The Euler fluid: zero viscosity and zero compressibility

As a result the shear stresses are zero and the density is constant

Euler fluid

(Acheson, chap. 1)

- For an Euler fluid the continuity equation implies that

and the inviscid (zero viscosity) condition implies that the stress tensor reduces to a scalar isotropic pressure, *p,* which may vary in space (pressure field).

$$\rho = Cte \Rightarrow \boxed{\nabla \cdot \boldsymbol{u} = 0}$$

- The surface forces acting on an element of fluid, per unit volume, are given

by $-\nabla p$ (recall that the force in the x direction is $-\frac{\partial p}{\partial x}$).

$p \rightarrow \frac{N}{m^2}$

$\nabla p \rightarrow \frac{N}{m^3}$

- The forces per unit mass are then $-\frac{\nabla p}{\rho}$.

- The total force may include body terms, such as gravity, $-\nabla gz$.

- The Euler equation is

$$\vec{a} = \boxed{\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \boldsymbol{g},}$$

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}$$

Euler fluid

- 4 equations (continuity + Euler) and 4 unknowns (u, v, w, p);

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

$$\nabla \cdot \vec{u} = 0 \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

- The gravitational force, being conservative, can be written as the gradient of a potential (=gz):

$$\mathbf{g} = -\nabla\chi. \quad = gz$$

Euler fluid

- Euler's equation becomes:

$$\underbrace{\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}}_{\frac{D\vec{u}}{Dt}} = -\nabla\left(\frac{p}{\rho} + \chi\right)$$

- From ~~slide 27~~,

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \underbrace{(\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u}} + \nabla(\tfrac{1}{2}\boldsymbol{u}^2)$$

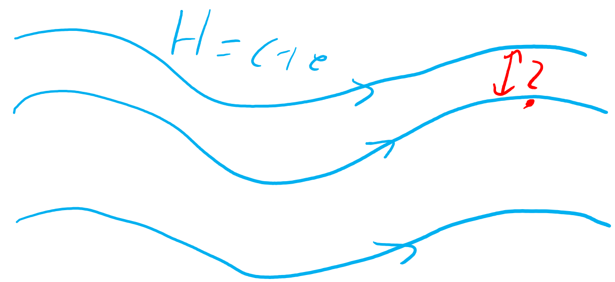
- We can write:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\underbrace{\nabla \wedge \boldsymbol{u}}_{\vec{\omega}}) \wedge \boldsymbol{u} = -\nabla\underbrace{\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right)}_{H}$$

Bernoulli streamline theorem

- If the fluid is steady, $= \frac{\partial \vec{u}}{\partial t} = 0$

$$\underbrace{\vec{u} \cdot (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u}}_{=0} = -\nabla H \cdot \vec{u}$$



where: $$\boxed{H = \frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi.}$$

- On taking the dot product with u, we obtain

$$(\boldsymbol{u} \cdot \nabla)H = 0,$$

$$\frac{DH}{Dt} = \cancel{\frac{\partial H}{\partial t}}^{\,0} + \vec{u} \cdot \nabla H$$

- If an ideal fluid is in steady flow, then H is constant along a streamline.
- The above theorem says nothing about H being the same constant on different streamlines.

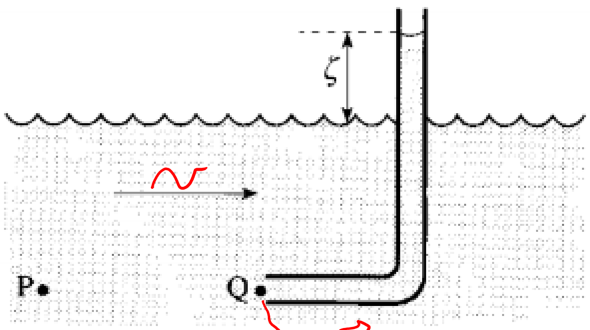
Bernoulli theorem for irrotational flow

1) $$\cancelto{0}{\frac{\partial \boldsymbol{u}}{\partial t}} + \underbrace{(\nabla \wedge \boldsymbol{u})}_{\vec{\omega}} \wedge \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \cancelto{gz}{\chi}\right)$$

2) $\boxed{\vec{\omega} = 0} \Rightarrow \nabla H = 0 \Rightarrow H = cte$ em todo o espaço

If an ideal fluid is in steady irrotational flow, then H is constant throughout the whole flow field.

Pitot tube



$\vec{u} = 0$

Location of pitot tubes on a Boeing 777

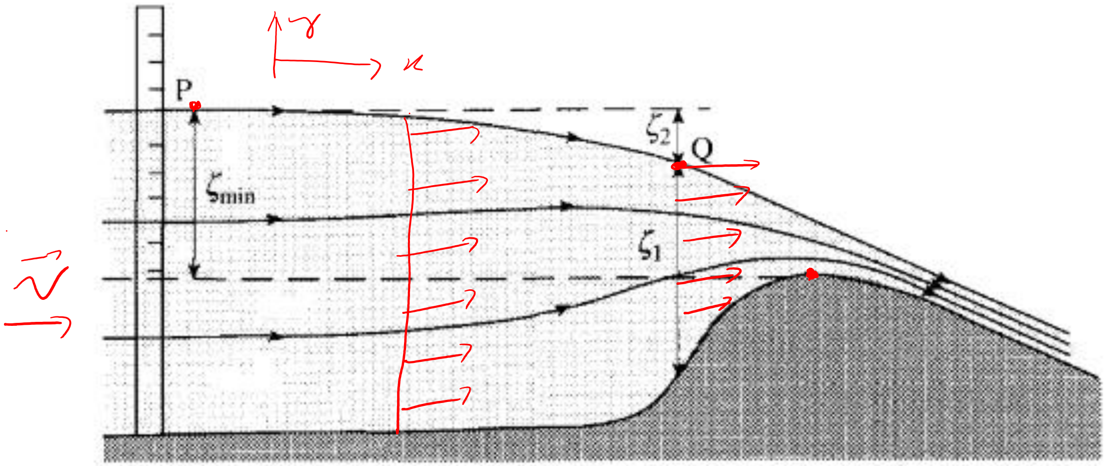
$$\Delta p = \rho g \zeta$$

$$H_P = H_Q$$

$$\frac{p_P}{\rho} + \frac{v_P^2}{2} + \cancel{g z_P} = \frac{p_Q}{\rho} + \cancel{\frac{v_Q^2}{2}}^{=0} + \cancel{g z_Q} \qquad z_Q = z_P$$

$$v^2 = 2\,\frac{(p_Q - p_P)}{\rho} = \frac{2}{\cancel{\rho}}\,\cancel{\rho} g \zeta = 2 g \zeta \Rightarrow \boxed{v = \sqrt{2 g \zeta}}$$

(Faber 2.10)



$q = V \cdot A$

$\frac{m}{s} \cdot m^2$

$H_P = H_Q$

$$\cancel{\frac{p_P}{\rho}} + \cancel{\frac{v_P^2}{2}}^{\approx 0} + g z_P = \cancel{\frac{p_Q}{\rho}} + \frac{v_Q^2}{2} + g z_Q$$

$\rightarrow v_P \ll v_Q$

$$\frac{v_Q^2}{2} = g(z_P - z_Q) \Rightarrow \boxed{v_Q = \sqrt{2 g \zeta_2}}$$

$\rightarrow v_{(y)} = C_{te}$, depende de x

$$q = V_\varphi \cdot S_1 = \sqrt{2 g S_2} \cdot S_1 \Rightarrow S_2 = \frac{q^2}{2 g S_1^2}$$

$$S = S_1 + S_2 = S_1 + \frac{q^2}{2 g S_1^2}$$

$$\frac{\partial S}{\partial S_1} = 0 = 1 - \frac{\cancel{2} q^2}{\cancel{2} g S_1^3} = 1 - \frac{q^2}{g S_1^3} = 0 \Rightarrow \boxed{S_1 = \sqrt[3]{\frac{q^2}{g}}}$$

$$S_{min} = \frac{3}{2} \frac{q^{2/3}}{g^{1/3}} \Rightarrow \boxed{q = \left(\frac{2}{3} \frac{g^{1/3}}{S_{min}} \right)^{3/2}}$$