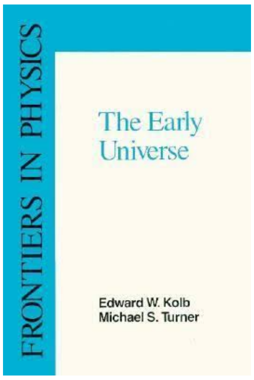
Universo Primitivo

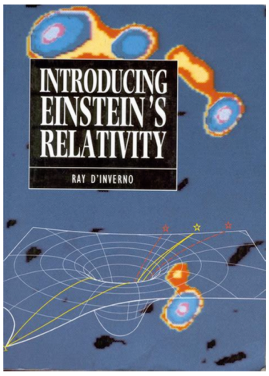
2018-2019 (1º Semestre)

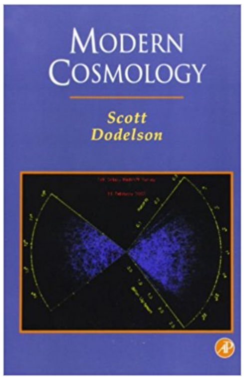
Mestrado em Física - Astronomia

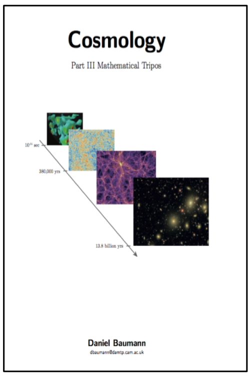
Chapter 2

2. The Standard Model of Cosmology (SMC)
    - Fundamental assumptions;
    - The GR equations and the Friedmann-Lemaitre-Robertson-Walker solution;
    - FLRW models:
        - Dynamic equations;
        - Energy-momentum conservation;
        - Fluid components and equations of state;
        - Cosmological parameters;
        - The Friedmann equation: the evolutionary phases of the Universe; exact solutions: age of the Universe;
        - Distances; horizons and volumes;
        - The accelerated expansion of the Universe;
    - Problems with the SMC: Horizon; Flatness; Relic particles; origin of perturbations; primordial Isotropy and homogeneity
    - The idea of Inflation

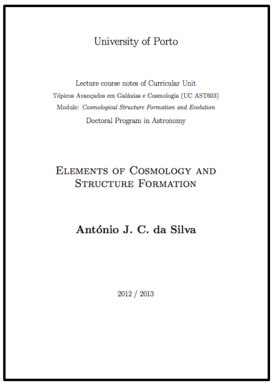








Ch. 1



Ch. 1

Standard Model of Cosmology

SMC: Mathematical framework

Fundamental assumptions:

• The Universe is homogeneous and isotropic when observed on large scales and expands uniformly with respect to any position

• The dynamics of space-time is described by Einstein's theory of general relativity (GR).

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab} + \Lambda g_{ab}$$

for the Universe to be homogeneous and isotropic the stress-energy tensor has to be that of a perfect fluid

$$T_{ab} = (\rho + \frac{p}{c^2}) U_a U_b - \frac{p}{c^2} g_{ab}$$

SMC: Mathematical framework

The cosmological constant in the GR equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$

(Λ as "cosmological constant")

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu} \right) = 8\pi G \tilde{T}_{\mu\nu},$$

(Λ as "vacuum energy")

The Einstein tensor, Riemann tensor and Ricci scalar are:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu}$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

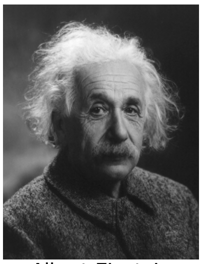
$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\nu} (g_{\alpha\nu,\lambda} + g_{\alpha\lambda,\nu} - g_{\nu\lambda,\alpha}) \qquad g_{\mu\nu,\lambda} \equiv \partial g_{\alpha\nu} / \partial x^{\lambda}$$

where,

$$\boxed{\mathrm{d}s^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \mathrm{d}X^{\mu} \mathrm{d}X^{\nu} \equiv g_{\mu\nu} \mathrm{d}X^{\mu} \mathrm{d}X^{\nu}}$$

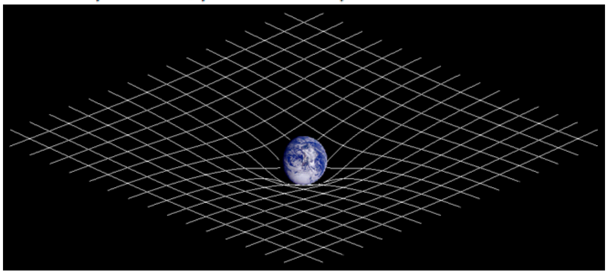
SMC: Mathematical framework

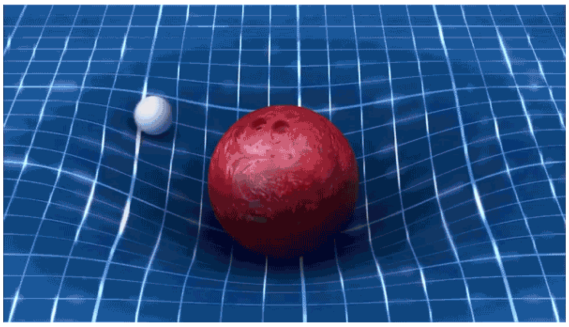
Einstein Equation:

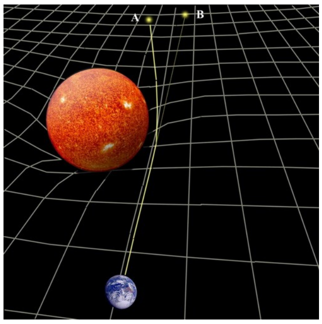


Albert Einstein
1879-1955

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$







SMC: Mathematical framework

Geodesic Equation:

In the absence of non-gravitational forces, free falling particles move along “geodesics”, described by the so called Geodesic equation.

$$\boxed{\frac{dU^\mu}{ds} + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta = 0}$$

where,

$$U^\mu \equiv \frac{dX^\mu}{ds}$$

four-velocity of the particle along its free-falling path $X^\mu(s)$

$$S = -m \int_A^B \mathrm{d}s \ .$$

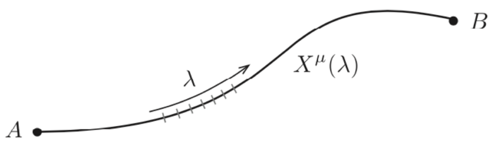


Figure 1.4: *Parameterisation of an arbitrary path in spacetime, $X^\mu(\lambda)$.*

SMC: Mathematical framework

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$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi G}{c^4}T_{ab} + \Lambda g_{ab} \qquad T_{ab} = (\rho + \frac{p}{c^2})U_a U_b - \frac{p}{c^2}g_{ab}$$

In these conditions **the solution of the Einstein equation** is the Friedmann-Lemaitre-Robertson-Walker (**FLRW**) metric:

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

SMC: Mathematical framework

- Dynamical equations:
  (result from the Einstein equations and govern the time evolution of $a(t)$)

$$\boxed{\begin{aligned}\left(\frac{\dot a}{a}\right)^2 &= \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2} \\ \frac{\ddot a}{a} &= -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3}\end{aligned}}$$

Friedmann equation

Raychaudhuri
(or acceleration) equation

- Energy momentum conservation: $\nabla_\mu T^\mu{}_\nu \equiv T^\mu{}_{\nu;\mu} = 0$

the covariant derivative reads: $\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\lambda}T^\lambda{}_\nu - \Gamma^\lambda_{\mu\nu}T^\mu{}_\lambda = 0$

the $\nu = 0$ (time) component of this equation gives:

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \quad \Rightarrow \quad d\left(\rho c^2 a^3\right) = -pd\left(a^3\right)$$

Energy conservation equation

$$p = w\rho c^2 \qquad -1 \leq w \leq 1$$

Equation of State (EoS)

for fluids with constant EoS parameter, w, the solution is:

$$\boxed{\rho(t) = \rho_i\left(\frac{a(t)}{a_i}\right)^{-3(1+w)}}$$

SMC: Mathematical framework

Covariant derivative:

Covariant derivative.—The covariant derivative is an important object in differential geometry and it is of fundamental importance in general relativity. The geometrical meaning of ∇_μ will be discussed in detail in the GR course. In this course, we will have to be satisfied with treating it as an operator that acts in a specific way on scalars, vectors and tensors:

- There is no difference between the covariant derivative and the partial derivative if it acts on a scalar
$$\nabla_\mu f = \partial_\mu f\ . \tag{1.3.83}$$
- Acting on a contravariant vector, V^ν, the covariant derivative is a partial derivative plus a correction that is linear in the vector:
$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda\ . \tag{1.3.84}$$
Look carefully at the index structure of the second term. A similar definition applies to the covariant derivative of covariant vectors, ω_ν,
$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda\ . \tag{1.3.85}$$
Notice the change of the sign of the second term and the placement of the dummy index.
- For tensors with many indices, you just repeat (1.3.84) and (1.3.85) for each index. For each upper index you introduce a term with a single $+\Gamma$, and for each lower index a term with a single $-\Gamma$:
$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} &= \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l}\\ &+ \Gamma^{\mu_1}{}_{\sigma\lambda}T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}{}_{\sigma\lambda}T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \cdots\\ &- \Gamma^\lambda{}_{\sigma\nu_1}T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} - \Gamma^\lambda{}_{\sigma\nu_2}T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots\ .\end{aligned} \tag{1.3.86}$$
This is the general expression for the covariant derivative. Luckily, we will only be dealing with relatively simple tensors, so this monsterous expression will usually reduce to something managable.

SMC: Mathematical framework

- EoS for different energy density components:
  - w=1/3 (radiation)

$$\rho_\gamma = \rho_{\gamma 0}\left(\frac{a_0}{a}\right)^4 \xrightarrow{(1)} \left(\frac{\dot a}{a}\right)^2 \propto \frac{1}{a^4} \longrightarrow a \propto t^{1/2}.$$

  - w=0 (matter)

$$\rho_{\rm m} = \rho_{\rm m0}\left(\frac{a_0}{a}\right)^3 \xrightarrow{(2)} \left(\frac{\dot a}{a}\right)^2 \propto \frac{1}{a^3} \longrightarrow a \propto t^{2/3}.$$

  - w=-1 (cosmological constant)

$$\rho_\Lambda = \Lambda/8\pi G = -P_\Lambda \xrightarrow{(3)} a \propto e^{\sqrt{\Lambda}t}.$$

(1) after integration of the Friedmann equation with $k=0,\ \Lambda=0,\ \ \rho=\rho_\gamma$.
(2) after integration of the Friedmann equation with $k=0,\ \Lambda=0,\ \ \rho=\rho_m$.
(3) after integration of the Friedmann equation with $k=0,\ \Lambda=8\pi G\rho_\Lambda,\ \rho=0$

SMC: FLRW models

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}$$

- Cosmological parameters:

$$\frac{8\pi G}{3H^2}\rho + \frac{\Lambda c^2}{3H^2} - \frac{kc^2}{a^2H^2} = 1 \quad \Leftrightarrow \quad \sum_i \Omega_i + \Omega_\Lambda + \Omega_k = 1$$

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

Hubble parameter

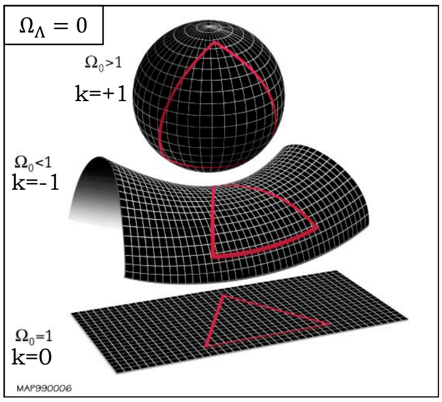
$$\Omega = \frac{\rho}{\rho_c}, \quad \Omega_\Lambda = \frac{\Lambda c^2}{3H^2}, \quad \Omega_k = -\frac{kc^2}{a^2H^2}, \quad \rho_c = \frac{3H^2}{8\pi G}$$

Critical energy density

total mater energy density parameter

Vacuum or dark energy density parameter

Curvature density parameter



$$\rho \equiv \sum_i \rho_i$$

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\text{crit}}}$$

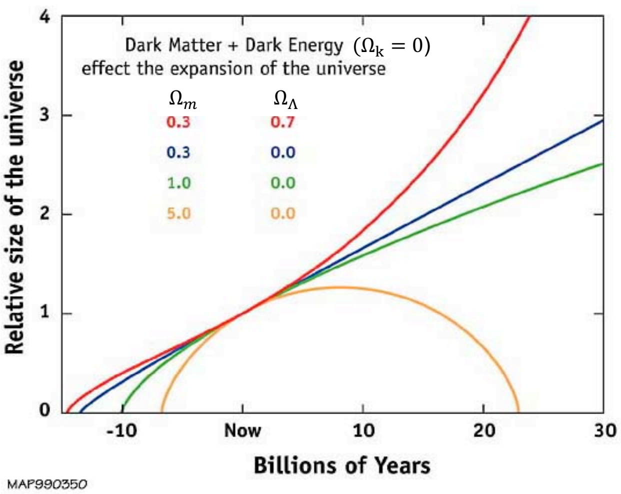
includes all matter and radiation components (baryons, dark mater, radiation, …)

SMC: FLRW models

- Friedmann equation revisited

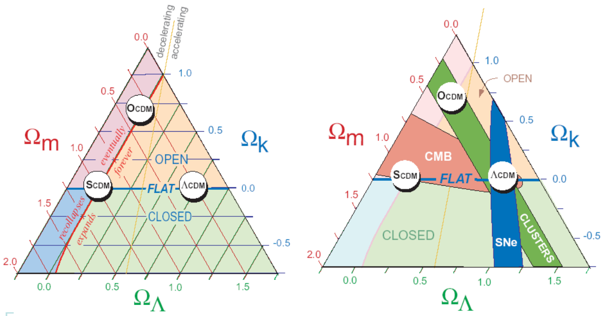
$$H^2(t) = \frac{8\pi G}{3}(\rho_r + \rho_m) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$
$$= H_0^2\left[\Omega_{r0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda 0}\right]$$

The evolutionary fate of the Universe is determined by cosmological parameters



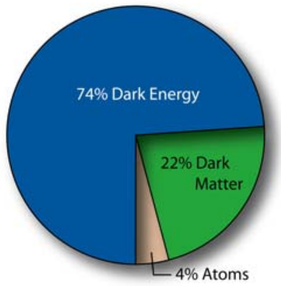
SMC: Concordance Cosmology

Combination of different observational datasets…



From:

… allow us to impose constraints on cosmological parameters



WMAP3 parameters

Parameter	Value	Description
	Basic parameters	
H_0	$70.9^{+2.4}_{-3.2}$ km s^{-1} Mpc^{-1}	Hubble parameter
Ω_b	$0.0444^{+0.0042}_{-0.0035}$	Baryon density
Ω_m	$0.266^{+0.025}_{-0.040}$	Total matter density (baryons + dark matter)
τ	$0.079^{+0.029}_{-0.032}$	Optical depth to reionization
A_s	$0.813^{+0.042}_{-0.052}$	Scalar fluctuation amplitude
n_s	$0.948^{+0.015}_{-0.018}$	Scalar spectral index
	Derived parameters	
ρ_0	$0.94^{+0.06}_{-0.09} \times 10^{-26}$ kg/m^3	Critical density
Ω_Λ	$0.732^{+0.040}_{-0.025}$	Dark energy density
z_{ion}	$10.5^{+2.6}_{-2.9}$	Reionization red-shift
σ_8	$0.772^{+0.036}_{-0.048}$	Galaxy fluctuation amplitude
t_0	$13.73^{+0.13}_{-0.17} \times 10^9$ years	Age of the universe

SMC: Exact solutions of the Friedmann equation

- Scale factor:

$$\frac{d}{dt}\frac{a(t)}{a_0} = H_0\sqrt{1-\Omega_0+\Omega_{m0}\left(\frac{a}{a_0}\right)^{-1}+\Omega_{r0}\left(\frac{a}{a_0}\right)^{-2}-\Omega_{\Lambda 0}\left[1-\left(\frac{a}{a_0}\right)^2\right]}$$

for a critical density ($\Omega_k = \Omega_\Lambda = 0$) universe, gives:

$$\frac{a(t)}{a_0} = \left(\frac{3(1+w)}{2}H_0 t\right)^{2/(3(1+w))}$$

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3(w+1)t}$$

- Age of the Universe:

$$t = H_0^{-1}\int_0^{\frac{a(t)}{a_0}=(1+z)^{-1}} \frac{1}{\sqrt{1-\Omega_0+\Omega_{m0}x^{-1}+\Omega_{r0}x^{-2}-\Omega_\Lambda\left(1-x^2\right)}}\,dx$$

SMC: distances, horizons and volumes

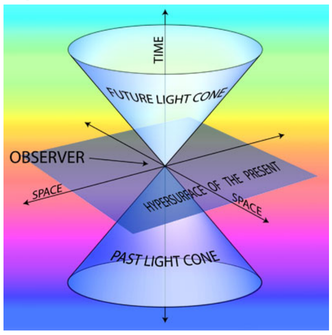
- Coordinate distance:
  ($can\ be\ computed\ using\ photons\ that\ travel\ along\ null\ geodesics$: $ds^2 = 0$)

$$\int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}} = c\int_0^t \frac{dt'}{a(t')} = c\,\eta(t) \quad \longleftarrow \quad ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

- Proper (physical) distance / particle horizon:

$$d_H(t) = \int_0^{r_e} \sqrt{|g_{rr}|}dr = a(t)\int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}} = c\,a(t)\eta(t)$$

From:



for a $\Omega_\Lambda = 0$ universe this gives:

$$d_H(t) \simeq \frac{2}{3w+1}\frac{c}{H_0}\Omega_{w0}^{1/2}\left(\frac{a}{a_0}\right)^{3(1+w)/2} = 3\frac{1+w}{1+3w}ct$$

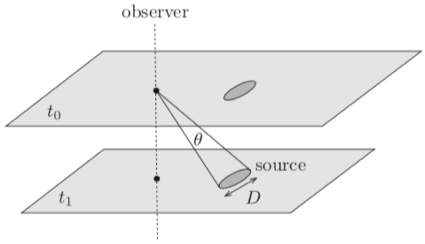
SMC: distances, horizons and volumes

- Angular size of a region at a given time:

$$\boxed{\theta = \frac{D}{d_H(t)}}$$

where

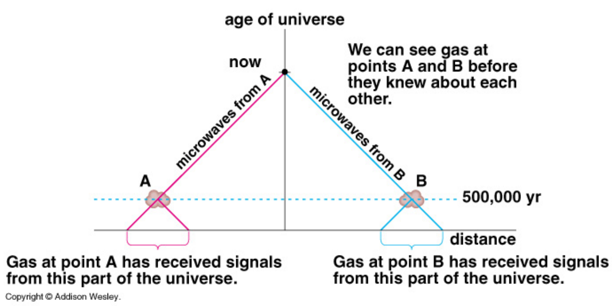
$$d_H(t) = \int_0^{r_e} \sqrt{|g_{rr}|}dr = a(t)\int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}} = c\,a(t)\eta(t)$$



Angular size of the particle horizon at a given time for a critical density universe ($\Omega_\Lambda = 0$)

From:

$$\boxed{\theta_H \simeq 2\tan\frac{\theta_H}{2} = \frac{\Omega_0^{3/2}\sqrt{1+z}}{\Omega_0 z + (\Omega_0 - 2)\left(\sqrt{1+\Omega_0 z} - 1\right)}}$$



SMC: distances, horizons and volumes

- Hubble length:

$$R_H(t) = \frac{c}{H(t)} = \frac{3(w+1)}{2} ct$$

where the last equality holds for a critical density universe Ω=1

- Physical volume element:

$$dV = \sqrt{|g|}\, dr\, d\theta\, d\phi = (ar)^2 \frac{a\, dr}{\sqrt{1-kr^2}} d\Omega$$

$$\frac{dV}{d\Omega\, dz} = \frac{c}{H(z)} \frac{(a_0 r)^2}{(1+z)^3} = \frac{c}{H_0} \frac{d_A^2}{\mathscr{H}(z)(1+z)}$$

where:

$$\mathscr{H}(z) = H(z)/H_0$$

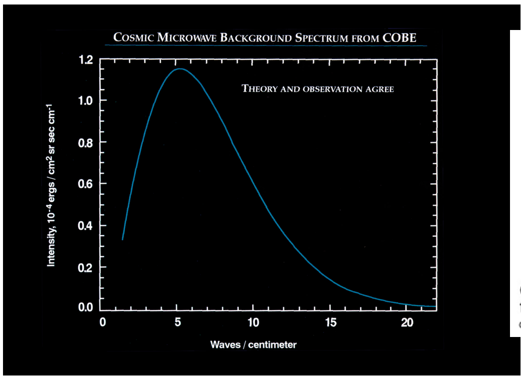
Problems of the FLRW models as the sole ingredient of the SMC

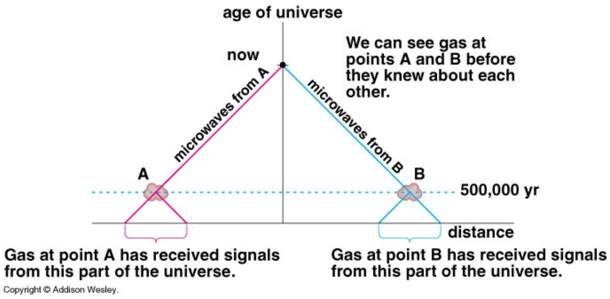
The Horizon Problem

At high redshift ($z \gg 1$):

$$\theta_H \simeq \frac{180}{\pi}\sqrt{\frac{\Omega_0}{z}}\,\mathrm{deg}$$

there are ~54000 causal disconnected areas in the CMB sky. Why the CMB has a thermal spectrum with a so uniform temperature in all directions (2.725 °K)





The Flatness Problem

From the Friedmann Equation, written at early times:

$$|\Omega(t) - 1| = \frac{|k|}{a^2(t)H^2(t)} = \frac{|k|}{\dot{a}^2(t)}$$

is a decreasing function of time

decreases as time approaches the big bang instant.

This means that as we go back in time the energy density of universe has to be extremely close to critical density. For t=1e-43 s (Planck time) Ω should deviate no more than 1e-60 from the unity.

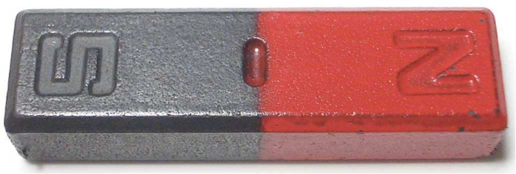
Why has the universe to start with $\Omega(t)$ so close to 1?

The Monopoles & other relics Problem

Particle physics predicts that a variety of **“exotic” stable particles**, such as the magnetic monopoles, should be produced in the early phase of the Universe and remain in measurable amounts until the present.

No such particles have yet been observed. Why?

This either implies that the predictions from particle physics are wrong, or their densities are very small and therefore there's something missing from this evolutionary picture of the Big Bang.

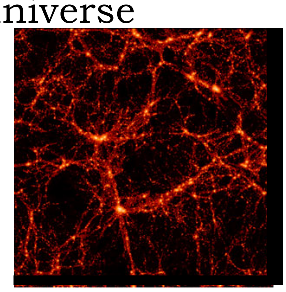


The Origin of Perturbations Problem

Locally the universe is not homogeneous. It displays a complex hierarchical pattern of galaxies, clusters and super clusters.

What’s the origin of cosmological structure?
Does it grew from gravitational instability?
What is the origin of the initial perturbations?

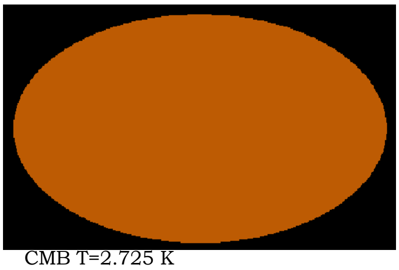
Without a mechanism to explain their existence one has to assume that they ``were born'' with the universe already showing the correct amplitudes on all scales, so that gravitaty can correctly reproduce the present-day structures?



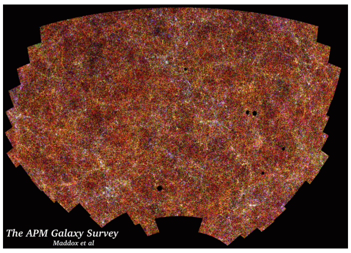
The homogeneity and isotropy Problem

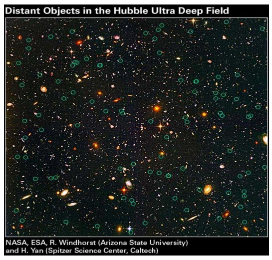
Why is the universe homogeneous on large scales? At early times homogeneity had to be even more “perfect”.

The FLRW universes form a very special subset of solutions of the GR equations. So *why nature “prefers” homogeneity and isotropy from the beginning as opposed to having evolved into that stage?*



CMB T=2.725 K





The Theory of Inflation…

Inflation can be defined as

$$\text{Inflation} \Leftrightarrow \ddot{a} > 0 \Leftrightarrow \frac{d}{dt}\left(cH^{-1}/a\right) < 0.$$

This happens when

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) \quad \Rightarrow \quad p < -\rho c/3$$