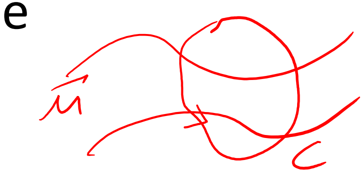
Kelvin's circulation theorem

- An ideal fluid that is vorticity free at a given instant is vorticity free at all times.
  $\vec{\omega} = 0$
- Demonstration: see Faber 120-122
- In three dimensions the conservation of vorticity (which corresponds to the conservation of angular momentum in mechanics) takes a somewhat subtle form.
- The circulation of a velocity field is defined to be

$$K\{t\} = \oint \boldsymbol{u}\{\boldsymbol{x}, t\} \cdot \mathrm{d}\boldsymbol{l},$$



where the line is a closed loop which moves with the fluid.

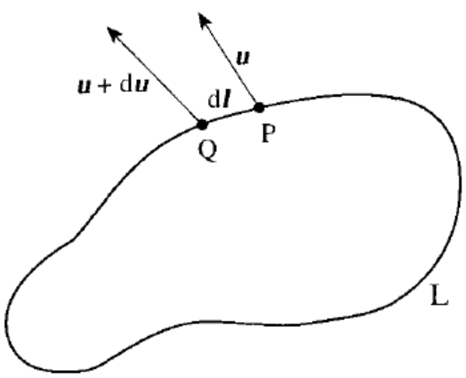
Circulation and vorticity

- By Stokes' theorem

$$K = \oint_{C(t)} \mathbf{u} \cdot d\mathbf{l} = \int_{S(t)} (\underbrace{\boldsymbol{\nabla} \times \mathbf{u}}_{\omega}) \cdot \mathbf{n} dS = \int_{S(t)} \Omega \cdot \mathbf{n} dS,$$

where S(t) is a surface whose edges connect with C(t).

K is zero for all loops if Ω is zero in the domain!



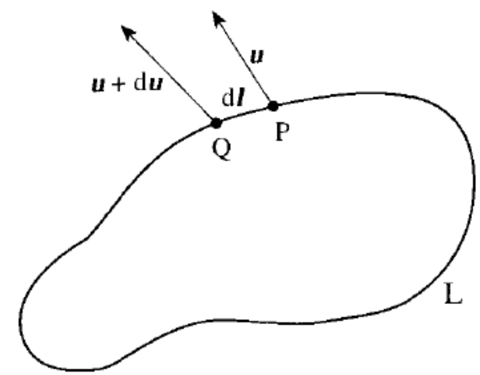
Kelvin´s theorem asserts that

$$\frac{DK}{Dt} = 0.$$

Demonstration

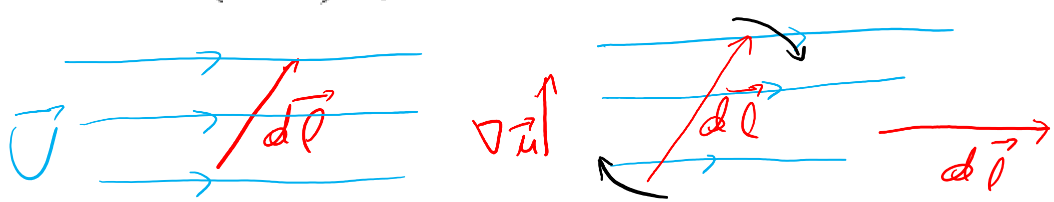
The loop moves with the flow and thus

$$\frac{DK}{Dt} = \oint_C \frac{D\vec{u}}{Dt}\cdot d\vec{l} + \vec{u}\cdot\frac{D(d\vec{l})}{Dt}$$



The second term is the relative velocity of two nearby points on the loop and can be written as $(\partial \boldsymbol{u}/\partial l)\mathrm{d}\boldsymbol{l}$.

$$\frac{D(d\vec{l})}{Dt} = d\vec{l}\cdot\nabla\vec{u}$$



Hence

$$\oint_C \vec{u}\cdot\frac{D(d\vec{l})}{Dt} = \oint_C \vec{u}\cdot\left[(d\vec{l}\cdot\nabla)\vec{u}\right] = \frac{1}{2}\oint_C \nabla(\vec{u}^2)\cdot d\vec{l}$$

$$\overset{\text{Stokes}}{=} \frac{1}{2}\int_S (\underbrace{\nabla\times\nabla\vec{u}^2}_{=0})\cdot d\vec{S} = 0$$

If the fluid is incompressible, using Euler:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}\left(\frac{p}{\rho} + gz\right),$$

$$\oint_C \frac{D\vec{u}}{Dt} \cdot d\vec{l} = -\oint_C \nabla\left(\frac{p}{\rho} + gz\right) \cdot d\vec{l}$$

$$\overset{\text{stokes}}{=} - \iint_S \Big[\underbrace{\nabla \times \nabla}_{=0}\left(\frac{p}{\rho} + gz\right)\Big] \cdot d\vec{S} = 0$$

Therefore:

$$\boxed{\frac{\mathrm{D}K}{\mathrm{D}t} = 0}$$

— fluido ideal
($\nu = 0$, $\rho = cte$)

— $\vec{\omega} = 0$

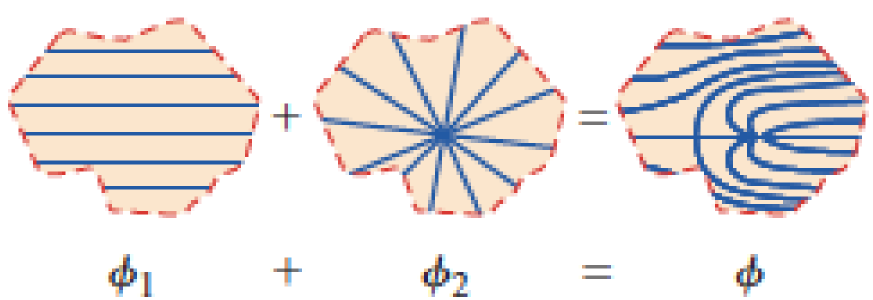
Superposition

$\vec{V} = \nabla \phi$

- Since the Laplace equation is a linear homogeneous differential equation, the linear combination of two or more solutions of the equation must also be a solution.
- For example, if ϕ_1 and ϕ_2 are each solutions of the Laplace equation, then $A\,\phi_1 + B\,\phi_2$ are also solutions, where A and B are arbitrary constants.

$\nabla^2 \phi_T = \nabla^2 (A\phi_1 + B\phi_2) = A\nabla^2\phi_1 + B\nabla^2\phi_2$

- By extension, you may combine several solutions of the Laplace equation, and the combination is guaranteed to also be a solution.



Uniform (free) stream

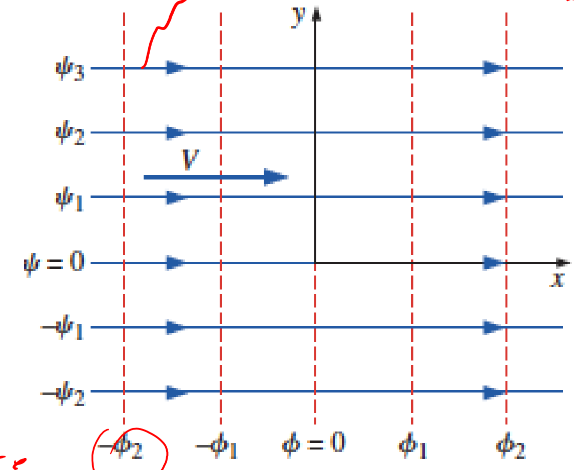
Pot. de escoamento

função corrente

Uniform stream: $u = \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y} = V \qquad v = \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x} = 0$

$\phi = Vx + f(y) \quad \rightarrow \quad v = \dfrac{\partial \phi}{\partial y} = f'(y) = 0 \quad \rightarrow \quad f(y) = \text{constant}$

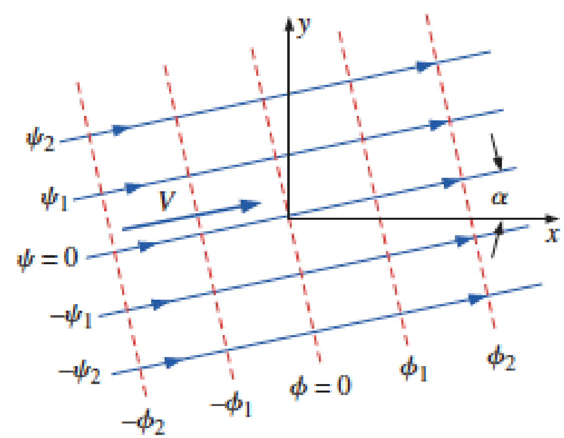
linha de corrente $\psi = C_{te}$



Velocity potential function for a uniform stream: $\phi = Vx = C_{te} \Rightarrow x = \dfrac{C_{te}}{V}$

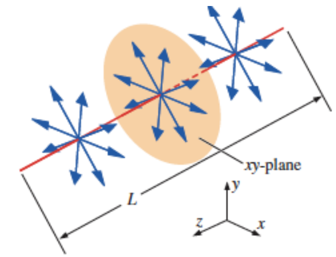
Stream function for a uniform stream: $\psi = Vy = C_2 \Rightarrow y = \dfrac{C_2}{V}$

Uniform stream: $\phi = Vr\cos\theta \qquad \psi = Vr\sin\theta$



Uniform stream inclined at angle α: $\begin{cases} \phi = V(x\cos\alpha + y\sin\alpha) \\ \psi = V(y\cos\alpha - x\sin\alpha) \end{cases}$

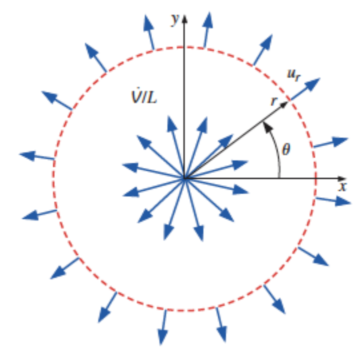
Line source or sink



Let the volume flow rate per unit depth, be the line source strength, m

$$\frac{\dot V}{L} = 2\pi r u_r \qquad u_r = \frac{\dot V/L}{2\pi r}$$

The components of the velocity are

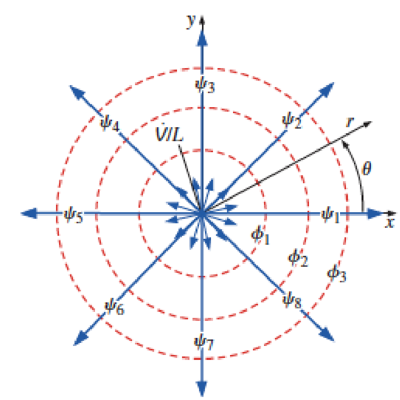


Line source: $$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\dot V/L}{2\pi r} \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0$$

$$\frac{\partial \psi}{\partial r} = -u_\theta = 0 \quad \rightarrow \quad \psi = f(\theta) \quad \rightarrow \quad \frac{\partial \psi}{\partial \theta} = f'(\theta) = r u_r = \frac{\dot V/L}{2\pi}$$

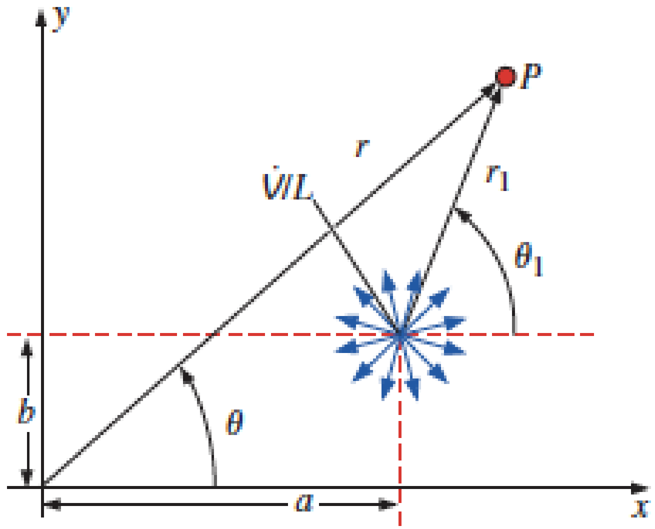
With solution

$$f(\theta) = \frac{\dot V/L}{2\pi}\theta + \text{constant}$$



Line source at the origin: $$\boxed{\phi = \frac{\dot V/L}{2\pi}\ln r \quad \text{and} \quad \psi = \frac{\dot V/L}{2\pi}\theta}$$

Line source or sink at an arbitrary point



$r_1 = \sqrt{(x-a)^2 + (y-a)^2}$

Line source at point (a, b):

$$\phi = \frac{\dot{V}/L}{2\pi} \ln r_1 = \frac{\dot{V}/L}{2\pi} \ln \sqrt{(x-a)^2 + (y-b)^2}$$

$$\psi = \frac{\dot{V}/L}{2\pi} \theta_1 = \frac{\dot{V}/L}{2\pi} \arctan \frac{y-b}{x-a}$$

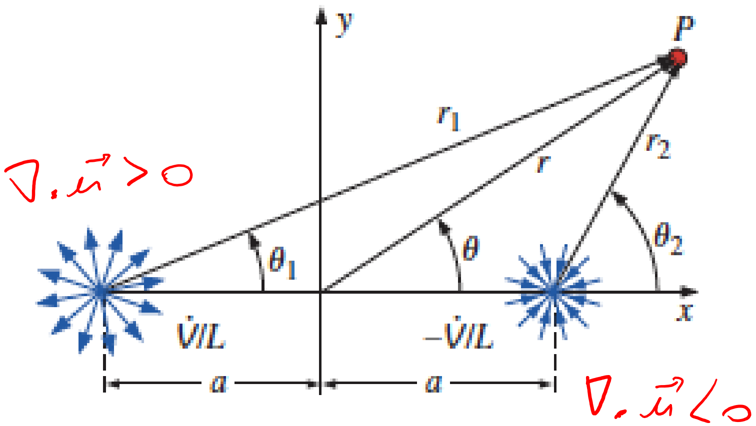
Superposition of a source and sink of equal strength

Line source at $(-a, 0)$: $\psi_1 = \frac{\dot{V}/L}{2\pi}\theta_1$ where $\theta_1 = \arctan\frac{y}{x+a}$

Similarly for the sink,

Line sink at $(a, 0)$: $\psi_2 = \frac{-\dot{V}/L}{2\pi}\theta_2$ where $\theta_2 = \arctan\frac{y}{x-a}$

Composite stream function: $\psi = \psi_1 + \psi_2 = \frac{\dot{V}/L}{2\pi}(\theta_1 - \theta_2)$



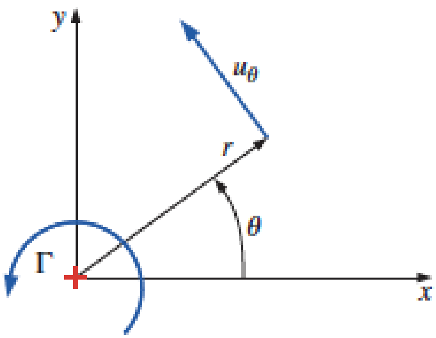
Final result, Cartesian coordinates: $\psi = \frac{-\dot{V}/L}{2\pi}\arctan\frac{2ay}{x^2 + y^2 - a^2}$

Final result, cylindrical coordinates: $\psi = \frac{-\dot{V}/L}{2\pi}\arctan\frac{2ar\sin\theta}{r^2 - a^2}$

Using

$$\arctan(u) \pm \arctan(v) = \arctan\left(\frac{u \pm v}{1 \mp uv}\right) \pmod{\pi}, \quad uv \neq 1.$$

Line vortex

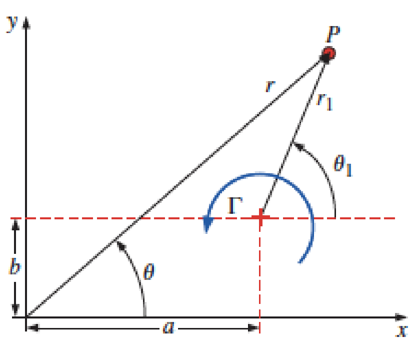


The radial component of the velocity is zero and

Line vortex: $$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

where $\Gamma = 2\pi r u_\theta$, is the circulation, around a loop of radius r.

Then,



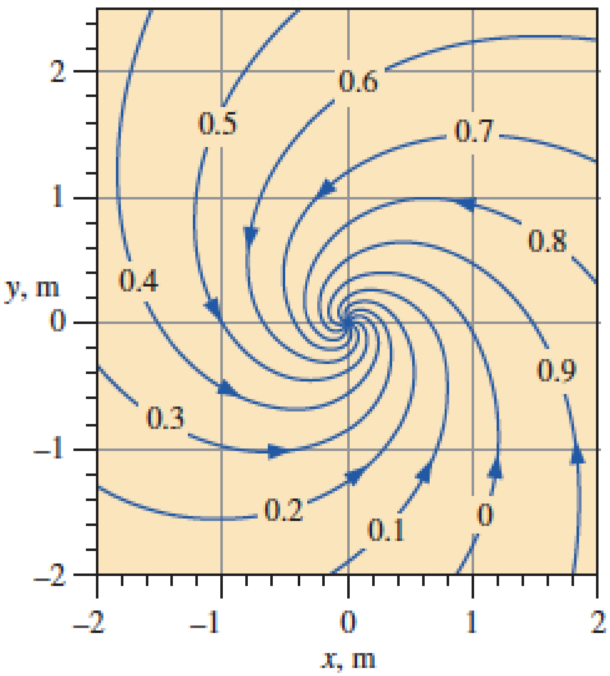
Line vortex at the origin: $$\phi = \frac{\Gamma}{2\pi}\theta \qquad \psi = -\frac{\Gamma}{2\pi}\ln r$$

Line vortex at point (a, b):
$$\phi = \frac{\Gamma}{2\pi}\theta_1 = \frac{\Gamma}{2\pi}\arctan\frac{y-b}{x-a}$$
$$\psi = -\frac{\Gamma}{2\pi}\ln r_1 = -\frac{\Gamma}{2\pi}\ln\sqrt{(x-a)^2+(y-b)^2}$$

Superposition of a line sink and a line vortex at the origin

The stream function is

Superposition: $$\psi = \underbrace{\frac{\dot{V}/L}{2\pi}\theta}_{\text{fonte}} - \underbrace{\frac{\Gamma}{2\pi}\ln r}_{\text{vórtice}}$$



with streamlines

Streamlines: $$r = \exp\left(\frac{(\dot{V}/L)\theta - 2\pi\psi}{\Gamma}\right)$$

Velocity components: $$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\dot{V}/L}{2\pi r} \qquad u_\theta = -\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

Note that velocity diverges at the origin, which is a singularity (unphysical).