# Cosmologia Física

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# Structure Formation

## Baryonic matter linear clustering

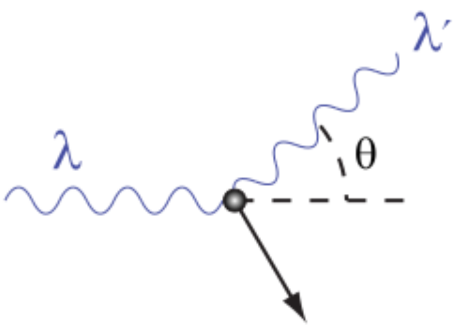
# Plasma perturbations

Before decoupling ($z > 1100$), the main components of the cosmological fluid are dark matter and the ionized **plasma of baryons + photons**.

(There are also other species that decouple earlier - like neutrinos)

In cosmology, **baryons** are free electrons and nuclear particles and all matter formed by them, i.e., atoms, molecules, stars, galaxies, clusters of galaxies, i.e., "normal matter" as opposed to dark matter.

In the early Universe, baryons are coupled to photons in a ionized plasma due to processes of interaction between radiation and matter, such as, **Compton scattering** and **Thomson scattering.**




(the electron reemits a photon of lower energy)

We want to study the evolution of the density contrast of the "plasma perturbations",

$$\bar{\rho}_{pl}\delta_{pl} = \bar{\rho}_r\delta_r = \bar{\rho}_b\delta_b$$

in the various epochs:

- radiation dominated

- matter dominated - that is divided in two sub-epochs:

  the **plasma epoch** ($z_{eq} > z > z_{dec}$)

  the **transparency epoch** ($z < z_{dec}$)

Unlike DM perturbations, in the case of plasma density perturbations we need to consider pressure in the fluid equations (radiation pressure w=1/3).

Pressure affects the fluid equations in 2 different ways:

- **The mean pressure contributes to the mean energy density**
  this is more relevant in the radiation epoch

- **The pressure also have perturbations**
  this is more relevant in the early matter epoch (the plasma epoch)

# Radiation-dominated epoch

In this epoch, **radiation pressure contributes to the mean energy density**

Let us see how this modifies the Newtonian fluid equations.

**One of the conditions for the validity of the Newtonian approximation is not verified anymore: pressure is no longer negligible and must be accounted for as a source of gravity.**

In Newtonian equations, pressure does not appear as a source of gravity. The correct calculation for energy conservation for relativistic particles in GR must be made with the Boltzmann equation.

## The Euclidean approximation

However it is still possible to use an approximation, the **Euclidean approximation** → no space-time curvature, valid for sub-Hubble scales (like the Newtonian approximation) but where Special Relativity applies and we may consider mass-energy equivalence.

So this approximation corresponds to a "**special relativistic Newtonian gravitational theory**", which is different from General Relativity.

Applying $\nabla_\mu T^\mu_\nu = 0.$ to a perfect fluid in a perturbed Minkowski metric (i.e. a metric with a potential Φ), we would get the special relativity continuity and Euler equations, which include density and pressure.

Continuity equation (it is now a conservation of total energy and not only mass conservation)

$$\frac{d\rho}{dt} + \nabla_r.(\rho\,\mathbf{u}) = 0 \quad \rightarrow \quad \frac{d(\rho+p)}{dt} - \frac{\partial p}{\partial t} + \nabla_r.(\rho+p)\,\mathbf{u} = 0$$

**There are two differences with respect to the Newtonian continuity equation:**

- the total energy density is $\boldsymbol{\rho}$ + p instead of $\boldsymbol{\rho}$

- the change of energy density at one point is not only due to the flow of the fluid, but also to the time-variation of the mean pressure.

Now, like we did before, we need to develop the total time derivative (to account for the expansion, i.e., to introduce comoving coordinates) and insert the total velocity

$$u = \dot{r} = \dot{a}x + v$$

The comoving equation is then, (remember ρ+p = 4/3 ρ)

$$\frac{4}{3}\frac{\partial \rho}{\partial t} - \frac{\dot{a}}{a}\vec{x}.\vec{\nabla}_x(\frac{4}{3}\rho) = \frac{\partial p}{\partial t} - \frac{1}{a}\left[(\dot{a}\vec{x} + \vec{v}).\vec{\nabla}_x\frac{4}{3}\rho + \frac{4}{3}\rho(3\dot{a} + \vec{\nabla}_x.\vec{v})\right]$$

which simplifies to

$$\frac{\partial \rho}{\partial t} + 4\frac{\dot{a}}{a}\rho + \frac{4}{3a}\vec{\nabla}.(\rho\vec{v}) = 0$$

Inserting the density perturbation: $\rho = \bar{\rho} + \delta\bar{\rho}$

the equation becomes, as usually, the sum between the zero-order continuity equation (in the presence of pressure) plus first and second-order terms.

The first-order terms give the **linearized perturbed comoving continuity equation in the presence of pressure**:

$$\boxed{\frac{\partial \delta}{\partial t} + \frac{4}{3}\frac{1}{a}\nabla.\mathrm{v} = 0}$$

## Euler equation

Now the source of 'force' is not only the potential but also the gradient of pressure, i.e., pressure perturbation.

$$\left(\frac{\partial u}{\partial t}\right)_r + (u.\nabla_r)u = -\nabla_r\Phi \quad \rightarrow \quad \left(\frac{\partial u}{\partial t}\right)_r + (u.\nabla_r)u = -\nabla_r\Phi - \frac{\nabla_r p}{\rho + p}$$

However, in the radiation epoch, the radiation energy density is high, and so is the mean pressure (1/3 of the energy density) → **pressure perturbations are negligible** compared to the mean pressure and the Euler force is dominated by the potential term → The Euclidean Euler equation remains identical to the Newtonian case.

So, the comoving linearized perturbed Euler equation remains the same as for dark matter:

$$\boxed{\frac{\partial v}{\partial t} + \frac{\dot{a}}{a}v = -\frac{1}{a}\nabla\Phi}$$

## Poisson equation

Note that the “special relativistic Newtonian theory” is not a fully developed real working theory, it is mostly a set of arguments to modify the Newtonian approximation.

So in principle we do not know what its Poisson-like equation is.
(The reason is that Poisson equation is a gravity equation, while continuity and Euler are energy conservation equations).

We can start by considering the zero-order Euler equation in comoving coordinates, which involves the “mean gravitational potential”:

$$\ddot{a}x = -\frac{1}{a}\nabla_x \bar{\Phi}$$

On the other hand, in the presence of pressure the second Friedmann equation is:

$$\frac{\ddot{a}}{a} + \frac{4}{3}\pi G(\bar{\rho} + 3p) = 0$$

Combining the 2 equations (with p = ρ/3), we get a Poisson-like equation:

$$\nabla^2\Phi = 4\pi\, G\, a^2 \bar{\rho}\, \delta \quad \rightarrow \quad \boxed{\nabla^2\Phi = 8\pi\, G\, a^2 \bar{\rho}\, \delta}$$

**So, the 3 fluid equations for plasma perturbations (without pressure perturbations) are only slightly different than those of dark matter perturbations.**

Combining the 3 equations as we did for dark matter, i.e., taking the time derivative of the continuity equation and combining with the divergence of the Euler equation (to cancel out the peculiar velocity terms),

we obtain **the evolution equation for the density contrast of the plasma**:

$$\boxed{\ddot{\delta} + \frac{2\dot{a}}{a}\dot{\delta} = \frac{32\pi}{3}G\bar{\rho}\delta}$$

instead of

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\,\bar{\rho}\,\delta = 0$$

**To solve it** we need to consider H(t) and ρ(t) in the radiation epoch.

From the Friedmann equation, dominated by radiation, we know that a (t) ~ $t^{1/2}$

and so, H(t) ~ $(2t)^{-1}$

From the zeroth-order continuity equation, the mean density evolution is

$$\frac{\bar{\rho}_{pl}}{\rho_c(a=1)} = \Omega_r a^{-4}$$

Remember that the plasma is a coupled fluid of photons and baryons, and at this point, $\rho_r$ or $\rho_b$ are indistinguishable.

Nevertheless, we cannot write $\rho_{pl} = \Omega_b\, a^{-4}$, because after decoupling $\rho_b$ does not continue to dilute with $a^{-4}$, and so the current value $\Omega_b$ cannot be used to parameterize the $a^{-4}$ behavior.

Inserting the Hubble function and mean density time-dependences results in an **equidimensional equation**:

$$\ddot{\delta} + \frac{1}{t}\dot{\delta} - \frac{1}{t^2}\delta = 0$$

The power-law solution has index: $n^2 - n + n - 1 = 0$

and so the growing solution is

$$\delta \propto a^2$$

There is a **growth of the plasma perturbations:** the plasma (or baryonic) perturbations grow fast in the short radiation epoch ($z >> z_{eq} \sim 3500$).

**Being coupled to the photons, baryonic matter is able to cluster faster than dark matter (during the brief period of the radiation epoch), because it is subject to the radiation pressure that increases the gravitational potential.**

*However, note that this result is valid deep in the radiation era in the approximation of zero pressure perturbations. The approximation breaks down as the radiation pressure dilutes, much before* $z_{eq}$*,*

# Matter-dominated epoch: the plasma epoch

In the matter epoch, the dominating fluid is CDM, and the mean radiation pressure that has been diluting with $a^{-4}$ gives now a negligible contribution to the total mean energy density.

However, pressure still exist in the plasma, and perturbations around a small mean pressure are relatively more important and have an impact in the plasma behavior → we need to consider **pressure perturbations**, $\delta_p$. **They will be responsible for a strong change in the growth of baryonic matter overdensities.**

For sub-Hubble scales we can use the Newtonian approximation (since there is no pressure contributing to the mean energy density). The equations are:

→ Continuity and Poisson equations: remain the same as in DM

→ Euler equation: is modified because the inhomogeneous pressure field introduces an extra force in the equation of motion of the fluid:

$$\left(\frac{\partial u}{\partial t}\right)_r + (u.\nabla_r)u = -\nabla_r \Phi - \frac{\nabla_r p}{\rho + p}$$

Inserting the comoving cordinates and keeping only the linear perturbed terms, the Euler equation is like the dark matter one with an additional term:

$$\frac{\partial v}{\partial t} + \frac{\dot{a}}{a} v = -\frac{1}{a}\nabla\Phi - \frac{1}{a}\frac{\nabla\bar{p}\delta_p}{\bar{\rho}}$$

Note that in the same way that the mean pressure may be related to the mean density (in the case of a **barotropic fluid**) defining an **equation-of-state** w = p/ρ,

the ratio of the (dimensional) pressure and density perturbations also defines an important property of the fluid: its **speed of sound $c_s$**:

$$c_s = \left(\frac{\delta_p\,\bar{p}}{\delta\,\bar{\rho}}\right)^{1/2}$$

Thus, we see that the new term in the Euler equation involves the square of the **speed of sound** in the plasma multiplied by the density contrast.

Inserting the speed of sound in the Euler equation, we write:

$$\frac{\partial v}{\partial t} + \frac{\dot{a}}{a} v = -\frac{1}{a} \nabla(\Phi + c_s^2 \delta)$$

**The Euler equation contains now a trade-off between 2 effects: attractive gravity (the potential, which is determined by the density contrast through Poisson equation) and repulsive pressure (the sound velocity), that works against the clustering.**

In addition, there is the usual Hubble drag term due to the expanding background.

Note that physically, this Euler equation is equivalent to the Jeans equation that describes the instability of a cloud of particles (gas) of density n as a mechanism for **star formation**:

$$\frac{\partial(n\langle v_j \rangle)}{\partial t} + n\frac{\partial \Phi}{\partial x_j} + \sum_i \frac{\partial(n\langle v_i v_j \rangle)}{\partial x_i} = 0$$

**In the case of a cloud of particles, the repulsive term is usually given by the dispersion velocity of the set of particles, while in the case of a fluid the repulsive term is given in terms of pressure gradients (the fluid sound speed).**
**The other terms of the original Jeans equation are the acceleration of the particles and the gravitational attraction, just like in Euler equation.**

Now, going to Fourier space, the spatial derivative is replaced by -ik, and this **introduces an explicit scale-dependence in the Euler equation**.

Combining the 3 comoving, linear and first order equations (continuity, Euler, Poisson) in the usual way, and written in Fourier space, we get the new equation for the evolution of $\delta_{pl}$ in the plasma epoch:

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\,\dot{\delta}_k - \left(4\pi G\bar{\rho} - \frac{c_s^2\,k^2}{a^2}\right)\,\delta_k = 0$$

This equation is known as the **Jeans equation.**

**Its solutions for δ will be scale-dependent** (due to the new k dependence)**:**

the equation defines a **threshold scale**, the **Jeans scale**, that separates **two regimes (gravity dominated vs pressure dominated):**

$$k_J = \sqrt{4\pi G\bar{\rho}}\,\frac{a}{c_s}$$ (the comoving Jeans scale)

It is also usual to define its reciprocal → the **Jeans length**:

$$\lambda_J(a) = \frac{2\pi}{k_J(a)}\, a = \frac{c_s(a)\sqrt{\pi}}{\sqrt{G\bar{\rho}(a)}}$$

(here written as the proper [non-comoving] Jeans length)

**The Jeans equation has two types of solutions:**

i) on **super-Jeans scales** ($k < k_J$) (i.e., the larger scales - but still sub-Hubble)

**gravity dominates →** the third term of the equation is negative.

This guarantees there is a **growing solution**:

- In the limiting (unrealistic) case of no expansion (no Hubble drag) → the growth would be exponential.

- In the limit of very large sub-Hubble scales $k << k_J$ (or also in the limit of $c_s = 0$ - which only happens after $z_{dec}$ - ) → the pressure term is zero and the equation is identical to the dark matter one, where we saw the solution is δ(a) ~a, or more precisely,

$\delta(a) \sim a^f$ (remember the growth parameter f accounts for different DE models)

ii) on **sub-Jeans scales** ($k > k_J$) (i.e., the smaller scales)

**pressure dominates** → the third term of the equation is positive.

This guarantees there is an **oscillating solution**. This means the clustering increases and decreases, oscillating in time, there is no net growth of the clustering → it is equivalent to a **frozen period** of no growth.

- In the limiting (unrealistic) case of no expansion (no Hubble drag) → the growth would be sinusoidal:

$$\delta = \sin\left(\sqrt{\frac{k^2\,c_s^2}{a^2} - 4\pi\,G\bar{\rho}}\,t\right)$$

- With background expansion, the Hubble drag term acts as friction → the solution is a **damped oscillation:**

$$\delta = \frac{A}{k^2\,c_s^2}\left[1 - \frac{\sin(kc_s\eta)}{kc_S\eta}\right]$$

(analytical solution written in terms of conformal time $\eta$)

## Properties of this solution

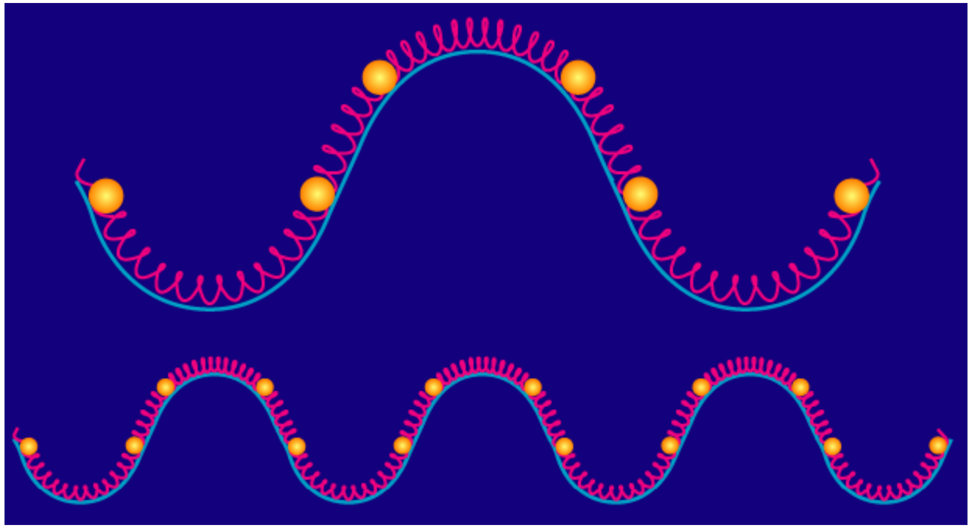
- the frequency in the solution is proportional to k → in **smaller scales the growth oscillates with larger frequencies.**

- the amplitude in the solution is inversely proportional to k → **larger scales reach higher clustering amplitudes** in their oscillating movement**.**

- the fact that the amplitudes of the perturbations oscillate in time in their spatial locations, creates a **propagation of the plasma overdensities in space, that propagates with the plasma sound velocity and creates spatial correlations in the plasma density field.**

- the scales oscillate in phase → at any given time, the clustering of any scale is slightly larger (or smaller) than the clustering of its neighboring scale → **the power spectrum of the plasma density will have an oscillatory shape → i.e., besides an oscillation in time there is also an oscillation in "space" (in scales).**

We can visualize Jeans equation with the following picture:



springs - photons

spheres - baryons

potential wells - dark matter

- The **gravity term** represents the potential (dominated by **dark matter potential wells**).

- The **pressure term** represents the movement of the coupled **baryon-photon plasma** (spheres with strings).

**We have thus encountered a second period where clustering is frozen**:

1 - For DM a freezing period occurs while scales are smaller than the Hubble radius during the (shorter) radiation epoch.

2 - For plasma (baryons) the freezing (with damped oscillations) period occurs while scales are smaller than the Jeans scale during the (longer) matter-plasma epoch.

## Evolution of the Jeans' scale

Like the Hubble radius, the Jeans scale also evolves with time. Its evolution depends on the evolutions of the mean density, scale factor, and **sound speed**.

$$k_J = \sqrt{4\pi G\bar{\rho}}\,\frac{a}{c_s}$$

In standard cosmologies, the expansion is **adiabatic** (even though other cases exist in alternative models), where the speed of sound is:

$$c_s^2 = \frac{\partial p}{\partial \rho}_{|\bar{\rho}}$$

So, since w=1/3 → p = 1/3 $\rho_\gamma$ $c^{2,}$ and $\rho = \rho_b + \rho_\gamma$ , the sound speed in the plasma is

$$\frac{\partial P}{\partial \rho}|_S = \frac{c^2}{3}\frac{\partial \rho_\gamma}{\partial(\rho_\gamma + \rho_B)}|_S = \frac{c^2}{3}\left(1 + \frac{\partial \rho_B}{\partial \rho_\gamma}|_S\right)^{-1}$$

We can get an expression for the derivative of $\rho_b$ with respect to $\rho_r$ by considering the homogeneous continuity equation for a fluid with 2 components,

$$\frac{d(\rho_b + \rho_\gamma)}{dt} + \frac{3}{a}\frac{da}{dt}(\rho_b + \rho_\gamma) = 0$$

$$d\rho_\gamma \left( \frac{d\rho_b}{d\rho_\gamma} + 1 \right) + 3\frac{da}{a}\rho_\gamma \left( \frac{\rho_b}{\rho_\gamma} + 1 \right) = 0$$

and inserting the solution from the continuity equation for radiation (there is energy conservation individually for each species),

$$\frac{d\rho_\gamma}{\rho_\gamma} = -4\frac{da}{a}$$

we get,

$$-4\left( \frac{d\rho_b}{d\rho_\gamma} + 1 \right) = -3\left( \frac{\rho_b}{\rho_\gamma} + 1 \right)$$

i.e.,

$$\frac{\partial \rho_B}{\partial \rho_\gamma}|_S = \frac{3}{4}\frac{\rho_B}{\rho_\gamma}$$

The speed of sound is then

$$c_s(a) = \frac{c}{\sqrt{3}} \left(1 + \frac{3}{4}\frac{\rho_b(a)}{\rho_\gamma(a)}\right)^{-1/2}$$

which can be written as,

$$c_s(a) = \frac{c}{\sqrt{3}} \left(1 + \frac{3}{4} a \frac{\Omega_b}{\Omega_\gamma}\right)^{-1/2} = \frac{c}{\sqrt{3}} \left(1 + \frac{3}{4} a\, f_b\, \frac{\Omega_m}{\Omega_\gamma}\right)^{-1/2} = \boxed{\frac{c}{\sqrt{3}} \left(1 + \frac{3}{4} f_b \frac{a}{a_{eq}}\right)^{-1/2}}$$

introducing the **baryonic fraction** $f_b = \Omega_b / \Omega_m \sim 0.15$

From this expression, we see that **the sound speed decreases as the Universe expands** (physically, this comes from the fact that the radiation density decreases faster than the matter density). In particular:

- **$a \ll a_{eq}$** → $\rho_\gamma \gg \rho_m$ → $c_s = c/\mathrm{sqrt}(3)$ → $c_s$ at its maximum value

- **$a = a_{eq}$** → $c_s = c/\mathrm{sqrt}(3.3)$

- **$a = a_{dec}$** → $c_s \sim c/3$ (using $a_{dec} \sim 3\, a_{eq}$)

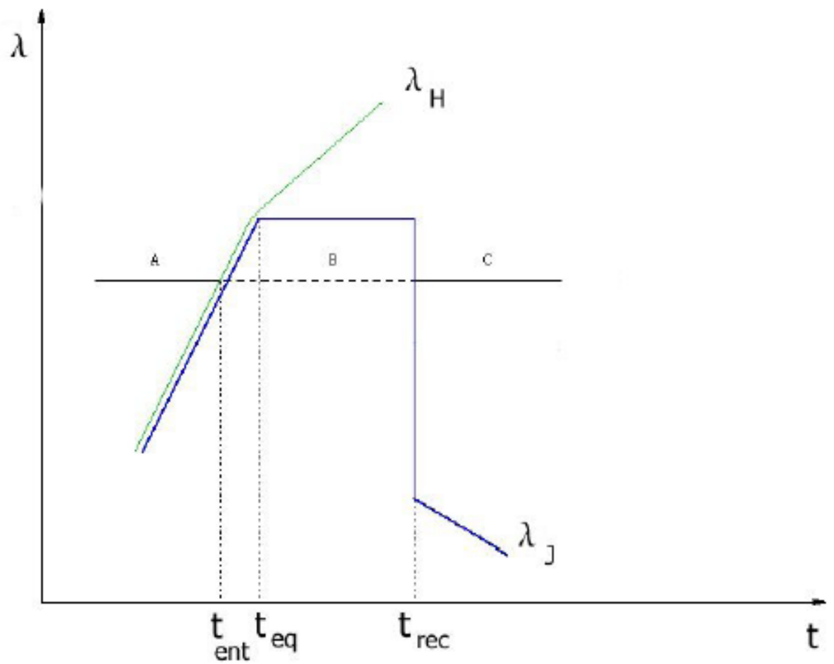
- **$a > a_{dec}$** → $c_s = 0$ (pressure drops to zero)

Now, the **comoving Jeans length** is $r_J \sim c_s / (a \, \mathrm{sqrt}(\rho))$

- **radiation epoch ($a < a_{eq}$)** → $c_s$ = constant = c/sqrt(3) , $\rho \sim a^{-4}$ → $r_J \sim a$

- **plasma epoch ($a_{eq} < a < a_{dec}$)** → towards the end of the epoch the factor 1 in $c_s$ is negligible $c_s \sim a^{-1/2}$, $\rho \sim a^{-3}$ → $r_J \sim$ constant

- **transparency epoch ($a > a_{dec}$)** → $c_s = 0$ → $r_J$ has a sudden drop to zero



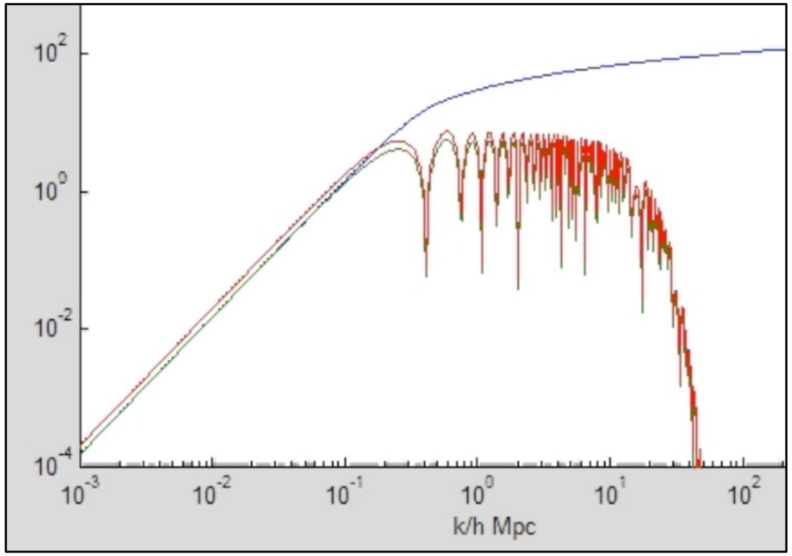

(evolution of Jeans and the Hubble lengths)

A → during this shorter period **baryon perturbations grow, while dark matter perturbations** either grow (if $r > r_H$ ) or **do not grow** (if $r < r_H$ ) → this is responsible for DM transfer function.

B → During this longer period all **DM perturbations grow, while the baryonic ones do not grow** (if $r < r_J$ ) → this is responsible for a **strong suppression of growth in baryonic matter (due the coupling with radiation → existence of pressure perturbations → speed of sound → Jeans equation).**

## Baryonic matter power spectrum

The resulting dimensionless **baryonic matter power spectrum** at the end of the plasma epoch looks like (green line):




It has a **lower amplitude** than the dark matter power spectrum (blue line) and strong oscillations on **intermediate scales**.

On **small scales** there are no overdensities due to yet another **free-streaming** effect. This time due to the fact that the last scattering surface has a finite thickness, i.e., **decoupling is not instantaneous → this is known as the Silk damping or diffusion damping.**

# Matter-dominated epoch: decoupling

## Silk damping

During this delta time, the plasma is "**partially coupled**" → photons start to stream out of the overdensities and drag baryons with them → **destroying the overdensity**.

On large perturbations, the photon crossing time is larger than the "thickness of the last-scattering surface" (i.e. the duration of the decoupling process) → they stream out of the perturbation already fully decoupled → no baryon dragging → **the overdensity remains in place.**

**The resulting effect is an exponential damping of the power spectrum amplitude** by a factor:

$$e^{-\ell^2/\ell_D^2}$$ (the effect is larger for large l - small scales)

here given in terms of angular scales l : l(z) = k $f_K$(z) = k $d_{A,com}$ (z)

$$\ell_D \sim k_D d_A^c(t_{\rm dec})$$

$k_D$ is the comoving diffusion scale, given by

$$k_D = \left( \frac{H(a_{\rm dec})}{\lambda_\gamma(a_{\rm dec})} \right)^{-1/2} a$$

The diffusion length $\lambda_\gamma$ is the size the photons with v=c can travel during the **thickness of the last scattering surface**

The result is $l_D$ ~ 1500 (concordance model), which corresponds to

$k_D$ (z = 1100) ~ 100 h $M_{pc}^{-1}$

## Matter-dominated epoch: the transparency epoch

**Baryonic acoustic oscillations** in the total matter power spectrum

**Once the plasma is dissolved, baryons become pressureless matter like dark matter** (as far as gravity is concerned; there is of course gas pressure once structures are formed, but that is not relevant for cosmological structure formation, only to the process of galaxy formation).

Its subsequent evolution is the same we found for DM, i.e, $\delta_b$ (a) ~ a.

However, the baryonic density field has a lower clustering amplitude since the baryon growth was suppressed for a long time (Jeans equation) → **$\delta_b$ is smaller than $\delta_{cdm}$ at $z_{dec}$**

**The value of $\delta_b$ at $z_{dec}$ is the amplitude of the CMB anisotropies** (as we will see later).

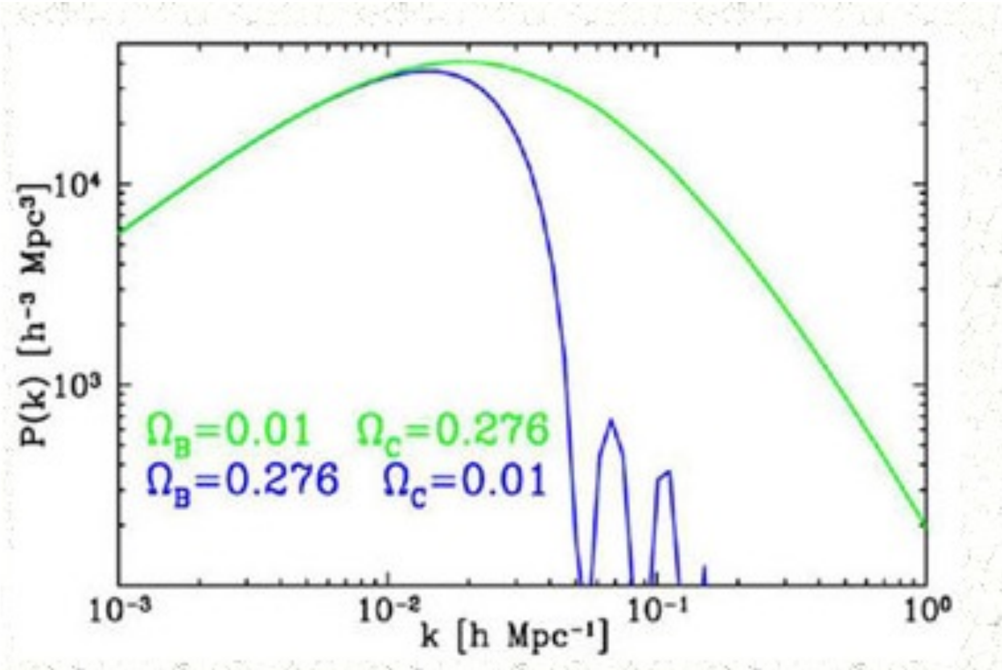
**These are $\delta T \sim 10^{-5}$ → the baryonic inhomogeneities are $\delta_b (z_{dec}) = 10^{-5}$**

**This means that from 1+z ~1100 until 1+z = 1 (today), $\delta_b$ can grow at most a factor of 1000, reaching $\delta_b$ (z=0) ~$10^{-2}$**

This does not agree with the observations → **there would exist no baryonic structures in the Universe**

(no astrophysical galaxies formed, only dark matter galactic halos).

Moreover, if baryons were the dominating matter species, the matter power spectrum would look very different than the one observed. For example:




Notice that any species that introduces a sound speed in the cosmological fluid creates oscillations in the overdensities.

This also happens in some dark energy models. The speed of sound needs to be kept to a very small value for those models to be viable.

These two arguments – larger amplitude and smooth shape of the matter power spectrum - led to the conclusion that structure formation needs to be driven by a fluid that does not couple with photons and is not subject to oscillations for such a long time → **This was the reason to introduce dark matter in the cosmological model.**

There is also another reason to introduce dark matter: to explain observed rotation curves of galaxies.

*These two reasons are different and deal with completely different scales.*
*In principle, dark matter does not need to be the same in both cases.*

**How does the presence of dark matter help in increasing baryonic clustering and in limiting the oscillations?**

After recombination, the matter perturbations evolve in the matter dominated background ($a \sim t^{2/3}$), with the system of equations:

$$\ddot{\delta}_{dm} + \frac{2\dot{a}}{a}\dot{\delta}_{dm} = 4\pi G(\rho_{dm}\delta_{dm} + \rho_b\delta_b)$$

$$\ddot{\delta}_b + \frac{2\dot{a}}{a}\dot{\delta}_b = 4\pi G(\rho_{dm}\delta_{dm} + \rho_b\delta_b)$$

i) Since $\rho_b\, \delta_b << \rho_{dm}\, \delta_{dm}$ let us neglect it in the third term of the equations:

dark matter: with $\rho_{dm}\, \delta_{dm} >> \rho_b\, \delta_b$, the equation for dark matter reduces to the usual one, with solution $\delta_{dm} \sim a$

baryonic matter: inserting the solution $\delta_{dm} = C\, a$ (where C is a constant) and neglecting $\rho_b\, \delta_b$ in the third term, the equation becomes,

$$\ddot{\delta}_b + 2\frac{\dot{a}}{a}\dot{\delta}_b - \frac{3}{2}\frac{H_0^2\Omega_m}{a^3}C\,a = 0$$

Changing the variable from t to "a" and with $a \sim t^{2/3}$, the equation becomes

$$\frac{d}{dt} = \dot{a}\frac{d}{da} = H_0 a^{-1/2}\frac{d}{da}$$

$$\delta_b'' + \frac{3}{2a}\delta_b' = \frac{3C}{2a}$$

(where the derivatives are taken with respect to the scale factor)

and it follows that $\delta_b = C\,a$ is a solution, i.e. the baryons not only start to cluster with the same rate as dark matter, but the solution has the same amplitude C.

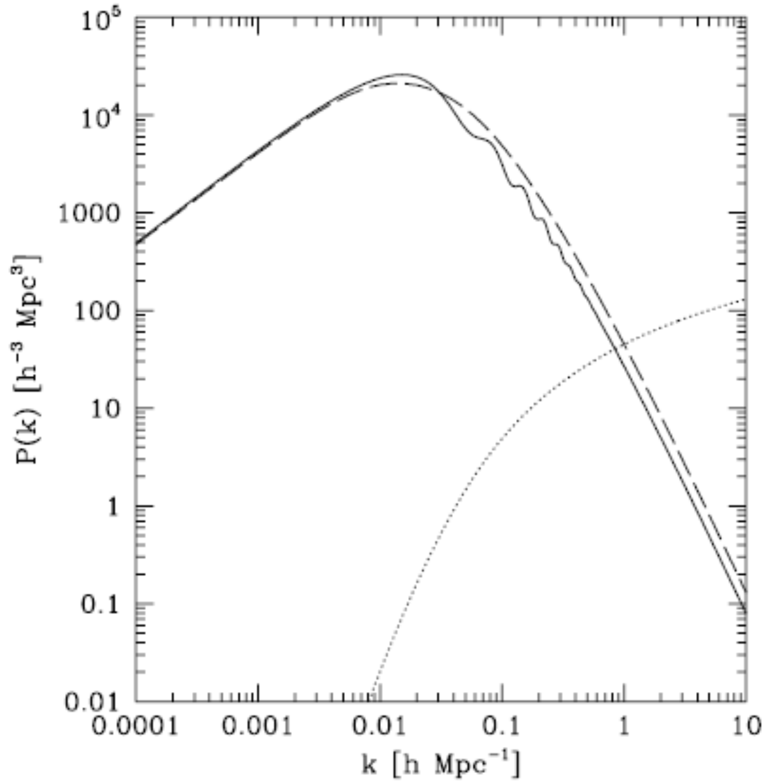
→ **This means that baryonic density contrast reaches the same amplitude as DM** soon after $z_{dec}$, i.e, **"baryons fall into the potential wells of dark matter"**, getting a large boost in amplitude → **it is the presence of dark matter that enables the formation of baryonic structures.**

Afterwards, $\delta_b$ continues to grow in the DM potential wells (the dark matter halos) with $\delta_b \sim a$. Their gravitational properties are now exactly the same as the ones of dark matter (dust) since they are no longer coupled to photons.

When $\delta_b \sim 1$, the baryonic perturbations are dense enough for the fluid to be in the form of gas → it starts to have important non-gravitational interactions. Pressure starts to be important again (not relativistic radiation pressure, but 'astrophysical pressure') → the process of galaxy formation starts, resulting in the formation of baryonic structures in the dark matter halos

ii) If we keep the term $\delta_b$ in the coupled equations (which contains the oscillating solution) a more precise solution is obtained:

the **total matter power spectrum** is a combination of the growth solution $\delta \sim a$ with the small amplitude oscillations of $\delta_b$ → **the small baryon oscillations become imprinted in the total matter power spectrum:**




(total matter power spectrum: solid
dark mater power spectrum: dashed)

The "relic" **baryon acoustic oscillations (BAO)** from the plasma epoch are still noticeable in the matter power spectrum at z=0, on intermediate scales.

On small scales there are no oscillations because of Silk damping.

The wiggles become less prominent in the matter power spectrum as time goes by (since the amplitude of the power spectrum increased by a factor $(1000)^2$ from the time it received the oscillations until today).

## Fitting function of the total matter power spectrum

It is sometimes useful to define **fitting functions** (fits to lengthy exact calculations or to numerical simulations) that directly model the physical effects. This also allows to speed up the calculations.

The total matter power spectrum is one case that gains by constructing a fitting function to capture all the relevant effects

$$T(k) = \frac{\Omega_b}{\Omega_0} T_b(k) + \frac{\Omega_c}{\Omega_0} T_c(k).$$

**Fitting functions:**

$$T_c(k) = \frac{\ln 1 + 2.34q}{2.34q} \left[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4\right]^{-1/4}$$

It is the transfer function used for the **dark matter power spectrum** (in the absence of baryons) but written with a modified scale variable,

$$q = \frac{k}{\Omega h^2} e^{2\Omega_b}$$

which leads to a suppression of power.

$$T_b = \left[\frac{\tilde{T}_0(k;1,1)}{1+(ks/5.2)^2} + \frac{\alpha_b}{1+(\beta_b/ks)^3} e^{-(k/k_{\rm Silk})^{1.4}}\right] j_0(k\tilde{s}).$$

(Eisenstein & Hu)

It is function of scale k, and depends on:

$$s = \int_0^{t(z_d)} c_s\,(1+z)dt$$ : sound horizon at $z_{dec}$

$$k_{\rm Silk} = 1.6(\Omega_b h^2)^{0.52}(\Omega_0 h^2)^{0.73}\left[1+(10.4\Omega_0 h^2)^{-0.95}\right]\ {\rm Mpc}^{-1}$$ : damping scale

$$\tilde{s}(k) = \frac{s}{(1+(\beta_{\rm node}/ks)^3)^{1/3}}.$$ : node shift $\quad \beta_{\rm node} = 8.41(\Omega_0 h^2)^{0.435}$

$j_0(x) = \sin(x)/x$ : Bessel function

$$\beta_b = 0.5 + \frac{\Omega_b}{\Omega_0} + (3 - 2\frac{\Omega_b}{\Omega_0})\sqrt{(17.2\Omega_0 h^2)^2 + 1}.$$

$$\alpha_b = 2.07 k_{\rm eq} s (1+R_d)^{-3/4} G\left(\frac{1+z_{\rm eq}}{1+z_d}\right) \qquad G(y) = y\left[-6\sqrt{1+y} + (2+3y)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right)\right]$$

$R_d = R\ (z_d)$ $\qquad R \equiv 3\rho_b/4\rho_\gamma$

$$\tilde{T}_0(k, \alpha_c, \beta_c) = \frac{\ln(e + 1.8\beta_c q)}{\ln(e + 1.8\beta_c q) + Cq^2},$$

$$C = \frac{14.2}{\alpha_c} + \frac{386}{1 + 69.9q^{1.08}}.$$

$$\alpha_c = a_1^{-\Omega_b/\Omega_0} a_2^{-(\Omega_b/\Omega_0)^3},$$

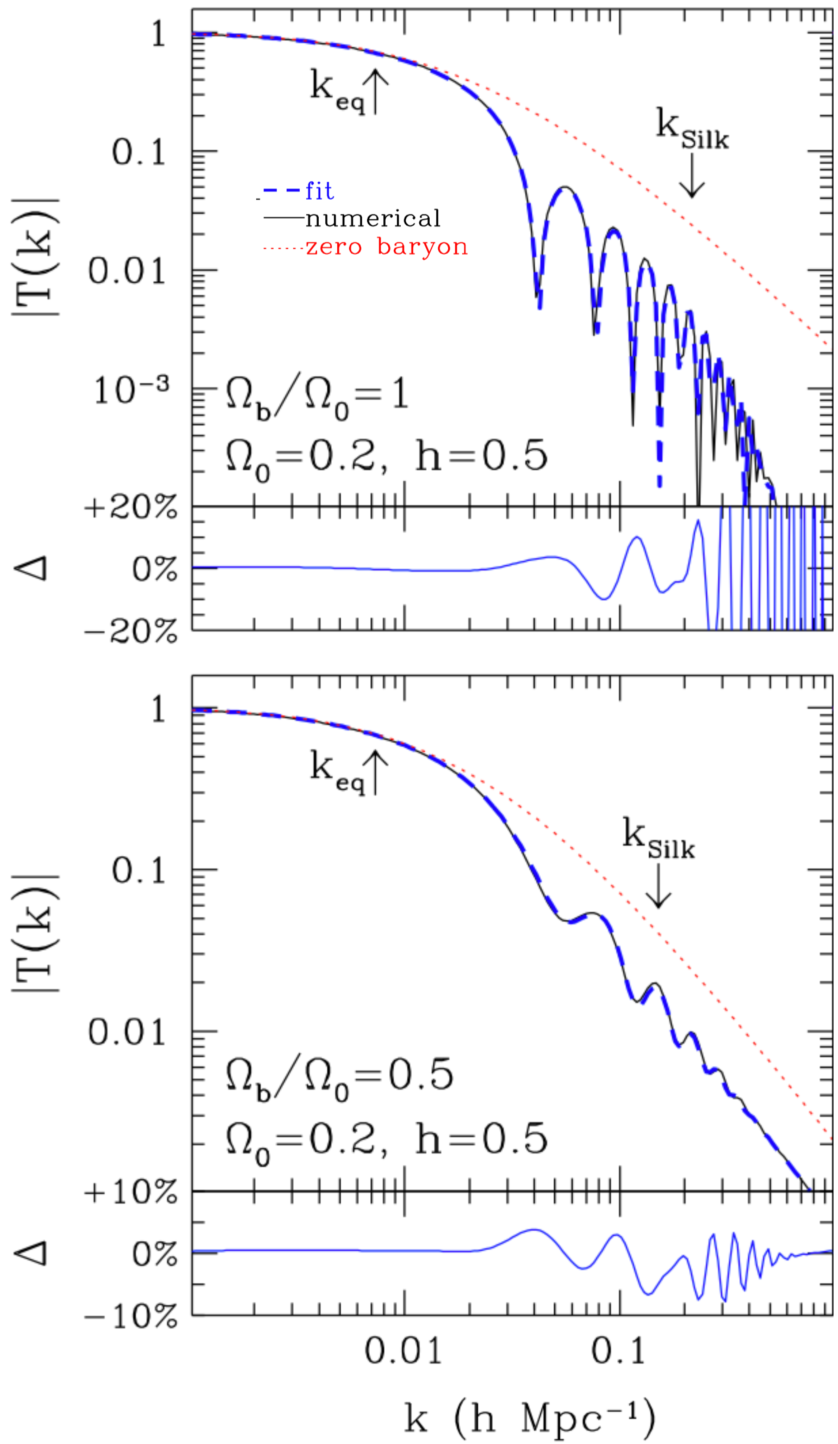
$$a_1 = (46.9\Omega_0 h^2)^{0.670}[1 + (32.1\Omega_0 h^2)^{-0.53}$$

$$a_2 = (12.0\Omega_0 h^2)^{0.424}[1 + (45.0\Omega_0 h^2)^{-0.58}$$

$$\beta_c^{-1} = 1 + b_1[(\Omega_c/\Omega_0)^{b_2} - 1],$$

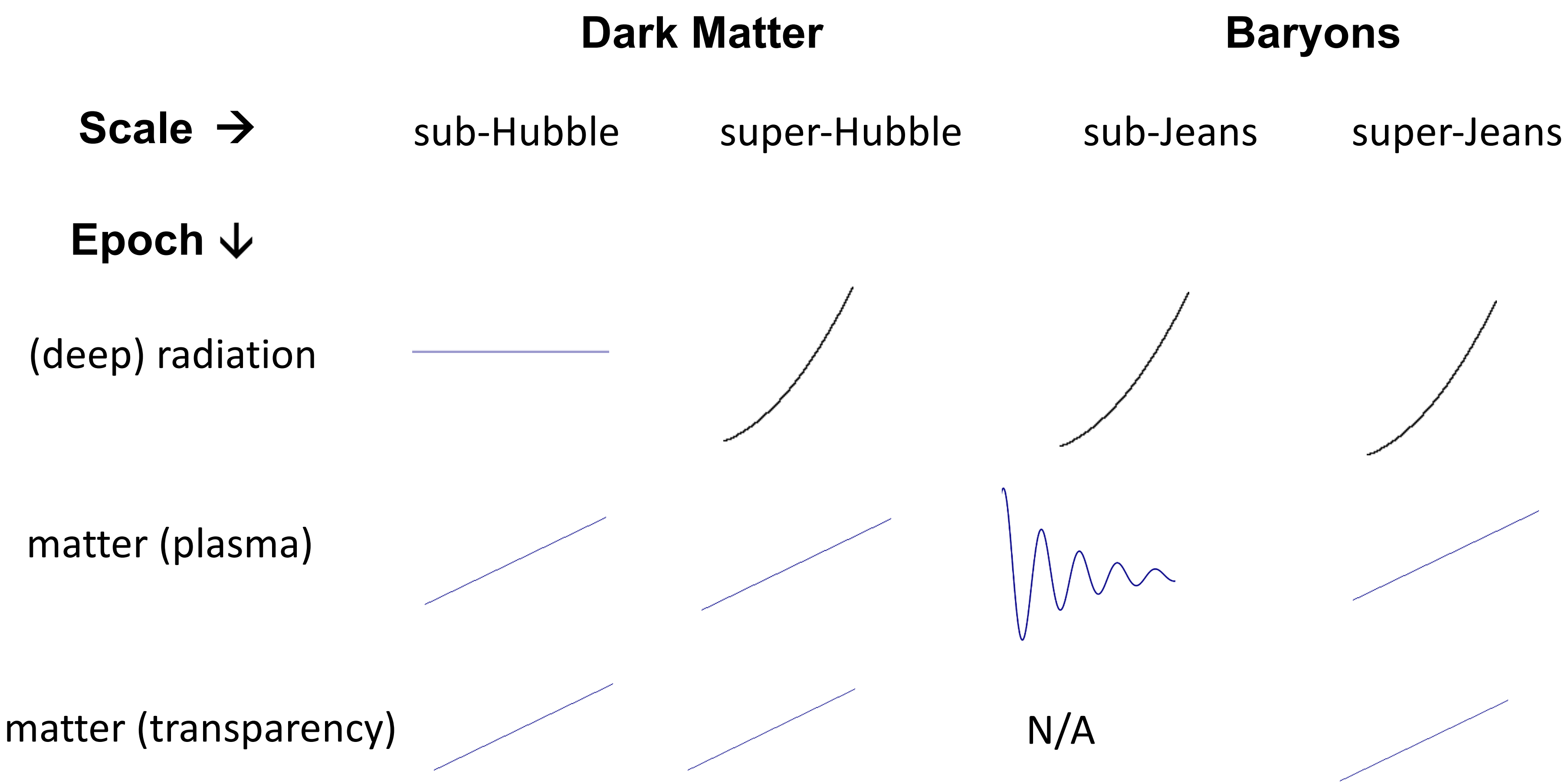
$$b_1 = 0.944[1 + (458\Omega_0 h^2)^{-0.708}]^{-1},$$

$$b_2 = (0.395\Omega_0 h^2)^{-0.0266}.$$



# Summary of dark matter and baryonic power spectra

evolution of linear dark and baryonic matter perturbations



## $z > z_{eq}$ (radiation epoch)

The animation seems to move to the left (a "transfer" effect, not all scales grow).

Dark matter perturbations **grow** until caught by the expanding Hubble radius and "**freeze**".

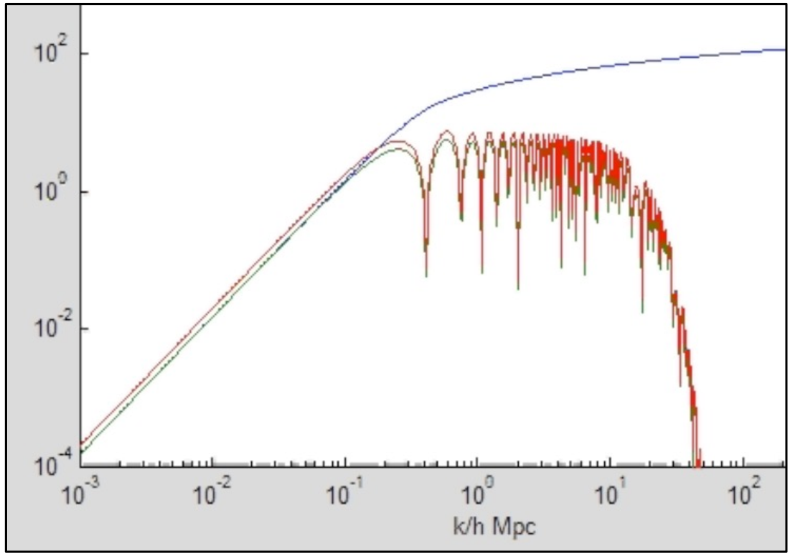
Plasma perturbations start to **oscillate** (the zero $\delta_p$ approximation is already not valid).

## z_eq > z > z_dec: (plasma matter epoch)

The animation seems to move upwards (scale-independent growth).

All dark matter scales **grow** $\delta_{DM}(a) \sim a$, while the plasma density contrast continues to **oscillate**.

*DM (blue), baryons (green), radiation (red)*




## z < z_dec: (transparency matter epoch)

Photons and baryons decouple at $z_{dec}$, photon perturbations wash out, while baryons fall into the DM potential wells. The oscillations look smaller as the perturbations grow.

The baryonic damping scale is larger than the dark matter one.

# Total matter correlation function: BAO peak

As we saw, the coherent scale-dependent time frequency of the oscillating clustering, creates a sinusoidal power spectrum, with a single wavelength oscillations.

What is the Fourier transform of a sine?

It is a Dirac delta → this implies that the Fourier transform of the power spectrum (the correlation function) will contain one single peak, at spatial separation given by that wavelength → this is known as the **Baryon Acoustic Oscillations (BAO) peak.**

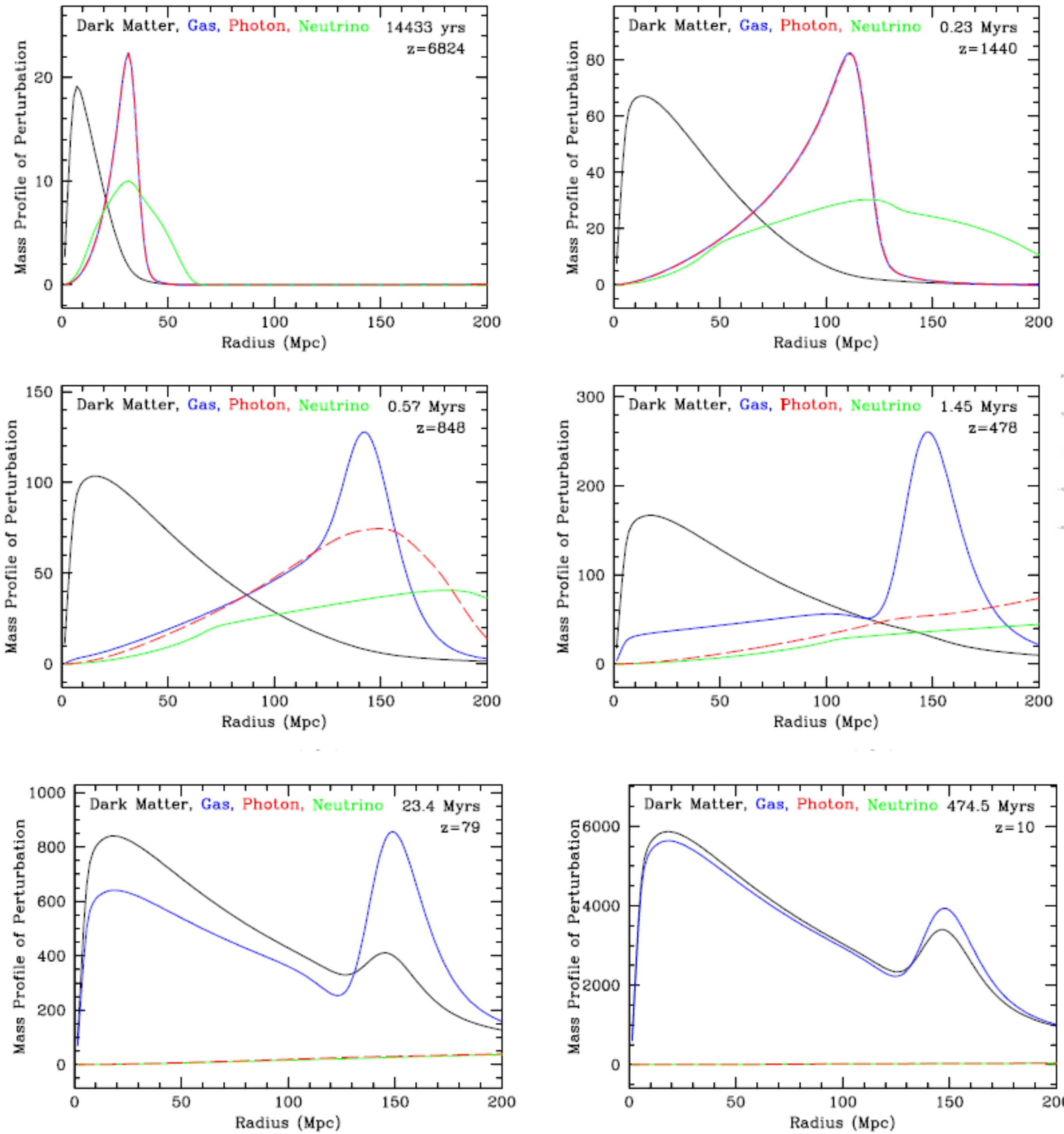
**What is the physical meaning of this peak ?**

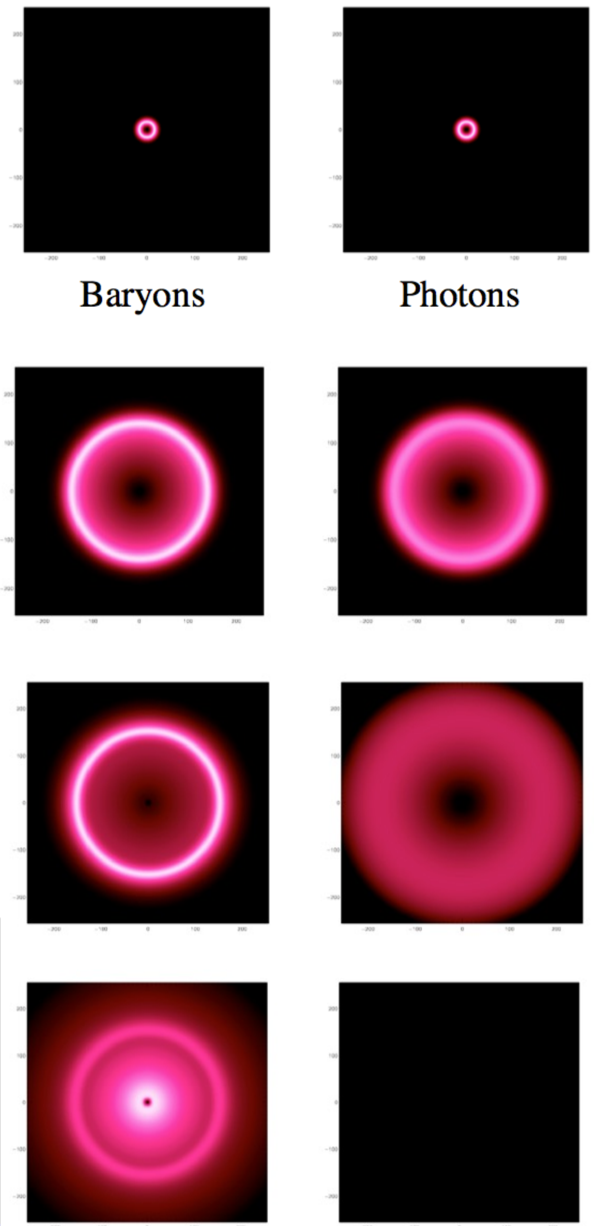
Let us go back to the plasma epoch. During that epoch, perturbations of different scales oscillate in phase at each spatial location. **The oscillation of a density field at a spatial location produces a density wave that propagates through the plasma with the plasma sound speed.**

This means that **secondary overdensities** travel through the plasma from every location where there is an overdensity. At each instant, the secondary overdensity is at a distance D(t) from the primary overdensity.

At decoupling, the plasma dissolves, the speed of sound goes to zero, and the wave stops. **The traveling overdensity then stops at the maximum distance that the wave could travel from $t=0$ to $t=t_{dec}$.**

# Formation of the BAO peak



During the **transparency epoch**, the overdensity remains frozen on that scale, as the density contrast continues to evolve linearly with ‘a’ for all scales.

That overdensity is still imprinted today in the statistical cosmological functions on real space → it appears as a **single peak** in the matter correlation function → the **BAO peak.**

**The location of this peak in the correlation function is the maximum distance the sound wave travelled up to decoupling time (z=1100) → it is the (comoving) sound horizon at decoupling.**

This is another **characteristic scale of the universe**.

Its value is given by the **sound comoving distance** from a=0 to $a_{dec}$

Using the sound speed, instead of the light speed, this distance is given by

$$r_s(z_*) = \int_0^{a_*} \frac{c_s(a')}{a'^2\, H(a')} da'$$

where $c_s$ (a) is a decreasing function of a.

The integration limit **$z_*$** is a more precise version of $z_{dec}$ = 1100, where the redshift is computed using a **fitting function** (function of the cosmological parameters), that accounts for the details of the recombination process:

$$z_* = 1048\left[1 + 0.00124(\Omega_b h^2)^{-0.738}\right]\left[1 + g_1(\Omega_m h^2)^{g_2}\right]$$

with $$g_1 = \frac{0.0783(\Omega_b h^2)^{-0.238}}{1 + 39.5(\Omega_b h^2)^{0.763}} \qquad g_2 = \frac{0.560}{1 + 21.1(\Omega_b h^2)^{1.81}}$$

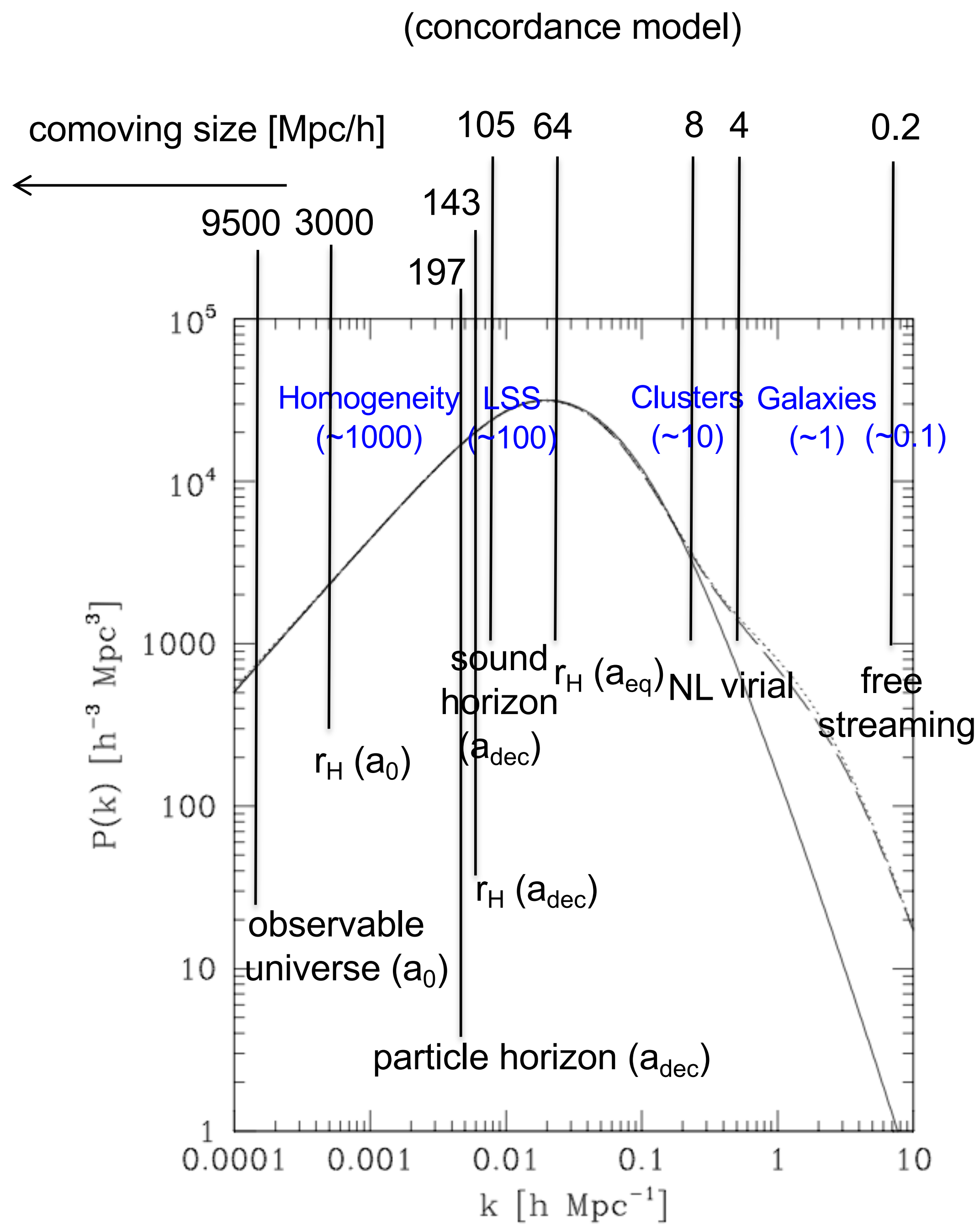
The resulting value for the **comoving sound horizon** is then:

**$r_s$ ($z_*$) = 105 Mpc/h = 150 Mpc** (concordance model)

(the proper size of the sound horizon at $z_{dec}$ is a factor of 1000 smaller ~ 0.1 Mpc/h)

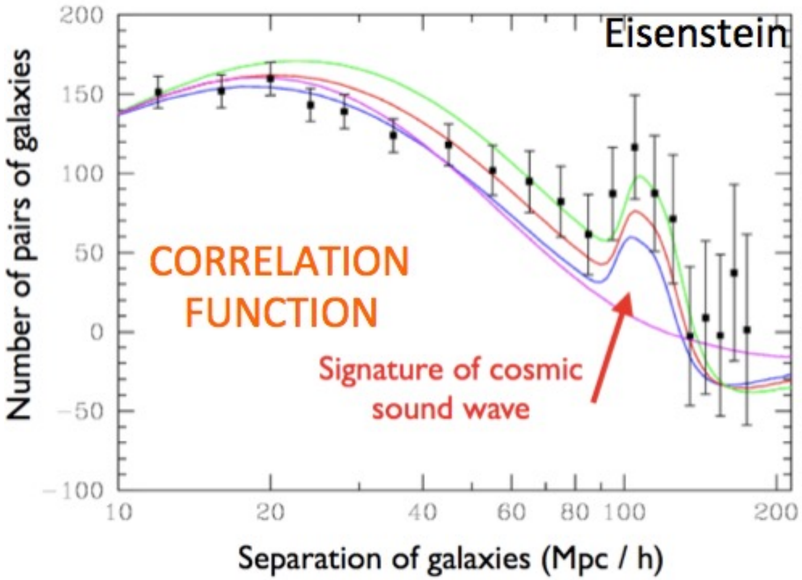
Notice that the sound horizon is smaller than the particle horizon, since the speed of sound is smaller than the speed of light:

$r_c$ ($z_*$) = 197 Mpc/h = 281 Mpc (concordance model)

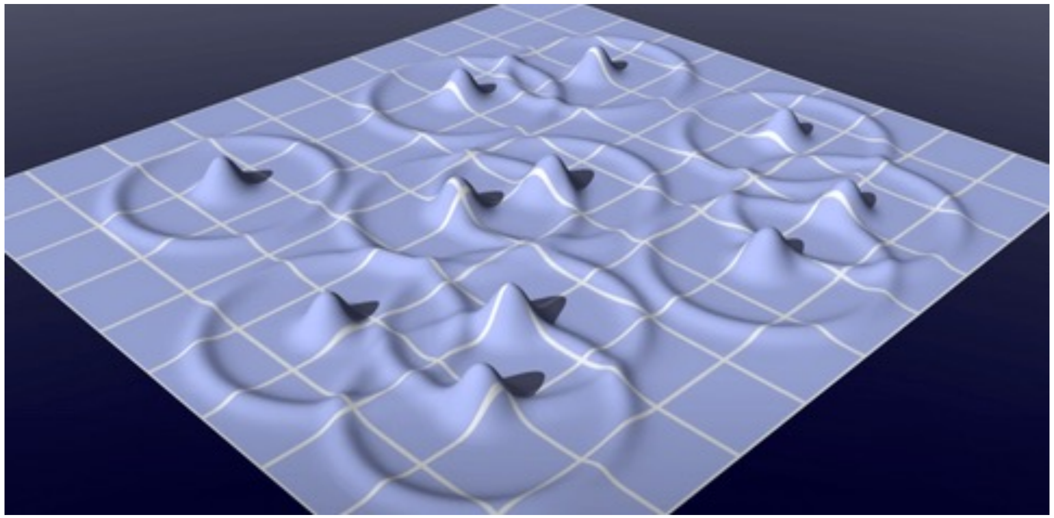


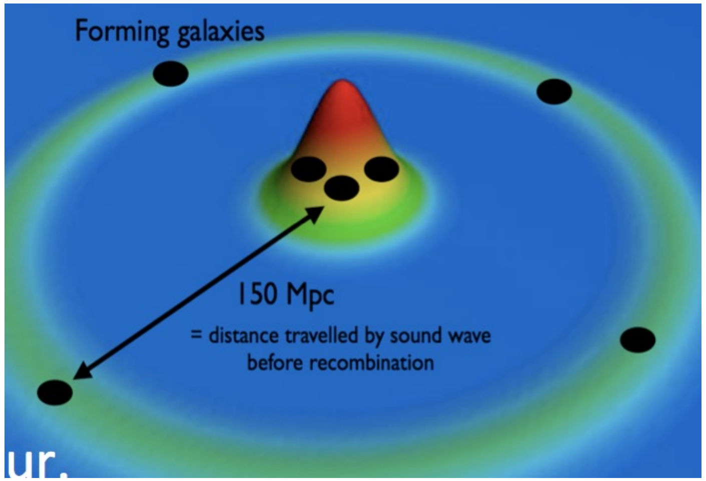
(concordance model)
comoving size [Mpc/h]
9500
3000
197
143
105
64
8
4
0.2
Homogeneity
(~1000)
LSS
(~100)
Clusters
(~10)
Galaxies
(~1)
(~0.1)
observable universe ($a_0$)
$r_H$ ($a_0$)
particle horizon ($a_{dec}$)
$r_H$ ($a_{dec}$)
sound horizon ($a_{dec}$)
$r_H$ ($a_{eq}$)
NL
virial
free streaming
P(k) [h$^{-3}$ Mpc$^3$]
$10^5$
$10^4$
1000
100
10
1
0.0001
0.001
0.01
0.1
1
10
k [h Mpc$^{-1}$]

**So, a secondary peak in the correlation function appears at the separation of 105 Mpc/h** (this peak was detected for the first time in 2005)




In other words, **on average there is a secondary overdensity at a separation of 150 Mpc from every primary overdensity:**





# Thermodynamics of the Universe

It is relevant to consider **thermodynamical** properties of the Universe, since pressure and density perturbations are **thermodynamically related**:

For example, if the universe expanded in an **isothermal** way, pressure would need to decrease as the density decreases: pV = nRT (to keep T constant).

This is the case in astrophysical processes that have time to thermalize (the heat transfer is fast compared to the sound speed).

However, in cosmology, the temperature of the cosmological fluid decreases with the expansion → **the evolution is not isothermal.**

A special case of a non-isothermal evolution is the **adiabatic evolution** (also called isentropic), where the temperature changes in a way that heat transfer compensates the entropy change → entropy is conserved.

In general, pressure perturbations is a non-isothermal Universe can be separated in 2 parts: **adiabatic and non-adiabatic**:

$$\delta_p = \delta_{\rm pad} + \delta_{\rm pnad}$$

**An adiabatic evolution has the following property:**

$$p(\bar{\rho}) = \bar{p}(\rho)$$ → **points where the density is the mean density have pressure identical to the mean pressure**

Taylor expanding p(ρ) we can write: $$p(\rho) = p(\bar{\rho}) + \frac{\partial p}{\partial \rho}_{|\bar{\rho}}(\rho - \bar{\rho})$$

$$\rho = \bar{\rho} + \delta\bar{\rho}$$

Inserting these definitions in the Taylor expansion, it follows:

$$p(\rho) = \bar{p} + \delta_p\bar{p}$$

$$\bar{p}(\rho) + \delta_p\bar{p} = p(\bar{\rho}) + \frac{\partial p}{\partial \rho}_{|\bar{\rho}}\delta\bar{\rho}$$

Using the adiabatic property, we get

$$\left(\frac{\partial p}{\partial \rho}\right)_s^{1/2} = \left(\frac{\delta_p\,\bar{p}}{\delta\,\bar{\rho}}\right)^{1/2}$$

This means that the **adiabatic speed of sound** is given by

$$c_s = \left(\frac{\partial p}{\partial \rho}\right)_s^{1/2}$$

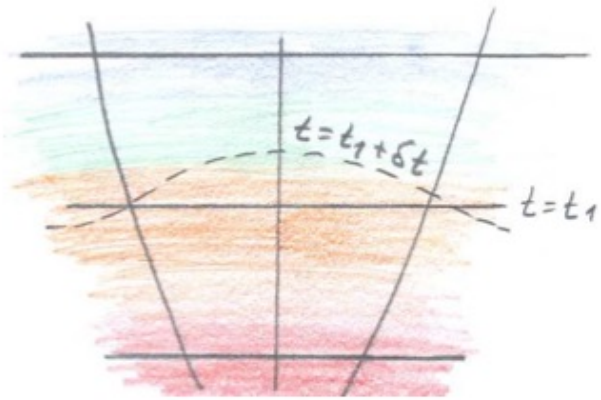
i.e., the speed of sound in an adiabatic fluid is given by **a derivative of the background quantities, not requiring the information on the perturbations.**

In an adiabatic evolution, perturbations can be written as derivatives of background quantities.

This property can be used to constrain the **relative amplitudes between the various components**.

Let us express the **adiabatic property** in a different way:

**the values of ρ and p in a region x at a time t are identical to the mean quantities at a different time t+δt → some parts of the universe are "ahead" and others are "behind"** in the evolution.




This implies that we can write:

$$\delta(t)\bar{\rho}(t) = \rho(t) - \bar{\rho}(t) = \bar{\rho}(t+dt) - \bar{\rho}(t) = \frac{d\bar{\rho}(t)}{dt}dt$$

i.e., **the density contrast (times the mean density) is equal to the derivative of the mean density.**

In an adiabatic evolution, similar relations can be written for each density and pressure component of the cosmological fluid → a given time element dt can be written from the previous equation applied to any species :

$$dt = \frac{\delta_m \bar{\rho}_m}{d\bar{\rho}_m/dt} = \frac{\delta_r \bar{\rho}_r}{d\bar{\rho}_r/dt}$$

So, the adiabatic condition allows us not only to relate perturbations with mean quantities, but also to relate the perturbations of various species. This property allows us to find **the relative initial condition amplitudes between the components**.

Indeed, from the **continuity equation** for the background, for the two species, we have:

$$\frac{d\bar{\rho}_m}{dt} = -3H\bar{\rho}_m \qquad \frac{d\bar{\rho}_r}{dt} = -4H\bar{\rho}_r$$

Inserting in the adiabatic "time equation", we get

$$\delta_r = \frac{4}{3}\delta_m$$

**meaning that clustering of radiation (and baryonic matter) starts with a larger amplitude than that of dark matter**.

The **thermodynamic type of the perturbations** is another property of a model of the universe.

**Inflation predicts that the primordial fluctuations are adiabatic** → this provides an initial condition for the relative amplitude of the various species of perturbations.

Other types of perturbations could be possible, but are ruled out by observations:

For example, in isocurvature perturbations the evolution is such that the total $\delta\rho$ is conserved → **the growth of one species implies the decrease of another** → not compatible with observations.