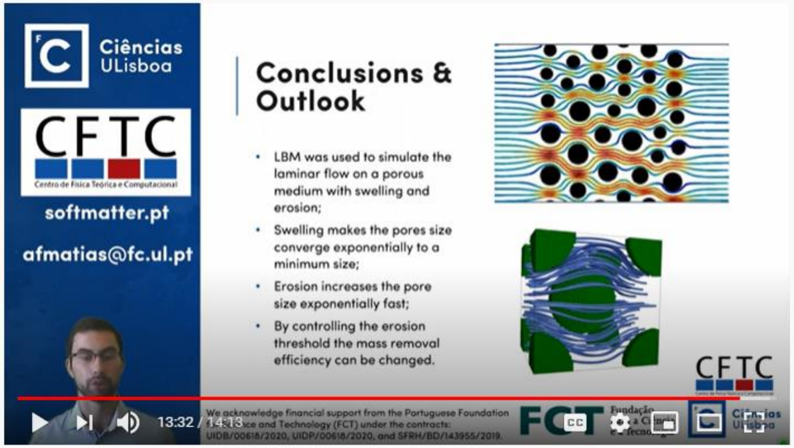
When the corrugations are no longer very small they do not all grow at the same rate, as is shown by fig. 8.4(b). One of the advancing *fingers* of the less viscous fluid tends to get ahead, whereupon it expands sideways and, by doing so, slows down the advance of its competitors. In due course only a single finger survives. It continues to advance at its tip, but it appears to stop expanding sideways when its width reaches half the width of the cell. The tip has a characteristically rounded shape, which Saffman and Taylor were able to explain.

Are the fingers stable and, if not, how do they split up? This question has proved in recent years to be of much greater complexity and interest than Saffman and Taylor could have guessed when their paper on this subject was published in 1958. A partial answer is provided by the two remarkable photographs of fingers spreading radially from a central source which are reproduced in figs. 8.5 and 8.6. The first one shows a number of fingers which are splitting in an irregular and unsurprising way, and one finger which has developed side branches of astonishing regularity; it differs from the others by having a defect at its tip, in the shape of a small gas bubble which has accidentally entered the apparatus and become entrained in the flow. The second photograph shows an even more regular pattern



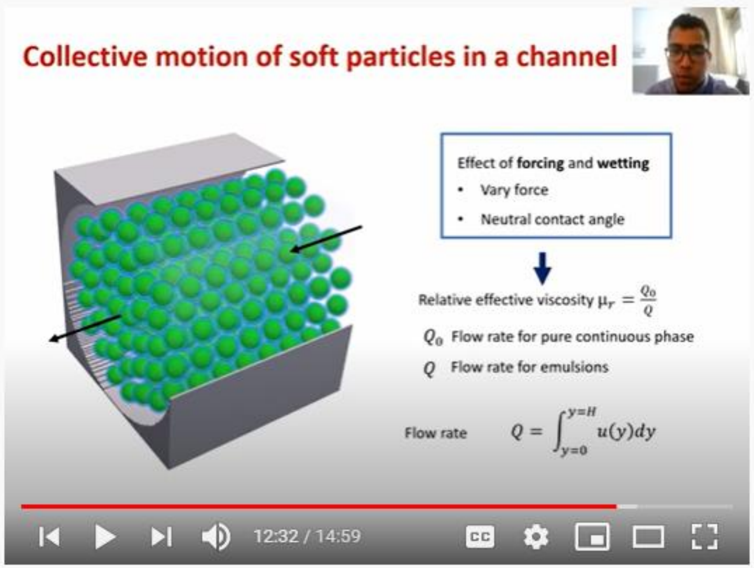
Presentations on fluid simulations

https://www.youtube.com/watch?v=pPYXBDXJ21Y



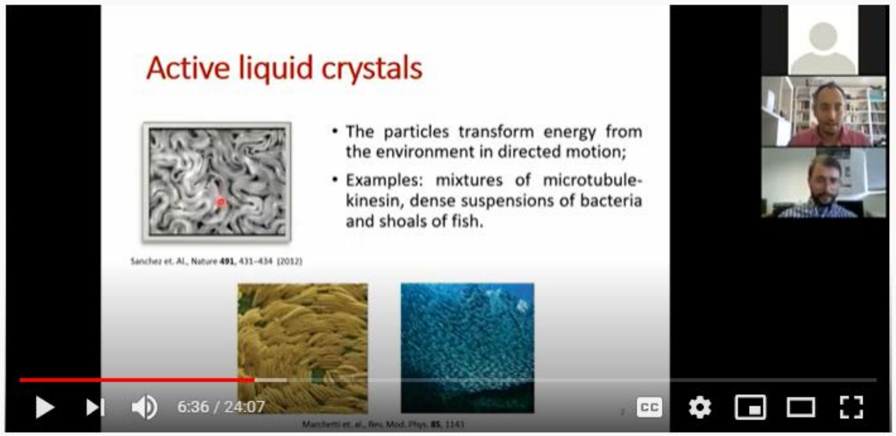
Mass extraction efficiency in a porous medium with swelling and erosion - A.F.V Matias

https://www.youtube.com/watch?v=1iAl8WrM8_c&list=PLWIVj90xdDE-67K1K-CR5bybhlHuAUfYq

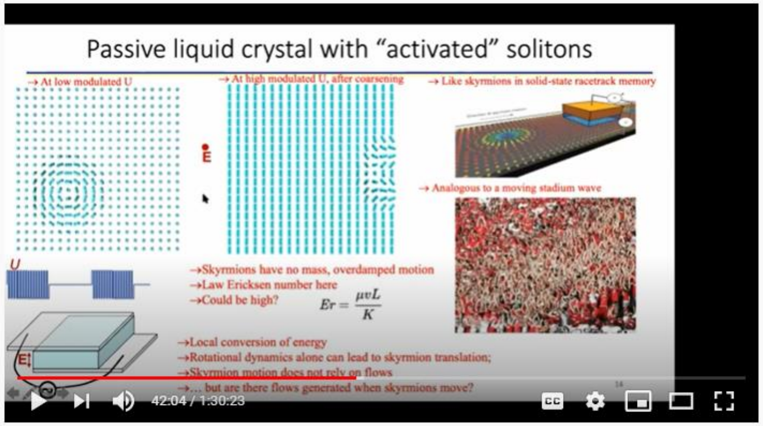


Discontinuous shear thinning of soft particles in a 3D microchannel - Danilo Silva

Liquid crystals



The active interfaces of swarming bacteria (Rodrigo Coelho)



CFTC seminar: Topological carnival

https://www.youtube.com/watch?v=UhQpb3iQOtM&t=84s

https://www.youtube.com/watch?v=rSL7NvFCAR8&t=2084s

Course review

Kinematics

Material derivative

$$\frac{D(\ldots)}{Dt} = \frac{\partial(\ldots)}{\partial t} + \vec{u}\cdot\nabla(\ldots)$$

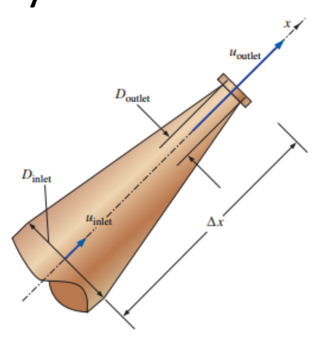
A material derivative is the time derivative of a property following a fluid particle.

Acceleration

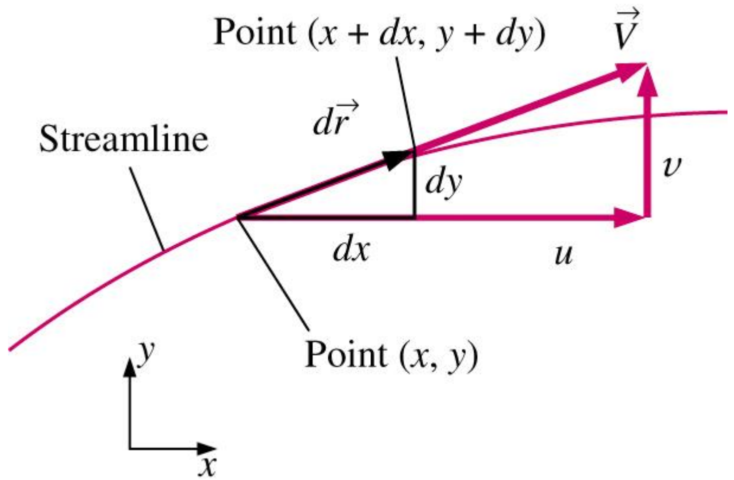
$$\vec{a} = \frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}$$

$\Rightarrow \frac{\partial \vec{u}}{\partial t} = 0$

Steady state does not mean necessarily **a**=0. Ex.:

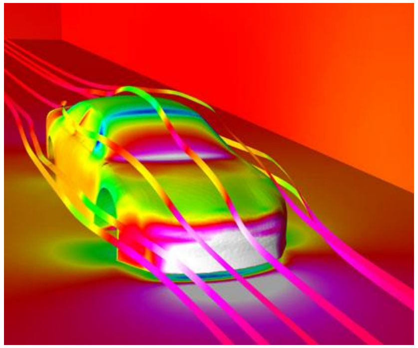


Streamline: is a curve that is everywhere tangent to the *instantaneous* local velocity vector.



$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

NASCAR surface pressure contours and streamlines



Other ways to visualize the flow:

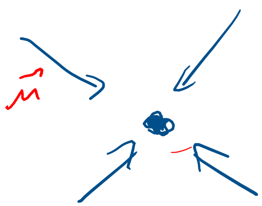
A **Pathline** is the actual path traveled by an individual fluid particle over some time period.

A **Streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.

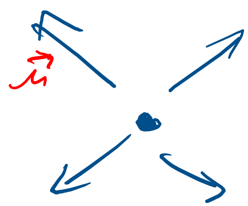
For **steady flow**, streamlines, pathlines, and streaklines are identical.

Continuity equation

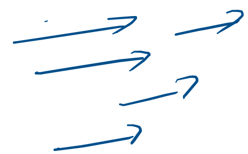
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \text{ , if } \underbrace{\rho = C_{te}}_{\text{incompressivel}} \Rightarrow \nabla \cdot \vec{u} = 0$$



$\nabla \cdot \vec{u} < 0$



$\nabla \cdot \vec{u} > 0$

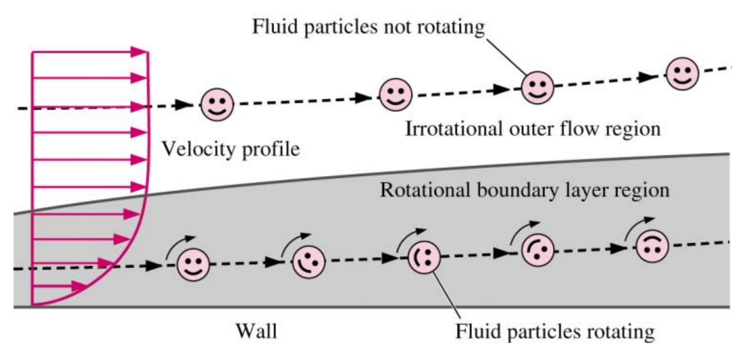


$\nabla \cdot \vec{u} = 0$

Vorticity

$$\vec{w} = \nabla \times \vec{u}$$

Boundary layer:



Euler equation: for incompressive and inviscid fluids.

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \boldsymbol{g},$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right)$$

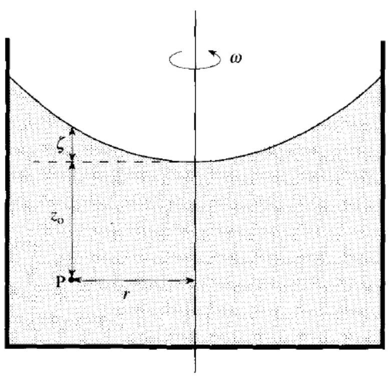
$(\nabla \wedge \boldsymbol{u}) = \vec{\omega}$

$\chi = \rho g z$

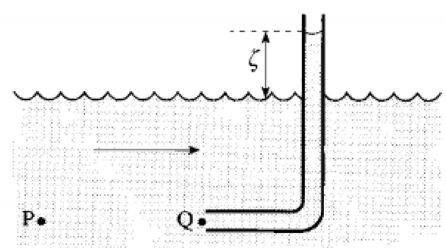
$\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right) = C_{te}$ if $\vec{\omega} = 0$

$\rho \frac{\partial \vec{u}}{\partial t} = 0$

$\vec{\omega} \neq 0$ (Euler)



$\vec{\omega} = 0$ (Bernoulli)



Potential flow. For irrotational flows in Euler fluids.

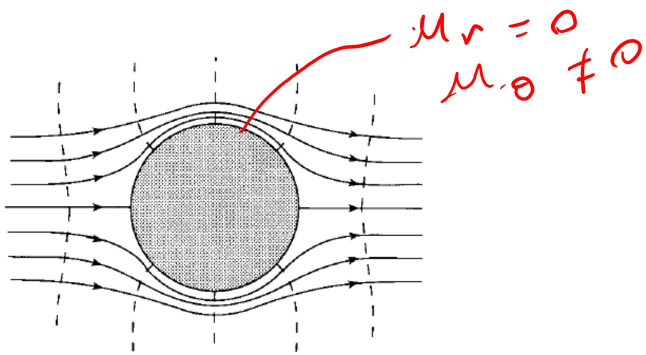
$$\vec{\omega} = \nabla \times \vec{u} = 0 \quad \Rightarrow \quad \vec{u} = \nabla \phi$$

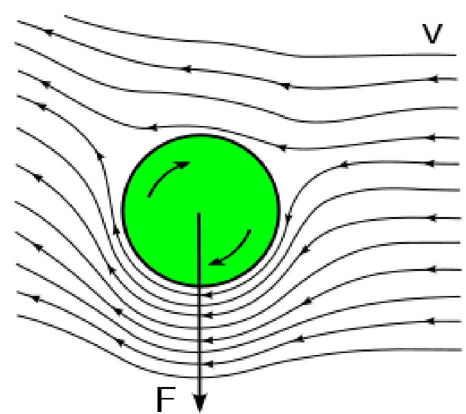
$$\nabla \cdot \vec{u} = 0 \quad \Rightarrow \quad \nabla \cdot \nabla \phi = \nabla^2 \phi = 0$$

In this case, the pressure is given by the Bernoulli equation.

Kelvin circulation theorem: An ideal fluid that is vorticity free at a given instant is vorticity free at all times.

Flow around a sphere: the drag and lift forces are zero for an ideal fluid.





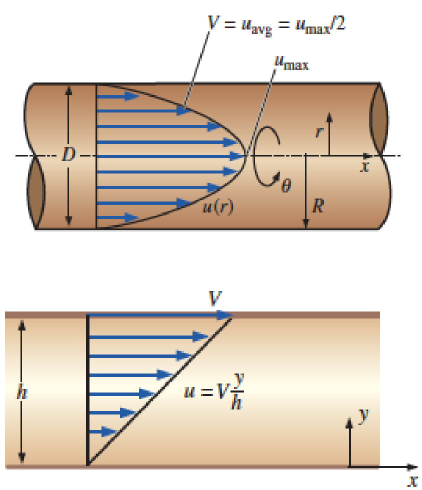
Navier-Stokes: incompressible viscous fluids.

Newtonian fluids, defined as fluids for which the shear stress is linearly proportional to the shear strain rate.

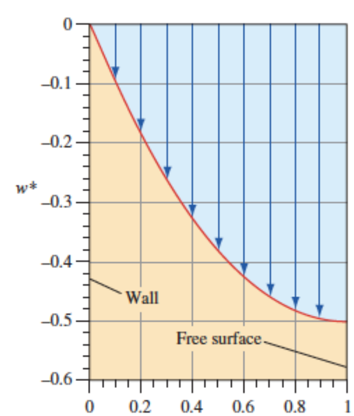
$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{\nabla p}{\rho} + \vec{g} + \nu \nabla^2 \vec{u}$$

Boundary conditions. 1) no-slip: at the surface, the velocity of the liquid and solid are the same. 2) Interface BC: at the interface, the velocity and the shear-stress of the two fluid are the same. 3) Free surface BC: at the free surface, the shear stress is zero.

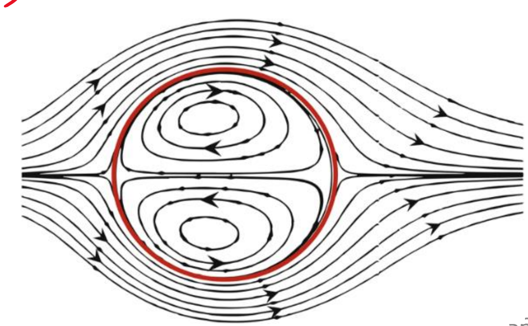
1)



3)

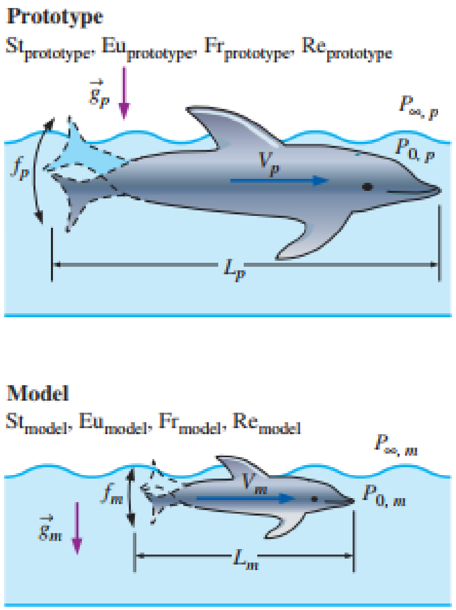


2)



Nondimensionalized Navier–Stokes:

$$[\mathrm{St}]\frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^* \cdot \vec{\nabla}^*)\vec{V}^* = -[Eu]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^*$$

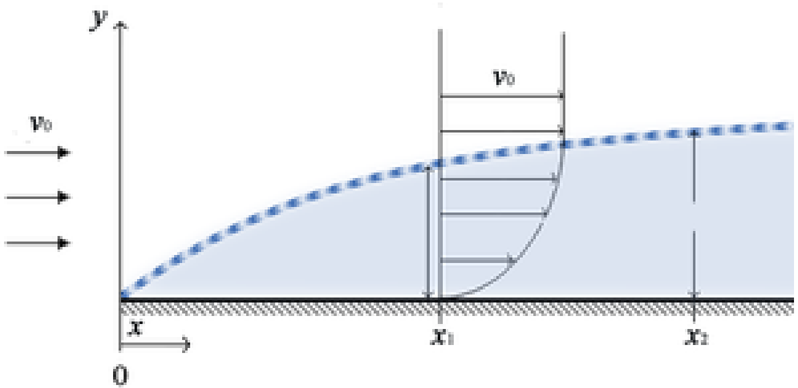


Since there are four dimensionless parameters, dynamic similarity between a model and a prototype requires all four of these to be the same for the model and the prototype ($\mathrm{St}_{model} = \mathrm{St}_{prototype}$, $\mathrm{Eu}_{model} = \mathrm{Eu}_{prototype}$, $\mathrm{Fr}_{model} = \mathrm{Fr}_{prototype}$, and $\mathrm{Re}_{model} = \mathrm{Re}_{prototype}$).

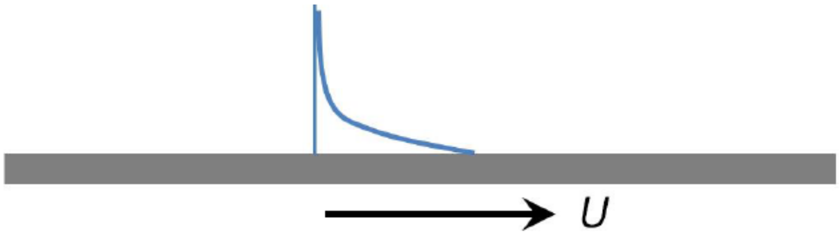
Approximate Navier–Stokes equation for creeping flow: $\vec{\nabla} P \cong \mu \nabla^2 \vec{V}$

Drag force on a sphere in creeping flow: $F_D = 3\pi\mu VD$

Boundary layer. Separates viscous and inviscid flows close to a solid surface.



Diffusion of vorticity



$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2},$$

Instabilities