Ciências ULisboa

Potential Flow

Margarida Telo da Gama

Rodrigo Coelho

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Overview

- Refs.: chap. 4 of Acheson, chap, 10 of Çengel, Faber.
- For irrotational flow, $\nabla \times \vec{V} = 0$, which implies that $\vec{V} = \pm \nabla\phi$.
- ϕ is a scalar field called the potential flow function.

- If the fluid is incompressible, then the continuity equation implies that $\nabla \cdot \vec{V} = 0$.
- In this case the $\underbrace{\text{potential flow}}_{\text{escoamento potencial}}$ function satisfies the Laplace equation

$$\nabla \cdot \vec{V} = \nabla \cdot \nabla\phi = \boxed{\nabla^2\phi = 0}$$

- We can obtain many velocity fields using the techniques used to solve Laplace's equation.

Flow potential

Consider $$\mathrm{d}\phi = u_1 \mathrm{d}x_1 + u_2 \mathrm{d}x_2 + u_3 \mathrm{d}x_3.$$

ϕ is a single valued function if $$\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \frac{\partial^2 \phi}{\partial x_2 \partial x_1},$$ and two similar equations by exchanging 1 or 2 by 3.

which is equivalent to $$\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = (\boldsymbol{\nabla} \wedge \boldsymbol{u})_3 = \Omega_3 = 0,$$ and similar equations for 1 and 2.

that is, the flow is irrotational (zero vorticity).

For irrotational flow, the velocity field is the gradient of a scalar flow potential ϕ:

$$\boldsymbol{u}\{\boldsymbol{x}, t\} = \boldsymbol{\nabla}\phi\{\boldsymbol{x}, t\},$$

Velocity field

Given the flow potential, the velocity field is obtained from its gradient:

Cartesian coordinates,

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad w = \frac{\partial \phi}{\partial z}$$

and in cylindrical coordinates,

$$u_r = \frac{\partial \phi}{\partial r} \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad u_z = \frac{\partial \phi}{\partial z}$$

Cartesian Coordinates (x, y, z)

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = \nabla \phi$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Cylindrical Coordinates (r, θ, z)

$$r^2 = x^2 + y^2, \ \theta = \tan^{-1}\left({}^{y}\!/\!_{x} \right)$$

$$\vec{V} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z = \frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{e}_\theta + \frac{\partial \phi}{\partial z}\hat{e}_z = \nabla \phi$$

$$\nabla^2 \phi = \underbrace{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}}_{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r} \right)} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

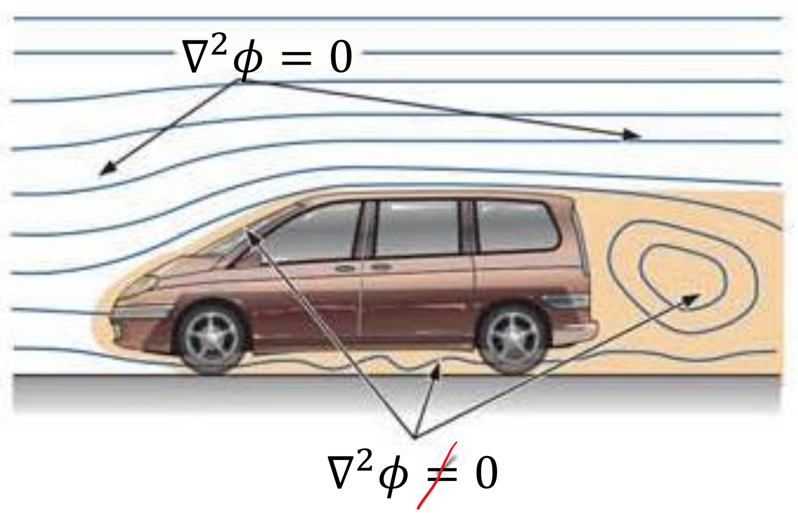
Spherical Coordinates (r, θ, φ)

$$r^2 = x^2 + y^2 + z^2, \ \theta = \cos^{-1}\left({}^{x}\!/\!_{r} \right), \text{ or } x = r\cos\theta, \varphi = \tan^{-1}\left({}^{z}\!/\!_{y} \right)$$

$$\vec{V} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_\varphi \hat{e}_\varphi = \frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \varphi}\hat{e}_\varphi = \nabla \phi$$

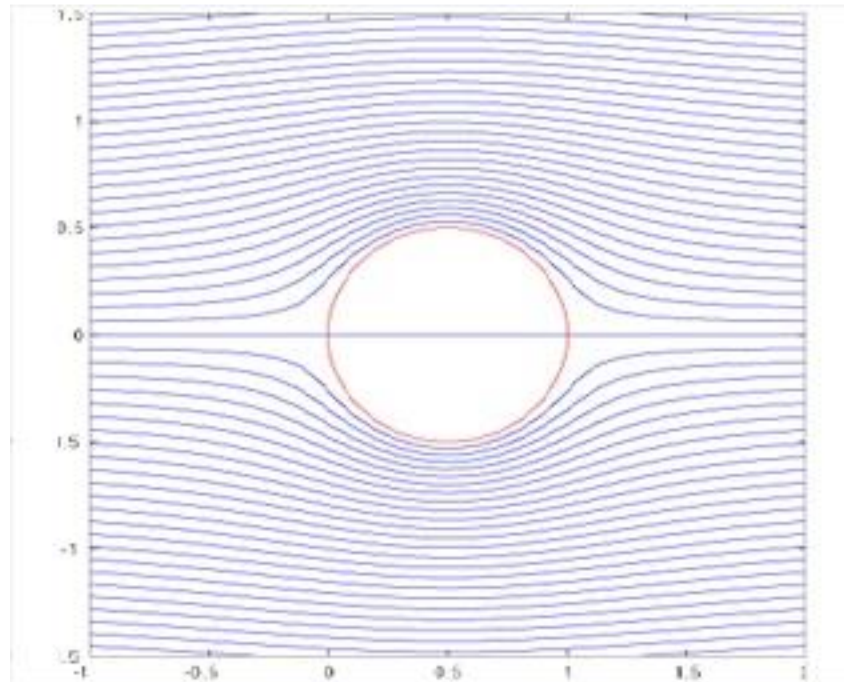
$$\nabla^2 \phi = \underbrace{\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r}}_{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r} \right)} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0$$

Example (schematic)

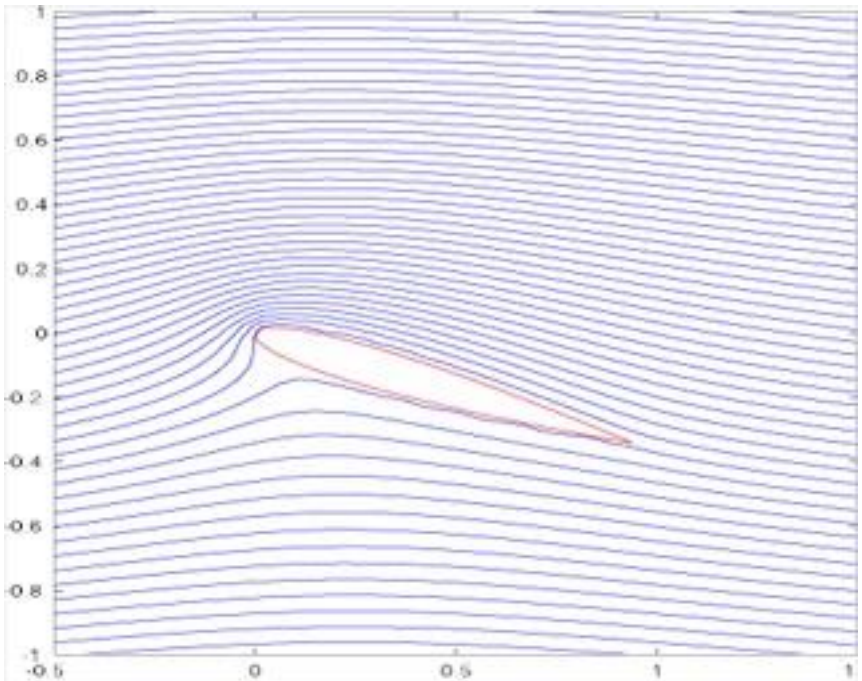


Examples (solutions of Laplace’s equation)

Cylinder in free stream



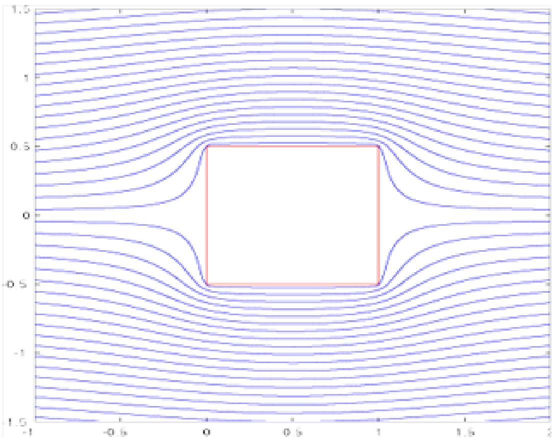
Airfoil in free stream

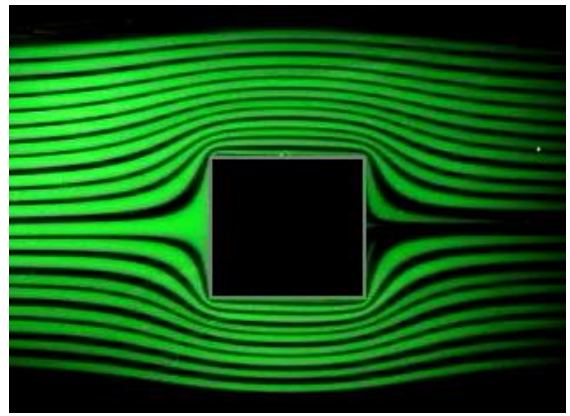


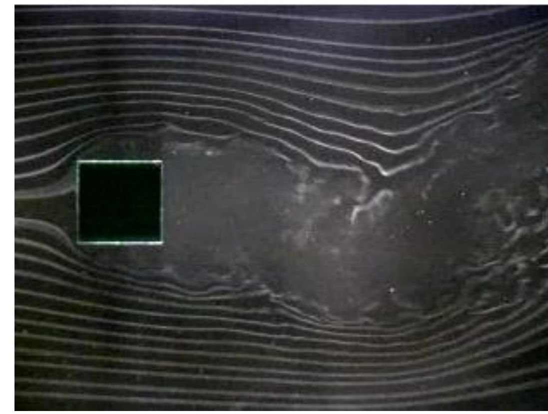
Examples

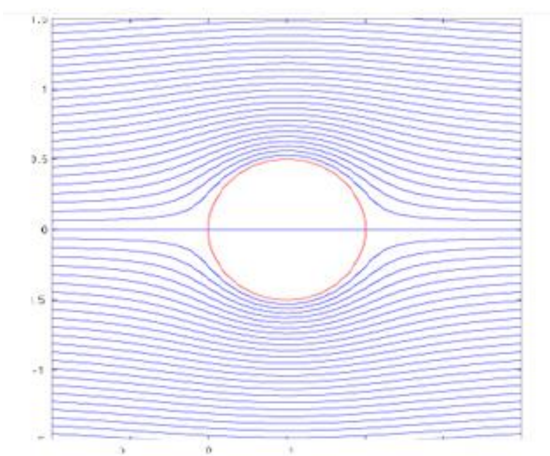
$Re < 1$

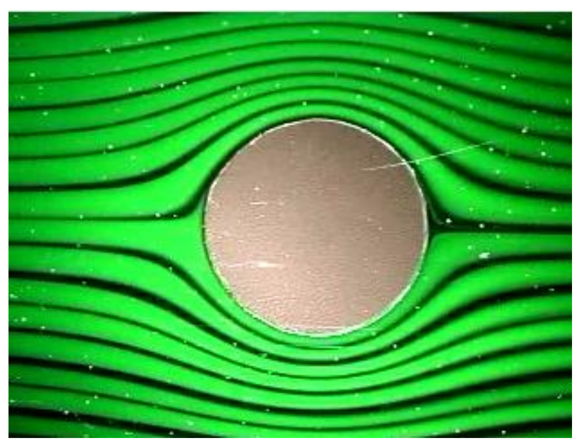
Re=10000

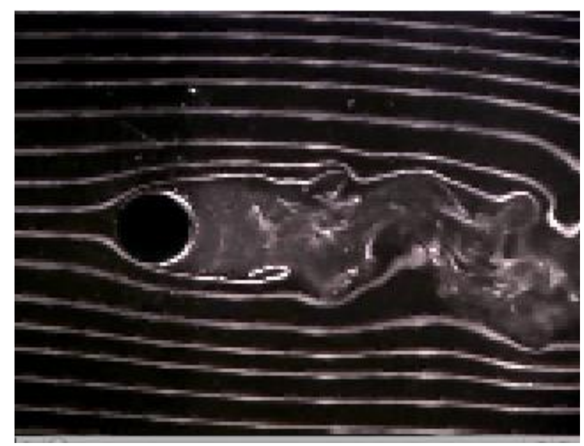












Ex.:

$$u = \alpha x, \qquad v = -\alpha y, \qquad w = 0$$

$$\vec{\omega} = \nabla \times \vec{V} = \vec{0}$$

$$u_x = \frac{\partial \phi}{\partial x} = \alpha x \Rightarrow \phi = \frac{\alpha x^2}{2} + C_1(y)$$

$$u_y = \frac{\partial \phi}{\partial y} = -\alpha y \Rightarrow \phi = -\frac{\alpha y^2}{2} + C_2(x)$$

$$\boxed{\phi = \frac{\alpha}{2}(x^2 - y^2) + C}$$

Back to Laplace's equation

For irrotational regions of flow: $\nabla^2\phi = 0$

In cartesian coordinates

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

In cylindrical coordinates

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Spherical and mixed coordinates may also be useful.

- The beauty of this is that we have combined three unknown velocity components (e.g., u, v, and w) into one unknown scalar field ϕ, eliminating two of the equations required for a solution.
- Once we obtain a solution, we can calculate all three components of the velocity field.

- The Laplace equation is well known since it shows up in several fields of physics, applied mathematics, and engineering. Various solution techniques, both analytical and numerical, are available in the literature.
- Solutions of the Laplace equation are dominated by the geometry (i.e., boundary conditions).

- The solution is valid for any incompressible fluid, regardless of its density or its viscosity, in regions of the flow in which the irrotational approximation is appropriate

Pressure

Of course we still need a dynamical equation to calculate the pressure field. This will be given by the Euler equation.

If gravity is the only body force, then

For irrotational regions of flow: $$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g}$$

Or in its integrated form, the Bernoulli equation

Steady, incompressible flow: $$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Since the flow is irrotational, we can apply Bernoulli to ANY two points in the flow domain.

Stream function

- For irrotational flows in 2D, the stream function obeys the Laplace equation:

$$\nabla^2 \psi = 0.$$

- In potential 2D flow, both the flow potential and the stream function are solutions of the Laplace equation.
- Lines of constant flow potential are perpendicular to the streamlines (check).
- In axisymmetric flows the stream function obeys a linear equation but that is no longer Laplace's equation.

Stream function

For incompressible 2D flows:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \quad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Important property: ψ is constant along a streamline.

$$\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + (\boldsymbol{u} \cdot \nabla)\psi = u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} = 0 \Rightarrow \frac{D\psi}{Dt} = 0$$

$\Rightarrow \psi = C_{te}$

em uma linha de corrente

Generic coordinate system (only in 2D)

$$\boldsymbol{u} = \nabla \wedge (\psi \boldsymbol{k})$$

Complex potential

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

The complex potential is also a solution of the Laplace equation

$$w = \phi + \mathrm{i}\psi$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$