UNIVERSO PRIMITIVO: INFLAÇÂO E ESTRUTURA DE LARGA ESCALA
Mestrado em Física Astronomia
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Exercise Sheet 1

1. In a FRLW universe, fundamental observers experience no external forces and have fixed coordinates in the comoving coordinate system. The proper distance between two of such observers scales as $r(t) = a(t)\, x$, where $a(t)$ is the scale factor and x is their comoving separation.
   1.1. Derivate this expression to prove the Hubble law, $v(t) = H(t)\, r(t)$, where $H = \dot{a}/a$.
   1.2. Derive a similar expression for a pair of non-fundamental observers that have a relative peculiar velocity, $v_p = \dot{x}$, in the commoving coordinate system.

2. Consider a homogeneous and isotropic fluid with an energy-stress tensor $T^{\mu}_{\nu} = (\rho + p)\, U^{\mu}\, U_{\nu} - p g^{\mu}_{\nu}$ of a perfect fluid.
   2.1. Prove that $\dot{\rho} = -3H(\rho + p)$ where $H = \dot{a}/a$ is the Hubble factor. [Hint: apply the conservation law, $T^{\mu}_{\nu;\mu} = 0$ to the $\nu = 0$ component].
   2.2. Use this continuity equation to prove that $dE = -pdV$, where $dE = d(\rho a^3 L^3)$ is the energy inside a volume element, $dV = a^3 L^3$, of comoving size L^3.
   2.3. Integrate the equation in 2.1 to prove that $\rho(t) = \rho_i \left(\frac{a(t)}{a_i}\right)^{-3(1+w)}$ where w is the equation of state parameter of a given fluid component and ρ_i, a_i are integration constants;
   2.4. Use the expression in 2.3 to derive the time dependence of the scale factor for the following components: radiation ($\mathrm{w} = 1/3$), collisionless matter ($\mathrm{w} = 0$) and cosmological constant ($\mathrm{w} = -1$) [Hint: assume no curvature term in the FRLW dynamic equations].

3. Consider the FLRW dynamic equations
   3.1. Use the Friedman equation and the continuity equation (2.1) to derive the acceleration equation.
   3.2. Use the definition of the cosmological density parameters to prove the following form of the Friedmann equation.

$$\begin{aligned} H^2(t) &= \frac{8\pi G}{3}(\rho_r + \rho_m) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \\ &= H_0^2 \left[\Omega_{r0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda 0}\right] \end{aligned}$$

   3.3. Estimate the value of a/a_0 at matter-radiation equality if the radiation and matter density parameters at present are Ω_{r0} =0.0001 and $\Omega_{m0} = 0.25$, respectively.

4. Use the Friedmann equation in 3.2 to compute the Age of the universe for
   4.1. A critical density universe, $\Omega = \Omega_{r0} + \Omega_{m0} = 1$, with $\Omega_{r0} \simeq 0$.
   4.2. A flat, Λ −Universe with $\Omega_{r0} \simeq 0$, $\Omega_{m0} = 0.25$, $\Omega_{\Lambda 0} = 0.75$, $H_0 = 70\ km\, s^{-1}\, Mpc^{-1}$

   [Hint: integrate the Friedmann equation with respect to the scale factor]