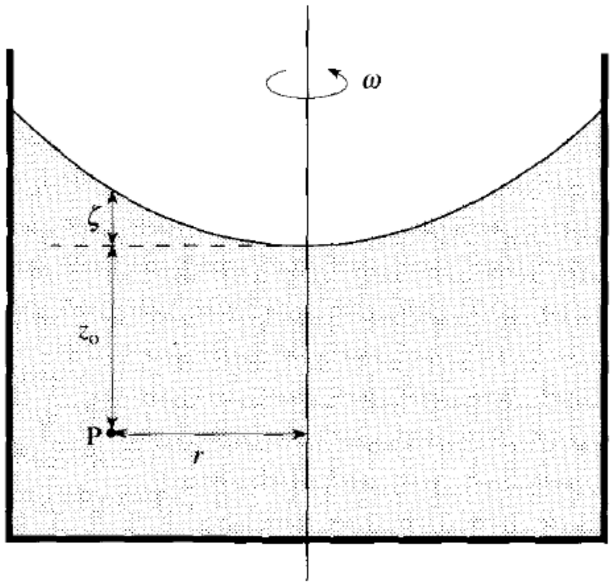
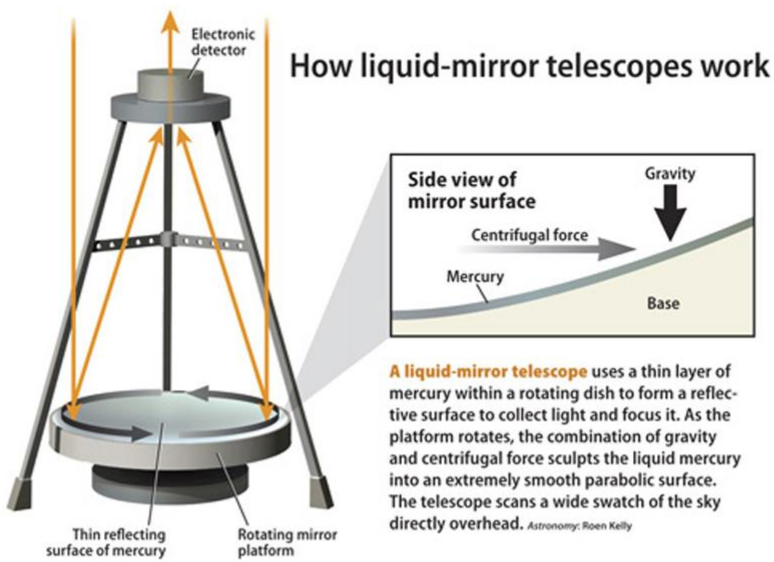
The bucket of liquid

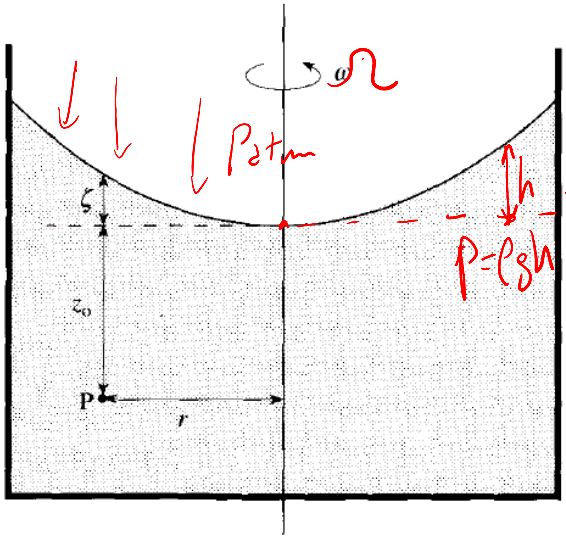
Faber, 2.5



Liquid mirror telescope:
https://www.youtube.com/watch?v=Q5Cr9P-Q88Y



In the steady state, the velocity profile becomes



$$\boxed{\vec{u} = \Omega r \hat{\varphi}} = -\Omega y \hat{x} + \Omega x \hat{y}$$

$$\hat{\varphi} = \frac{-y\hat{x} + x\hat{y}}{\sqrt{x^2+y^2}}$$

Vorticy

$$\vec{\omega} = \nabla \times \vec{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\Omega y & \Omega x & 0 \end{vmatrix} = \hat{z}\left(\frac{\partial}{\partial x}(\Omega x) + \frac{\partial}{\partial y}(\Omega y)\right)$$

$$= 2\Omega\hat{z}$$

Why Euler? The velocity field above leads to zero off-diagonal components of the stress tensor (check). Thus, the viscous effects are not relevant and we can use the ideal fluid approximation.

$$\sigma_{ij} = -P\delta_{ij} + \eta\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

Euler’s equation

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \boldsymbol{g},$$

or

$$\cancelto{0}{\frac{\partial \boldsymbol{u}}{\partial t}} + \underbrace{(\nabla \wedge \boldsymbol{u})}_{\omega} \wedge \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \cancelto{gh}{\chi}\right)$$

$$\vec{\omega} \times \vec{u} = -\nabla H$$

$$\Rightarrow \vec{\omega} \times \vec{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 2\Omega \\ -\Omega y & \Omega x & 0 \end{vmatrix} = -\hat{x}\,(2\Omega^2 x) + \hat{y}\,(-2\Omega^2 y)$$

$$= -2\Omega^2\,(x\,\hat{x} + y\,\hat{y})$$

$$\Rightarrow \vec{u}^2 = \Omega^2 y^2 + \Omega^2 x^2$$

$$\nabla \vec{u}^2 = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(\Omega^2 x^2 + \Omega^2 y^2\right)$$

$$= 2\Omega^2 x\,\hat{x} + 2y\Omega^2\,\hat{y}$$

$\Rightarrow \nabla p = 0$ (Superficie à p_{atm})

$$\Rightarrow \nabla(gh) = g\nabla h$$

<u>Componente x</u>

$$-2\Omega^2 x = -\frac{1}{\cancel{2}}\cancel{2}\Omega^2 x - g\frac{\partial h}{\partial x} \Rightarrow \frac{\partial h}{\partial x} = \frac{\Omega^2 x}{g}$$

$$h(x) = \frac{\Omega^2 x^2}{2g} + C_1(y, z)$$

<u>Componente y</u>

$$-2\Omega^2 y = -\frac{1}{\cancel{2}}\cancel{2}y\Omega^2 - g\frac{\partial h}{\partial y} \Rightarrow \frac{\partial h}{\partial y} = \frac{\Omega^2 y}{g}$$

$$h(y) = \frac{\Omega^2 y^2}{2g} + C_2(x, z)$$

$$\Rightarrow \boxed{h = \frac{\Omega^2}{2g}(x^2 + y^2) + C_{1e}} = \frac{\Omega^2}{2g} r^2 + C_{1e}$$

Calcular "p" para $h = cte$

Componente x

$\Rightarrow \nabla h = 0 \quad , \quad \nabla p \neq 0$

$$-2\Omega^2 x = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{\cancel{2}}\cancel{2}\Omega^2 x \Rightarrow \frac{\partial p}{\partial x} = \rho\,\Omega^2 x$$

$$p = \frac{\rho\Omega^2 x^2}{2} + C_1(y,z)$$

Componente y

$$-2\Omega^2 y = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{1}{2}2\Omega^2 y \Rightarrow p = \frac{\rho\Omega^2 y^2}{2} + C_2(x,z)$$

$$p = \frac{\rho \Omega^2}{2}(x^2 + y^2) + G_z = \frac{\rho \Omega^2}{2} r^2 + p_0$$

$$\rightarrow h = \frac{\Omega^2 r^2}{2g} \Rightarrow \Omega^2 r^2 = 2gh$$

$$p = \frac{\rho}{2} 2gh + p_0 = \boxed{\rho g h + p_0}$$