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Module: *Cosmological Structure Formation and Evolution*

Doctoral Program in Astronomy

Elements of Cosmology and Structure Formation

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Chapter 1

The Standard Model of Cosmology

In the following sections we review the basic elements of standard cosmology needed for the remainder of the course. The majority of the results presented here can be found in many textbooks of cosmology, e.g. in Refs. [18, 61, 65, 75, 76, 79, 80].

1.1 The basis of the standard model of Cosmology

The standard model (**SM**) of cosmology is presently based on the Hot Big Bang paradigm. The universe is expanding and cooling from an initial ultra dense state, where matter and radiation were prisoners in a gaseous hot plasma of fundamental particles. The basic mathematical framework of the SM is set by the following assumptions:

- The universe is homogeneous and isotropic when observed on large scales.
- The dynamics of space-time is described by Einstein's theory of general relativity.

The first of these assumptions derives from the cosmological principle, which states that the universe on large scales should have no privileged positions or directions and therefore at a given time should look the same to all observers. The most general homogeneous and isotropic solution of the Einstein equations of GR is the FLRW line element,

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{1.1}$$

where c is the speed of light, t is the universal time, r, θ, ϕ are the comoving spatial coordinates and $a(t)$ is the *scale factor*, which describes the overall expansion (or contraction) of the three-dimensional space. The constant k gives the spatial curvature of the universe. It can be negative, zero or positive depending on whether the universe is open, closed or flat. If $k \leq 0$ the universe has an infinite extension and the r coordinate ranges from zero to infinity. If k is positive the universe is finite in size and r ranges from 0 to $1/\sqrt{k}$. With an appropriate rescaling of coordinates it is always possible to make k to take the values: -1, 0, $+1$. Observers with fixed coordinates in the comoving coordinate system experience no external forces and are said to be fundamental comoving observers as they move with the expansion (or contraction) of the cosmological fluid. The proper distance between two of these observers scales as:

$$\ell(t) = \frac{a(t)}{a_0}\ell_0, \tag{1.2}$$

where a_0 and ℓ_0 are the scale factor and the proper distance at some initial time t_0. Taking the derivative of this expression with respect to time we obtain the relative speed between fundamental observers:

$$v(t) = \frac{d\ell}{dt} = \frac{\dot{a}(t)}{a_0}\ell_0 = \frac{\dot{a}(t)}{a(t)}\ell(t) \equiv H(t)\ell(t). \tag{1.3}$$

This expression shows that in an expanding universe ($\dot{a}(t) > 0$) the further away any two observers are, the faster they recede from one another due to the cosmic expansion. This is known as the **Hubble law**, first derived from observations by Edwin Hubble in 1929. The proportionality factor between velocity of recession and distance,

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \tag{1.4}$$

gives the expansion rate of the universe at a given time. Its present value is usually parametrized as $H_0 = 100\,h$ km s^{-1} Mpc where h is the so-called *Hubble parameter*. H_0 is also known as the **Hubble constant** and has units of an inverse of time.

The cosmological expansion produces a *redshift* in the spectrum of the light emitted from distant objects. This redshift is defined as the change of energy of a photon measured by fundamental observers between the epochs of emission of the radiation, at t, and the present time, t_0,

$$z = \frac{E - E_0}{E_0} = \frac{\nu}{\nu_0} - 1 = \frac{\lambda_0}{\lambda} - 1 = \frac{a\,(t_0)}{a\,(t)} - 1. \tag{1.5}$$

Here E, ν and λ are the photon's energy, frequency and wavelength at t and the subscript '$_0$' denotes the same quantities at the present. To obtain this expression we use the quantum mechanics' proportionality between the energy and frequency of a photon $E = h\nu = hc/\lambda$ (h is the Planck's constant) and the fact that the wavelength of a free photon stretches as the other lengths with $a(t)$.

1.2 Fundamental equations

In GR the dynamics of space-time is set by Einstein's field equations,

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi G}{c^4}T_{ab} + \Lambda g_{ab}, \tag{1.6}$$

where G and Λ are the gravitational and cosmological constants, g_{ab} is the metric of space-time, G_{ab} is the Einstein tensor, R_{ab} is the Ricci tensor and R is the Ricci scalar (this last two quantities result from successive contractions of the Riemann tensor and are functions of the metric and its derivatives). The inclusion of the cosmological constant term in Eq. (1.6), originally introduced by Einstein in order to describe a static universe, is presently supported by observations of distant Type Ia supernovae [83, 84, 89]. Finally T_{ab} is the stress–energy tensor, which describes the gravitational contributions from all forms of energy in the universe. To satisfy the requirements of homogeneity and isotropy implied by the cosmological principal, the stress–energy tensor has to be that of a perfect fluid:

$$T_{ab} = (\rho + \frac{p}{c^2})U_a U_b - \frac{p}{c^2} g_{ab} \tag{1.7}$$

where $\rho = \rho(t)$ and $p = p(t)$ are the fluid's energy density and pressure and U_a is the four velocity field of a fundamental observer. In this case the solution of Einstein's field equations is the metric (1.1). The energy–momentum conservation law is expressed by the condition

$$T^{ab}{}_{;b} = 0 \tag{1.8}$$

where the symbol ';' denotes covariant derivative. This is a set of four equations giving the conservation of the energy density and the 3-momentum. The 'temporal component' equation gives,

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \quad \Rightarrow \quad d\left(\rho c^2 a^3\right) = -pd\left(a^3\right). \tag{1.9}$$

This expression has a simple physical interpretation. It translates the first law of thermodynamics applied to a comoving volume element: in an adiabatical process the variation of energy is given by the work produced by the pressure forces. It is often assumed that during several periods of the history of the universe the energy density and pressure can be related by an *equation of state* with a simple form:

$$p = w\rho c^2 \quad -1 \leq w \leq 1, \tag{1.10}$$

where w is a constant. In this case the integration of (1.9) gives

$$\rho(t) = \rho_i \left(\frac{a(t)}{a_i}\right)^{-3(1+w)}, \tag{1.11}$$

where $\rho_i = \rho(t_i)$, $a_i = a(t_i)$ are the energy density and the scale factor at some initial time t_i. The cases $w = 0$, $w = 1/3$ and $w = -1$ are classical examples which are appropriate to describe the phases when the universe is dominated by a fluid of (non relativistic) matter, radiation (and relativistic matter) and vacuum energy associated with the Λ term in (1.6).

1.2.1 The dynamical equations

Using Eqs. (1.1) and (1.7) in (1.6) one can derive two equations which give the expansion and acceleration rates of the universe as a function of its matter contents and geometry:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2} \tag{1.12}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{1.13}$$

These are known as the Friedmann and Raychaudhuri (or acceleration) equations, respectively. If we set $\Lambda = 0$ we find that $\ddot{a}$ is negative whenever $p > -\rho c^2/3$ ($a(t)$ is always positive). In this case both radiation ($w = 1/3$) and matter ($\omega = 0$) dominated epochs are periods of decelerated expansion (if the universe has a flat or open geometry) or decelerated contraction (only possible with closed geometries).

If we take as an observational fact that the universe is expanding ($\dot{a} > 0$) and assume that in the past $\ddot{a}$ was always negative ($\ddot{a} < 0$) then going backwards in time there's a moment, t_i, where $a(t_i) = 0$. This is the Big Bang event. The instant t_i is usually redefined as the origin of time, $t_i = 0$. The FLRW models are Big Bang universes for many combinations of energy densities, cosmological constant and geometries. However one should note that there are FLRW models (with $\Lambda > 0$, $k = 1$), which do not "start" from a Big Bang. This is the case of the Eddington–Lemaitre and Einstein universes (see e.g. Ref. [22]). Because equations (1.12), (1.13) and (1.9) are related by the Bianchi identities (see e.g. Ref. [97]) we only need to consider two of these equations to describe the dynamics of FLRW models. These are usually taken to be the Friedmann and energy conservation equations (the later usually in the form Eq. (1.11)).

We can re-write the Friedmann equation as a "conservation law of densities",

$$\frac{8\pi G}{3H^2}\rho + \frac{\Lambda c^2}{3H^2} - \frac{kc^2}{a^2H^2} = 1 \quad \Leftrightarrow \quad \Omega + \Omega_\Lambda + \Omega_k = 1 \tag{1.14}$$

where,

$$\Omega = \frac{\rho}{\rho_c}, \qquad \Omega_\Lambda = \frac{\Lambda c^2}{3H^2}, \qquad \Omega_k = -\frac{kc^2}{a^2H^2}, \qquad \rho_c = \frac{3H^2}{8\pi G} \tag{1.15}$$

are the matter (Ω), vacuum (Ω_Λ) and curvature (Ω_k) density parameters and ρ_c is the critical energy density of the universe. All these quantities evolve in time satisfying Eq. (1.14). Therefore only two of the above density parameters are independent. For $\Lambda = 0$ ($\Omega_\Lambda = 0$) models, the universe is geometrically closed ($k = 1$), flat ($k = 0$) or open ($k = -1$), depending on whether its total energy density, ρ, is greater, equal or smaller than the critical density, ρ_c. The acceleration rate of the universe is often expressed in terms of the *deceleration parameter* which is defined as:

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{a}\frac{1}{H^2} = \frac{1+3w}{2}\Omega - \Omega_\Lambda \tag{1.16}$$

where the last equality results from combining Eq. (1.15) with Eqs. (1.13) and (1.11).

1.2.2 Epochs

In general the cosmological fluid can be regarded as a mixture of ideal fluids describing different matter components, each of these having a particular energy density, ρ_i, and pressure, p_i. Considering that we have only two components consisting of non-relativistic matter, ρ_m, and radiation, ρ_r, ($\rho = \rho_r + \rho_m$) we can write the Freedmann equation as:

$$H^2(t) = \frac{8\pi G}{3}(\rho_r + \rho_m) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

$$= H_0^2 \left[\Omega_{r0} \left(\frac{a_0}{a} \right)^4 + \Omega_{m0} \left(\frac{a_0}{a} \right)^3 + \Omega_{k0} \left(\frac{a_0}{a} \right)^2 + \Omega_{\Lambda 0} \right]. \qquad (1.17)$$

The second equality results from using Eq. (1.11) for matter ($w = 0$) and radiation ($w = 1/3$), with $t_i = t_0$ being the present time ($a(t_i) = a(t_0) = a_0$). The quantities H_0, Ω_{r0}, Ω_{m0}, Ω_{k0} and $\Omega_{\Lambda 0}$ are respectively the Hubble constant, $H_0 = H(t_0)$, the matter, radiation, vacuum and curvature density parameters evaluated at the present time ($\Omega_{r0} = 8\pi G \rho_r (t_0) / 3{H_0}^2$, $\Omega_{m0} = 8\pi G \rho_m (t_0) / 3{H_0}^2$).

Equation (1.17) shows the relative contribution from the different fluid components to the expansion rate of the universe. In a Big Bang scenario $a \to 0$ in the limit $t \to 0$. The expansion rate is therefore initially dominated by the radiation component, Ω_r. As the universe expands, the contribution from the other terms becomes progressively important and can dominate the expansion. When matter becomes the dominant component, $H(t)$ is driven by the Ω_m term in Eq. (1.17). If the universe keeps on expanding and has a non-zero cosmological constant the dynamics of the expansion becomes dominated by the vacuum term, Ω_Λ. By comparing the first with second term inside the square brackets of Eq. (1.17) we obtain the redshift at which matter and radiation contribute equally to the expansion rate. This is

$$1 + z_{\text{eq}} = \Omega_{m0} / \Omega_{r0} \simeq 2.4 \times 10^4 \, \Omega_0 h^{-2},$$

where the last equality derives from the observed energy density of radiation (see next section). The redshift $1 + z_{\text{eq}}$ gives the time when the universe changes from being radiation dominated to became dominated by matter. It is usually referred to as the matter–radiation equality redshift.

Going back enough in time we find the universe in a stage where photons, electrons and baryons are tightly coupled in a collisional plasma. When the temperature dropped to about 3600 Kelvin ($z_{\text{rec}} \simeq 1300$) [61], electrons and baryons recombined to form the first neutral Hydrogen atoms. Soon after this short period, known as *recombination*, the number of free electrons drops dramatically and the scattering between the remaining free electrons and photons is no longer sufficient to keep matter and radiation in contact. At this point ($z_{\text{dec}} \simeq 1100$) the CMBR photons decoupled from the fluid. Present observations of Ω_0 and h indicate that the matter–radiation equality happens before recombination ($z_{\text{eq}} \simeq 14300$, see next section).

1.2.3 The observed universe

Our understanding of the universe relies ultimately on our ability to make measurements and to compare those measurements with theoretical models. As (1.17) indicates, the key observational parameters in the FLRW Big Bang models are the Hubble parameter and the present-day densities of the mass–energy contents of the universe. Despite great progress in the past decades there is still a considerable amount of uncertainty regarding the majority of these parameters.

- **Hubble parameter**: The quest for the measurement of the Hubble parameter, h, is the oldest among the cosmological parameters. It started soon after the discovery of the universal expansion and today it is still not known to high accuracy (see Refs. [47, 54] for reviews). There are two basic ways of measuring h. One involves the measurement of distances to nearby galaxies, typically by observing the periods and luminosities of Cepheid stars within them, and then use these determinations to calibrate other methods of measuring distances to more distant galaxies. This strategy is known as the *cosmic distance ladder* [92]. The second way consists of using *fundamental physics* methods, which permit the direct measurement of distances to faraway objects without using the cosmic distance ladder approach. This is the case of methods involving Type Ia or Type II supernovae, gravitational lensing, and the Sunyaev–Zel'dovich effect (see Section **??**). Although different approaches can still lead to different results, the range of h determinations has been shrinking with time. Presently different observational techniques seem to start to converge inside the range $h \in [0.5, 0.9]$ to a mid-value of $h \simeq 0.7$. This is the case of the methods based on the observation of distant Cepheids using the Hubble Space Telescope [36, 37] and the observation of Type Ia supernovae [56, 90].

- **Total matter/energy density**: The total matter or energy density of the universe, $\Omega_{tot,0} = \Omega_{r0}+\Omega_{m0}+\Omega_{\Lambda 0}+\Omega_{k0}$, is presently most accurately constrained from observations of the angular scale (or multipole) of the first acoustic peak in the angular power spectrum of the CMB anisotropies. The position of the peak is highly sensitive to $\Omega_{tot,0}$. According to recent determinations from different CMB experiments its position is located at a multipole value of $l_p \sim 210$. Assuming Gaussian adiabatic initial perturbations, the Boomerang, MAXIMA and DASI experiments provide the following constraints on the total matter/energy density: $\Omega_{tot,0} = 1.02^{+0.06}_{-0.03}$ [21, 74]; $\Omega_{tot,0} = 0.9^{+0.18}_{-0.16}$ [103]; and $\Omega_{tot,0} = 1.04 \pm 0.06$ [88], respectively. These estimates are remarkably consistent with an $\Omega_{tot,0} = 1$ (flat) universe.

- **Radiation density**: The energy density in all forms of electromagnetic radiation, $\Omega_{\gamma 0}$, is dominated by the contribution of the CMB, $\Omega_{CMB,0}$ (see e.g. Ref. [96]). This can be computed accurately from the observed CMB mean temperature, $T_{CMB} = 2.725 \pm 0.001$ K [33], by using the Stefan-Boltzmann law, $\Omega_{\gamma 0} \simeq \Omega_{CMB,0} = 2.48 \times 10^{-5}\, h^{-2}$. To estimate Ω_{r0} we need also to consider the contribution from the other relativistic species. Of particular importance is the *neutrino background*. If in addition to radiation we assume the existence of three families of massless neutrinos we obtain [18], $\Omega_{r0} \simeq 4.17 \times 10^{-5}\, h^{-2}$.

- **Matter density**: Many different techniques have been used to infer constraints on the present day value of the matter density parameter, Ω_{m0}. These include methods based on observations from CMB anisotropies, Type Ia supernovae, gravitational lensing, the evolution of the abundance of X-ray clusters with redshift, gas mass fraction in galaxy clusters, observational fits to the matter power spectrum of extra galactic objects, and measurements of large scale peculiar velocities of galaxies (for an overview on these and other methods see e.g. Ref. [91] and references therein). Results indicate that we are still far from having an accurate determination of Ω_{m0}. In some cases different techniques can even show some degree of inconsistency.[1] However, combined data analysis using results from many of these methods give evidence in favor of a matter density parameter of about $\Omega_{m0} = 1/3$. For example, the authors in Ref. [44] have performed a likelihood analysis using results from six independent data sets and found $\Omega_{m0} = 0.31 \pm 0.04 \pm 0.04$, assuming a flat Universe. In this determination the first error is mainly statistical and the second is systematical. The observational data included constraints from recent CMB observations made with Boomerang and MAXIMA, Type Ia supernovae, double radio galaxies, lensing and large scale structure formation data.

- **Cosmological constant**: The strongest evidence for a positive cosmological constant derives from observations of high-redshift Type Ia supernovae, which indicate that the universe is currently evolving in accelerated phase of expansion. Independently, a positive Λ is also supported from a combination of observations indicating that $\Omega_{m0} < 1$ in conjunction with the CMB results which show that the universe is approximately flat. Under the assumption of a flat Universe, with matter density $\Omega_{m0} \sim 0.3$, the energy density associated with a cosmological constant term is $\Omega_{\Lambda 0} \sim 0.7$.

Although there's still a considerable amount of uncertainty in the determination of many of the above parameters, the combination of data from present observations seem to indicate that the Universe is consistent with being flat, and the ratio between the vacuum and matter densities, $\Omega_\Lambda/\Omega_{m0}$, is of the order of 2. For discussions on the current status of measurements of cosmological parameters see e.g. Refs. [35, 87].

1.3 Exact solutions

The time dependence of the scale factor and the age of the universe result from the integration of Eq. (1.17). The usual way to proceed is first to multiply Eq. (1.17) by $(a/a_0)^2$ and use $\Omega_{k0} = 1 - \Omega_0 - \Omega_{\Lambda 0}$,

$$\frac{d}{dt}\frac{a(t)}{a_0} = H_0 \sqrt{1 - \Omega_0 + \Omega_{m0}\left(\frac{a}{a_0}\right)^{-1} + \Omega_{r0}\left(\frac{a}{a_0}\right)^{-2} - \Omega_{\Lambda 0}\left[1 - \left(\frac{a}{a_0}\right)^2\right]}. \tag{1.18}$$

1.3.1 The age of the universe

The integration of this expression, with the condition $a(t = 0) = 0$, gives

$$t = H_0^{-1} \int_0^{\frac{a(t)}{a_0} = (1+z)^{-1}} \frac{1}{\sqrt{1 - \Omega_0 + \Omega_{m0} x^{-1} + \Omega_{r0} x^{-2} - \Omega_\Lambda \left(1 - x^2\right)}}\, dx, \tag{1.19}$$

[1] For example, preliminary constraints from the Cosmic Lens All-Sky Survey (CLASS) [46] appear to be in strong conflict with the results from the Type Ia supernovae data [83, 84, 89].

where we have put $x = a(t)/a_0 = 1/(1+z)$. This gives the age of the universe as a function of the scale factor and the present day density parameters, $t = f(a, \Omega_0, \Omega_{m0}, \Omega_{r0}, \Omega_{\Lambda 0})$. Observationally we know that $\Omega_{r0} \ll \Omega_{m0} \simeq \Omega_0$ but at early times the radiation term dominates Eq. (1.19). Using current estimations of the density parameters it is easy to see that the radiation-dominated period is very short when compared to the present age of the universe, t_0. This means that in practice t_0 can be calculated to a very good approximation by setting $\Omega_{r0} = 0$ in Eq. (1.19). Analytical expressions for the age of the universe can be found in many textbooks for a range of cosmologies. Three cosmological scenarios of historical interest are the flat universe with cosmological constant ($\Omega_0 + \Omega_{\Lambda 0} = 1$), the critical density universe ($\Omega_0 = 1$) and the open universe without Λ ($\Omega_0 < 1$; $\Omega_{\Lambda 0} = 0$). The integration of Eq. (1.19) for these models is also analytical and can be found for example in Ref. [61].

In some situations of interest is useful to define time in terms of the so-called conformal time, $d\eta = dt/a$. This gives,

$$\eta(t) = \int_0^t \frac{dt'}{a(t')} = \int_0^{a(t)} \frac{da}{a(t')\dot{a}(t')}. \tag{1.20}$$

1.3.2 Scale factor

The inversion of $t = f(a, \Omega_0, \Omega_{m0}, \Omega_{r0}, \Omega_{\Lambda 0})$ with respect to a gives the dependence of the scale factor with time. However since the expansion rate of the universe is dominated at different phases by different fluid components (see Eq. (1.17)) it's quite useful to examine the solutions of the Friedmann equation for each of these phases. Restricting ourselves to the case $\Lambda = 0$, the Friedmann equation for a single component fluid reads,

$$\dot{a}^2 = a_0^2 H_0^2 \left[1 - \Omega_{w0} + \Omega_{w0} \left(\frac{a}{a_0} \right)^{-(1+3w)} \right], \tag{1.21}$$

where Ω_{w0} is the present-day density of the fluid and w is the equation of state parameter in Eq. (1.10). The solution of Eq. (1.21) is straightforward in the case of $\Omega_{w0} = 1$ (flat geometry)

$$\frac{a(t)}{a_0} = \left(\frac{3(1+w)}{2} H_0 t \right)^{2/(3(1+w))} \tag{1.22}$$

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3(w+1)t} \tag{1.23}$$

$$q(t) = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1+3w}{2} = \text{const}\,. \tag{1.24}$$

These expressions are particularly useful for fluids dominated by matter ($w = 0$) and radiation ($w = 1/3$). General solutions of Eq. (1.21) for this type of fluids are also not difficult to derive and can be found in many cosmology textbooks (see e.g. Ref. [61]).

1.3.3 Distances, horizons and volumes

Of particular importance to the study of physical processes acting on different cosmological scales is to determine the size of the largest causally connected regions at a given time. In a universe described by Eq. (1.1) the regions in causal contact with an observer of coordinates $O = (t, r_0, \theta_0, \phi_0)$ are those for which light rays emitted at the instant t_e reach O before or at the instant t. Light rays arriving at (r_0, θ_0, ϕ_0) later than t are beyond the *horizon* of O. Since light rays travel along null geodesics ($ds^2 = 0$), the **coordinate distance** travelled by light between $t_e = 0$ and t is easily obtained from Eq. (1.1)

$$\int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}} = c \int_0^t \frac{dt'}{a\,(t')} = c\,\eta(t), \tag{1.25}$$

where r_e is the radial coordinate at emission. Without loss of generality we set $r_0 = 0$ and assumed radial $d\theta = d\phi = 0$ geodesics. The corresponding (physical) **proper distance** is

$$d_H(t) = \int_0^{r_e} \sqrt{|g_{rr}|} dr = a(t) \int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}} = c\,a(t)\eta(t). \tag{1.26}$$

This forms a spherical surface centered at (r_0, θ_0, ϕ_0) known as the **particle horizon** of O. For any observer, $d_H(t)$ separates the regions which can establish causal contact with the observer at t (regions within the horizon) from those which cannot (regions beyond the horizon). Using Eqs. (1.20) and (1.21) in Eq. (1.26) one obtains the following expression, valid for $1 + z \gg (1/\Omega_{w0} - 1)^{1/(1+3w)}$ and $w > -1/3$ (see Ref. [18])

$$d_H(t) \simeq \frac{2}{3w+1} \frac{c}{H_0} \Omega_{w0}^{1/2} \left(\frac{a}{a_0}\right)^{3(1+w)/2} = 3\frac{1+w}{1+3w} ct\ . \tag{1.27}$$

For $\Omega_{w0} = 1$ this is in fact an exact solution of Eq. (1.26), whenever $w > -1/3$ (see e.g. Ref. [61]). For universes with $w < -1/3$, the distance to the particle horizon becomes infinite. This is the case of the vacuum-dominated de Sitter universe ($w = -1$), for which there's no particle horizon.

Another important length scale is the **Hubble length**, R_H (also referred to as *Hubble radius* or *speed of light sphere*). This is defined as the distance to the spherical surface (centered in O) made by all points that at the time t have cosmological recessional velocities equal to the speed of light. Setting $v = c$ in the Hubble expansion law (1.3) one finds,

$$R_H(t) = \frac{c}{H(t)} = \frac{3(w+1)}{2} ct, \tag{1.28}$$

where the last equality results from Eq. (1.23) and therefore it is only true for Einstein–de Sitter universes. The Hubble length can be thought as the proper distance travelled by light during the characteristic time scale of expansion, H^{-1}. Comparing Eq. (1.28) with Eq. (1.27) we see that R_H and d_H only differ by a factor of the order of the unity (in particular for $w = 1/3$, $R_H = d_H$). This explains why both of these quantities are often used interchangeably and referred to as the horizon. In practice the largest distance one can observe with electromagnetic radiation is limited by what is called the **visual horizon**. This is defined as the distance to the surface where the Cosmic Microwave Background Radiation was last scattered. Beyond this *last scattering surface* (LSS) the universe becomes opaque due to the strong interaction (Compton scattering) between matter and radiation. The CMB photons we observe today suffered their last scattering at $z \sim 1000$ when they were at a distance of about $6h^{-1}$ Mpc. This corresponds to a present distance to the LSS of $\sim 6000\, h^{-1}$ Mpc.

Let us now briefly examine how angular sizes and distances to faraway objects are defined. Light emitted from the edges of an object (e.g. a galaxy cluster) located at a coordinate position r and time t, occupies an angular size in the sky given by

$$\theta = \frac{D}{d_A} = \frac{D}{a(t)r} = \frac{D(1+z)}{a_0 r}, \tag{1.29}$$

where D is the object's proper diameter and $d_A(t) = a(t)r$ is the proper distance to the object at the moment of light emission. The second equality results from the fact that the object is observed presently with redshift z. The quantity d_A is called the **angular diameter distance**. The present distance to the object is,

$$a_0 r = a_0 \int_{r_e}^{0} \frac{dr}{\sqrt{1-kr^2}} = a_0 c \int_{a(t)}^{a_0} \frac{da}{a\dot{a}} = \frac{c}{H_0} \int_{a(t)}^{a_0} \frac{da}{a\mathscr{H}(a)}. \tag{1.30}$$

where $\mathscr{H} = H/H_0 = (1-\Omega_0+\Omega_{m0}x^{-1}+\Omega_{r0}x^{-2}+\Omega_{\Lambda 0}(1-x^2))^{0.5}$ and $x = a/a_0 = 1/(1+z)$ (see Eq. (1.18)). The relation between the angular size of objects and their redshifts is therefore dependent on the underlying cosmological model and can in principle be used to constrain cosmological parameters, provided one can find "standard rulers" (see e.g. Ref. [96] for a review on "the standard tests of classical cosmology"). The evaluation of Eq. (1.30) is of particular interest for matter-dominated universes (the phase we believe structures like galaxies and clusters form). For $\Lambda = 0$ the expression Eq. (1.30) can be computed analytically (it was first derived by Mattig in 1958 [70]). One obtains,

$$a_0 r = \frac{2c}{H_0} \frac{\Omega_0 z + (\Omega_0 - 2)\left(\sqrt{1+\Omega_0 z} - 1\right)}{\Omega_0^2 (1+z)}. \tag{1.31}$$

In the case $\Lambda \neq 0$, the evaluation of Eq. (1.30) has to be done numerically. A widely used fitting formula for flat cosmologies with Λ was derived by Ref. [82]:

$$a_0 r = \frac{c}{H_0} \left[\eta(0, \Omega_0) - \eta(z, \Omega_0)\right], \tag{1.32}$$

where

$$\begin{aligned} \eta(z, \Omega_0) \quad = \quad & 2\sqrt{s^3+1}\left[(1+z)^4 - 0.1540s(1+z)^3 + 0.4304s^2(1+z)^2 \right. \\ & \left. + 0.19097s^3(1+z) + 0.066941s^4\right]^{-1/8} \end{aligned} \tag{1.33}$$

and $s^3 = 1/\Omega_0 - \Omega_0$. This fitting formula shows an accuracy better then 0.4% for $0.2 < \Omega_0 < 1$ and any z.

Also useful is the definition of the volume element in the FLRW universes. This is,

$$dV = \sqrt{|g|}\, dr\, d\theta\, d\phi = (ar)^2 \frac{a\, dr}{\sqrt{1-kr^2}}\, d\Omega \tag{1.34}$$

where $|g|$ is the determinant of the metric (1.1) and $d\Omega = \sin\theta\, d\theta\, d\phi$ is the solid angle element. Using Eq. (1.1) and the definition of redshift, one obtains the (physical) volume element per unit of solid angle and unit of redshift,

$$\frac{dV}{d\Omega\, dz} = \frac{c}{H(z)} \frac{(a_0 r)^2}{(1+z)^3} = \frac{c}{H_0} \frac{d_A^2}{\mathscr{H}(z)(1+z)} \tag{1.35}$$

where $\mathscr{H}(z) = H(z)/H_0$ and $a_0 r$ is given by Eq. (1.30).

Combining Eqs. (1.27) and (1.31) we can write an expression for the angular size of the horizon scale at a given time in matter-dominated universes with $\Lambda = 0$. Setting $D = d_H$ we find,

$$\theta_H \simeq 2 \tan\frac{\theta_H}{2} = \frac{\Omega_0^{3/2}\sqrt{1+z}}{\Omega_0 z + (\Omega_0 - 2)\left(\sqrt{1+\Omega_0 z} - 1\right)}. \tag{1.36}$$

Although Eq. (1.27) is an approximate expression for non-flat universes, it's possible to show that Eq. (1.36) is in fact exact for all geometries (see e.g. Refs. [18, 106, 111] for general expressions of d_H and θ). At high redshifts Eq. (1.36) reduces to

$$\theta_H \simeq \frac{180}{\pi}\sqrt{\frac{\Omega_0}{z}}\ \text{deg}. \tag{1.37}$$

This expression tells us that the size of the horizon at the time of last scattering ($z \sim 1000$) occupies today an angular area in the sky no larger then ~ 2 degrees.

1.4 Initial conditions and Inflation

As a mathematical framework of the Big Bang theory, the FLRW models have the great virtue of describing well the dynamical properties of the observable universe (in particular its expansion rate and age). They also give the correct temperature dependence on redshift, which allows Big Bang nucleosynthesis to reproduce the observed light element abundances so well. However, as we will see next, these models alone show an extreme sensitivity to the "initial conditions" required to explain why the universe is the way we observe today. This extreme "fine tuning" of the initial conditions raises a number of important questions which led to the development of the theory of inflation.

1.4.1 Problems with the Big Bang

Some of the main problems regarding the initial conditions of the Big Bang theory are:

- **Horizon problem**: According to Eq. (1.37) there are about 14000 to 65000 causally disconnected regions in the CMB sky, assuming $\Omega_0 \in [0.2\,,\,1]$. If this is true why is the CMBR very isotropic (showing blackbody spectrums with so similar temperatures in all directions)? This is very difficult to understand if these regions were never in thermal contact. Without any other mechanism to explain why the whole sky presents such similar properties one is forced to impose this as an initial condition.

- **Flatness problem**: At early times the Friedmann equation can be written as:
$$|\Omega(t) - 1| = \frac{|k|}{a^2(t)H^2(t)} = \frac{|k|}{\dot{a}^2(t)} \quad , \tag{1.38}$$
where the quantity $a^2 H^2 = \dot{a}^2(t)$ is a decreasing function of time in all matter or radiation dominated Big Bang FLRW universes.[2] This means that as we go back in time the energy density of universe has to be very close to the critical density, $\Omega(t) \to 1$. Dividing Eq. (1.38) by itself written at the present we find that in order to get $\Omega_0 \sim 1$, the energy density needs to be extremely fine tuned in the past. For instance, when the universe was $t = 1$ second of age (nucleosynthesis period) $|\Omega - 1|$ as to be of the order of $\sim 10^{-18}$. This becomes even more drastic at earlier times. At the Planck epoch ($t \sim 10^{-43}$ – a time scale beyond which the classical GR equations should not be used) Ω can deviate from the unity only one part in 10^{60}! This shows $\Omega = 1$ as an unstable critical point, from which any initial deviation larger then what's allowed by Eq. (1.38) leads to a universe much different from that we observe today. So why has the universe to start with an energy density so close to one?

- **Monopoles and other relics problem**: According to particle physics, the standard Big Bang model meets the necessary conditions for variety of "exotic" particles (such as the magnetic monopole, a very stable and massive particle) to be produced during the early radiation dominated phase of the universe. Since these particles are diluted by the expansion as a^{-3} they can very easily become the dominant component of the universe. However no such particles have yet been observed. This either implies that the predictions from particle physics are wrong, or their densities are very small and therefore there's something missing from this evolutionary picture of the Big Bang.

- **Origin of structure problem**: Locally the universe is not homogeneous. It displays a complex hierarchical pattern of galaxies, clusters and super clusters. The general view is that structure forms via gravitational instability from very small "initial" density perturbations. But, what is the origin of these initial perturbations? Without a mechanism to explain their existence one has to assume that they "were born" with the universe already showing the correct amplitudes on all scales, so that gravitational instability can correctly reproduce the present-day structures.

- **Homogeneity and isotropy problem**: Why is the universe homogeneous on large scales? At early times this "homogeneity" had to be even more "perfect". The homogeneous and isotropic FLRW universes form a very special subset of all types of solutions

[2] This can be easily verified by noting that for both matter and radiation dominated universes the second member of Eq. (1.13) is always negative.

of the GR equations. So why would nature "prefer" homogeneity and isotropy from the beginning as opposed to, e.g., having evolved into that stage?

1.4.2 The theory of inflation

The theory of inflation (or simply inflation) was originally proposed by Guth [40] in an attempt to solve some of the above difficulties of the standard Big Bang model. This theory does not replace the Big Bang scenario. It's rather an additional mechanism attached to the early phases of the universe (prior to the radiation-dominated period), which liberates the Big Bang model from its extreme sensitivity to the initial conditions. The mechanism proposed by Guth solves the horizon, flatness and monopole problems. With time it would turn out that inflation is a powerful falsifiable theory for the origin of cosmic structure, which makes predictions that can be confirmed or ruled out against observations.

Inflation is simply defined as any period of the universe's history during which the scale factor $a(t)$ is accelerating,

$$\text{Inflation} \Leftrightarrow \ddot{a} > 0 \Leftrightarrow \frac{d}{dt}\left(cH^{-1}/a\right) < 0. \tag{1.39}$$

These are all equivalent ways of defining inflation. From the last equality we see that during an inflationary phase the comoving Hubble length always decreases. This is a necessary and sufficient condition for inflation to happen. As we noted before, in the early phases of the standard Big Bang scenario $aH = \dot{a}$ is bound to decrease continuously. This leads to the flatness problem. So if we reverse this situation ($aH = \dot{a} > 0$) i.e if the universe experiences a period of accelerated expansion, the energy density parameter in Eq. (1.38) is forced to move away from the critical value $\Omega = 1$ instead of approaching it. This would solve the flatness problem. But when and in what way can inflation occur within the Big Bang scenario?

In its simplest form, inflation is sourced by a homogeneous scalar field φ, known as the *inflaton*, with a stress energy tensor given by $T_{ab} = \varphi_{;a}\ \varphi_{;b} - g_{ab}\mathscr{L}(\varphi)$ and a Lagrangian $\mathscr{L}(\varphi) = \frac{1}{2}\varphi_{;a}\ \varphi_{;b}g^{ab} - V(\varphi)$. The shape of the potential $V(\varphi)$ depends on the details of the model of inflation under consideration. Scalar fields play a central role in describing spontaneous symmetry breaking phenomena in particle physics theories. They usually represent particles with spin zero, such as the Higgs field which is the particle responsible for electroweak symmetry breaking. At this point and for the reminder of the chapter we adopt a unit system where $c = \hbar = 1$.

When we include the contribution of the inflaton field into the second term of Eq. (1.6), the FLRW metric is still the solution of the Einstein equations of GR. The Friedmann and acceleration equations are now modified to:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho + \rho_\varphi\right) - \frac{k}{a^2} \tag{1.40}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \rho_\varphi + 3(p + p_\varphi)\right). \tag{1.41}$$

where ρ_φ and p_φ are the density and pressure of the field, respectively given by $\rho_\varphi = \dot{\varphi}^2/2 + V(\varphi)$ and $p_\varphi = \dot{\varphi}^2/2 - V(\varphi)$. One should note that although we are not assuming the existence of the Λ term in Eq. (1.6), the above equations possess the same form of Eqs. (1.12) and (1.13) if we have $\Lambda = -8\pi G p_\varphi = 8\pi G \rho_\varphi \Leftrightarrow p_\varphi = -\rho_\varphi$.

By either applying the conservation law (1.6) to $T_{ab}(\varphi)$ or using the Euler–Lagrange equations on $\mathscr{L}(\varphi)$, we can derive the following equation of motion for the inflaton field,

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + \frac{dV}{d\varphi} = 0. \tag{1.42}$$

This tells us that whenever the first two terms are negligible ($\ddot{\varphi}$, $\dot{\varphi} \simeq 0$) the field experiences a *slow-roll* period, during which its pressure and density are related by $p_\varphi = -\rho_\varphi \simeq V = \text{const.}$ Moreover since ρ_φ is approximately constant and initially ρ decreases as $\rho \propto a^{-4}$, after some time t_i from the Big Bang the energy density of the inflaton field dominates the dynamics of the expansion. This is the beginning of the inflationary period, during which the solution of the Friedmann equation (1.40) is,

$$\frac{a(t)}{a_i} = \begin{cases} \cosh\left[H(t-t_i)\right] & \text{if} \quad \Omega > 1 \\ \exp\left[H(t-t_i)\right] & \text{if} \quad \Omega = 1 \\ \sinh\left[H(t-t_i)\right] & \text{if} \quad \Omega < 1 \end{cases} \tag{1.43}$$

where $a_i = a(t_i) = (3/8\pi G V)^{1/2}$ and the Hubble expansion rate $H = (8\pi G V/3)^{1/2} \simeq$ const. This is the same dynamical behaviour we find in the de Sitter universe. Regardless of the geometry, after a time interval $t - t_i \gg (3/8\pi G V)^{1/2}$ the scale factor begins to grow exponentially and the universe rapidly behaves as flat ($\Omega = 1$). This is sufficient to remove the unpleasant fine-tuning condition on Ω required by the standard Hot Big Bang theory. However inflation cannot continue indefinitely. At some point the universe needs to re-enter in a decelerating phase which allows primordial nucleosynthesis to reproduce the correct light elements abundances. With an exponential expansion we would now be inhabitants of an "empty" universe.

Inflation ends when the scalar field approaches the minimum of the potential, V. At this point φ ends its slow-roll motion and starts to oscillate around the minimum, V_{min}. As it oscillates the particles described by the field annihilate and the resulting energy is transferred to the cosmological fluid, which experiences a (very rapid) temperature increase. This process is known as *re-heating.* After the re-heating phase the universe becomes radiation dominated and returns to its standard evolution. In a typical inflationary scenario, inflation starts just before the GUT (Grand Unified Theory) phase transition, for $t_i \sim 10^{-34}$s, and finishes soon after this at $t_f \sim 10^{-32}$s. The "amount of inflation" generated during this very short period is usually expressed in terms of number of *e-foldings*,

$$N = \ln\frac{a(t_f)}{a(t_i)} \simeq \int_{t_i}^{t_f} H dt \simeq -8\pi G \int_{\varphi_i}^{\varphi_f} V/V' d\varphi, \tag{1.44}$$

where the last equality results from the use of Eqs. (1.40) and (1.42) under the slow-roll approximation and φ_i and φ_f are the values of the inflaton field at the beginning and the end of the inflationary phase. The amount of inflation produced during this period is therefore dependent on the inflationary potential. The distance travelled by light during the same time interval is then

$$d(t_f, t_i) = a(t_f) \int_{t_i}^{t_f} \frac{dt}{a(t)} = H^{-1} \left(e^{H(t_f - t_i)} - 1 \right) \simeq H^{-1} e^N \quad , \tag{1.45}$$

which shows that a causally-connected region with size equal to the Hubble volume is exponentially expanded by e^N at the end of inflation, whereas the Hubble radius itself remains approximately constant during the same period, $R_H = H^{-1} \simeq$ const. After the end of inflation the Hubble radius starts to grow again and eventually may enclose at some point regions which were beyond the Hubble volume before inflation started. Another way of restating this is to think in terms of comoving coordinates. During inflation the comoving Hubble length decreases proportionally to $\sim e^{-N}$. Comoving scales of the size of the Hubble radius and smaller are therefore pushed outside the comoving Hubble sphere. After inflation these scales re-enter progressively the Hubble volume as the comoving Hubble radius starts to increase. If the number of e-foldings is sufficiently large, scales that didn't have time to establish causal contact before inflation are still today beyond our observable horizon. This would explain the high degree of isotropy and homogeneity we observe today.

This can also explain why magnetic monopoles and other Big Bang relics are unobservable today. If inflation happens before or during the phase when these particles are created, their density at the end of the inflationary period will decrease by a maximum factor of e^{3N}. Again, if N is big enough the density of these particles can still be very small today and therefore, in practice, unobservable. Of course, this only works if inflation has enough time to dilute these relics (see Ref. [65] for a discussion on the conditions required for this argument to hold for different kinds of relics). It can be shown that the minimum amount of inflation needed to solve these and other problems of the standard Big Bang model is about $N \sim 60$ (see e.g. Refs. [64, 65]).

Despite inflation's success to bring the primordial universe towards homogeneity and a flat geometry (as required by the standard Big Bang model), inflation's most remarkable feature is that it provides a theory for the origin of the primordial inhomogeneities. In the inflationary scenario these inhomogeneities arise from quantum fluctuations about the vacuum state of the inflaton field, which are always present due to the *Uncertainty Principle* (see Refs. [4, 41, 45, 65, 101]). The resulting irregularities can be of scalar (density perturbations) or tensorial (gravitational waves) nature. Their amplitudes on a given scale can be fully specified by the value of inflationary potential, and its derivative with respect to the inflaton field, at the time the scale crosses outside the Hubble radius during the inflationary phase (see e.g. Refs. [63, 64]).

Nowadays the title "standard model of cosmology" usually refers to the Hot Big Bang scenario plus the theory of inflation as the mechanism responsible for the origin of cosmic structure. Inflation generated perturbations are solely produced during the inflationary period and their subsequent evolution is governed by the effects of gravity and cosmic expansion alone. For this reason inflation-generated perturbations are said to be *passive*. Non-gravitational effects, such as gas cooling and heating, only become important at later times when highly non-linear structures (e.g. galaxy clusters) form. An alternative paradigm to inflation are the theories which consider *topological defects* as the main source of cosmic structure (see e.g. Refs. [25, 49, 60, 110]. In this case, perturbations are said to be *active* as they can arise at any time and evolve also under the effect of non-gravitational forces. This significantly complicates their treatment. However topological defect theories have been recently excluded from being the main source of cosmic structure as they fail to predict the observed features (namely a pronounced first peak and the indication for the existence of secondary acoustic peaks) in the CMB anisotropy spectrum (see Ref. [25]).

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