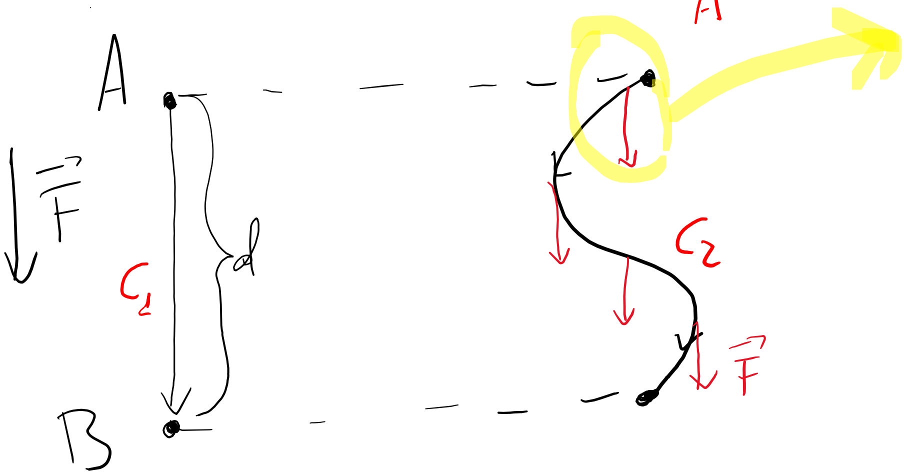
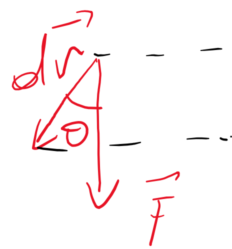
Line integral

$$W = \int \vec{F} \cdot d\vec{r} = \int_A^B F \, dy = \boxed{F \cdot d}$$

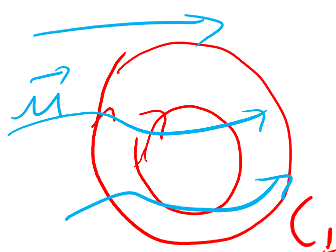




$$d\vec{r} \cdot \vec{F} = |\vec{F}| \, dy$$

Ex.: circulation

$$\boxed{\Gamma = \oint \vec{u} \cdot d\vec{r}}$$

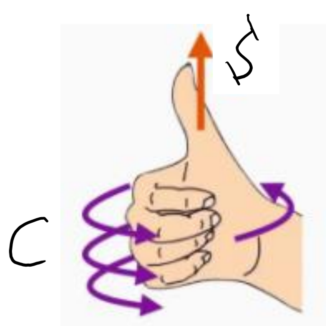


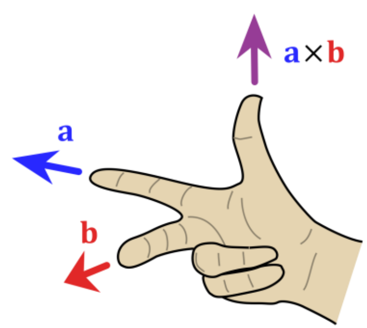
Stokes theorem

Acheson (appendix)

$$\oint_C \vec{F} \cdot d\vec{l} = \int_S \nabla \times \vec{F} \cdot d\vec{S}$$

The surface S is connected and limited by C. Use the right-hand rule to determine the direction of S according to C.



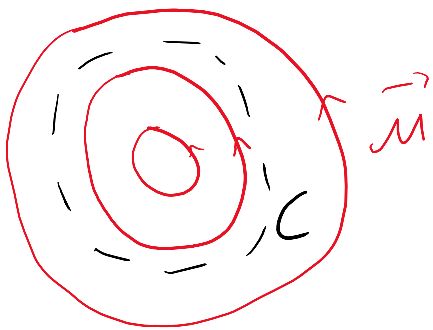


Other useful rule for cross product

$$\vec{c} = \vec{a} \times \vec{b}$$

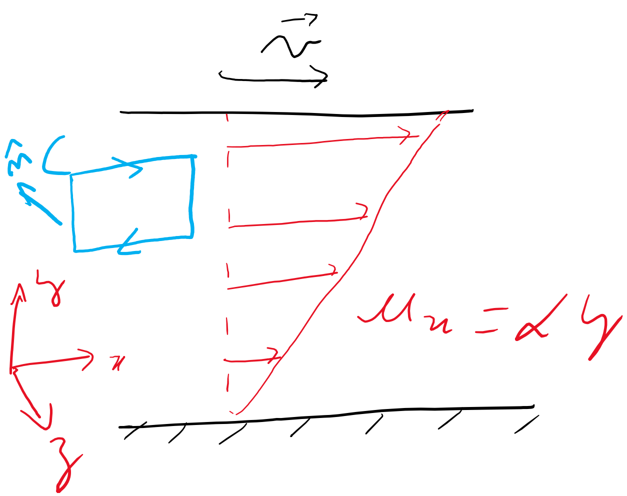
Stokes theorem

Ex.: vorticity $\vec{\omega} = \nabla \times \vec{u}$



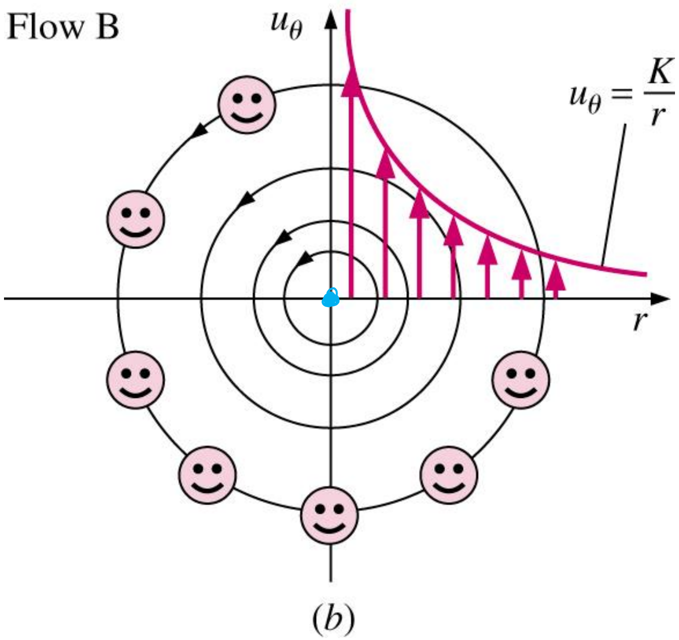
$$\boxed{\int_S \nabla \times \vec{u} \cdot d\vec{S} = \oint_C \vec{u} \cdot d\vec{l} = \Gamma}$$

Lid driven cavity



$$\vec{\omega} = \nabla \times \vec{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \alpha y & 0 & 0 \end{vmatrix}$$

$$= \boxed{-\alpha \hat{z}}$$



$\Gamma \neq 0$, $u(r=0) \to \infty$

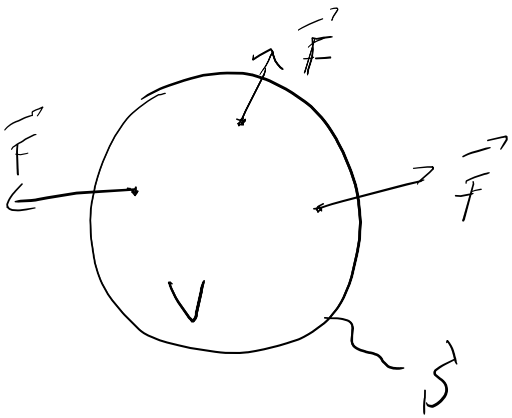
$$\int_S \nabla \times \vec{u} \cdot d\vec{S} \neq \int_C \vec{u} \cdot d\vec{l} = \Gamma$$

$$u_r = 0, u_\theta = \frac{K}{r}$$

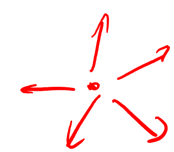
$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(K)}{\partial r} - 0\right)\vec{e}_z = 0\vec{e}_z$$

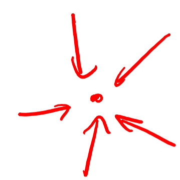
Gauss theorem

$$\int_S \vec{F}\cdot d\vec{S} = \int_V \nabla\cdot\vec{F}\, dV$$



Ex.: $\begin{cases} \nabla\cdot\vec{u} > 0 \\ \nabla\cdot\vec{u} < 0 \end{cases}$





Kronecker Delta

$j = \{x, y, z\}$

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Einstein notation

$$A_j B_j = \sum_{j=1}^{3} A_j B_j$$

$$= A_x B_x + A_y B_y + A_z B_z$$

$$= \vec{A} \cdot \vec{B}$$

Ex.:

$$B_j = A_i \delta_{ij} = A_x \delta_{xj} + A_y \delta_{yj} + A_z \delta_{zj} = A_j$$

$$C = A_i B_j \delta_{ij} = A_j B_j = \vec{A} \cdot \vec{B}$$

Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3), \\ \;\;0 & \text{if } i=j, \text{ or } j=k, \text{ or } k=i \end{cases}$$

$x\,y\,z \quad y\,z\,x$

ex.:

$\varepsilon_{xyz} = 1$

$\varepsilon_{xzy} = -1$

$\varepsilon_{zxy} = 1$

$\varepsilon_{zxx} = 0$

Properties

$\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij} = \ldots$

$$\varepsilon_{ijk} \cdot \varepsilon_{imn} = \underbrace{\delta_{jm}\delta_{kn}}_{\text{índices na ordem}} - \underbrace{\delta_{jn}\delta_{km}}_{\text{Trocados}}$$

Ex.: $\vec{A} \times \vec{B} = \varepsilon_{ijk} A_j B_k$

índice livre

(Acheson)

A.1. Vector identities

$$(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}, \tag{A.1}$$

$$\nabla \wedge \nabla\phi = 0, \qquad \nabla \cdot (\nabla \wedge \boldsymbol{F}) = 0, \tag{A.2, A.3}$$

$$\nabla \cdot (\phi\boldsymbol{F}) = \phi\nabla \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \nabla\phi, \tag{A.4}$$

$$\nabla \wedge (\phi\boldsymbol{F}) = \phi\nabla \wedge \boldsymbol{F} + (\nabla\phi) \wedge \boldsymbol{F}, \tag{A.5}$$

$$\nabla \wedge (\boldsymbol{F} \wedge \boldsymbol{G}) = (\boldsymbol{G} \cdot \nabla)\boldsymbol{F} - (\boldsymbol{F} \cdot \nabla)\boldsymbol{G} + \boldsymbol{F}(\nabla \cdot \boldsymbol{G}) - \boldsymbol{G}(\nabla \cdot \boldsymbol{F}), \tag{A.6}$$

$$\nabla \cdot (\boldsymbol{F} \wedge \boldsymbol{G}) = \boldsymbol{G} \cdot (\nabla \wedge \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \wedge \boldsymbol{G}), \tag{A.7}$$

$$\nabla(\boldsymbol{F} \cdot \boldsymbol{G}) = \boldsymbol{F} \wedge (\nabla \wedge \boldsymbol{G}) + \boldsymbol{G} \wedge (\nabla \wedge \boldsymbol{F}) + (\boldsymbol{F} \cdot \nabla)\boldsymbol{G} + (\boldsymbol{G} \cdot \nabla)\boldsymbol{F}, \tag{A.8}$$

$$(\boldsymbol{F} \cdot \nabla)\boldsymbol{F} = (\nabla \wedge \boldsymbol{F}) \wedge \boldsymbol{F} + \nabla(\tfrac{1}{2}\boldsymbol{F}^2), \tag{A.9}$$

$$\nabla^2\boldsymbol{F} = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla \wedge (\nabla \wedge \boldsymbol{F}). \tag{A.10}$$

Show that:

$$\nabla \times (\nabla \times \vec{u}) = \nabla (\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$$

$$= \varepsilon_{ijk}\, \partial_j (\nabla \times \vec{u})_k = \varepsilon_{ijk}\, \partial_j\, \varepsilon_{klm}\, \partial_l\, u_m$$

$$= \underbrace{\varepsilon_{ijk}\, \varepsilon_{klm}}\, \partial_j \partial_l\, u_m$$

$$-\varepsilon_{ikj}\, \varepsilon_{klm} = \varepsilon_{kij}\, \varepsilon_{klm} = \delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl}$$

$$= (\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl})\, \partial_j \partial_l\, u_m$$

$$= \delta_{il}\, \delta_{jm}\, \partial_j \partial_{\cancel{l}}^{\,i}\, u_{\cancel{m}}^{\,j} - \delta_{im}\, \delta_{jl}\, \partial_j \partial_{\cancel{l}}^{\,j}\, u_{\cancel{m}}^{\,i}$$

$$= \partial_j \partial_i\, u_j - \partial_j \partial_j\, u_i = \nabla (\nabla \cdot \vec{u}) - \nabla^2 \vec{u} \quad \boxed{/}$$