Example 4.3 Determine the current response of the series *RLC* circuit, Fig. 4-4(*a*), when a voltage *V* is suddenly applied by closing the switch at $t = 0$. Initial current and initial charge on *C* are zero.

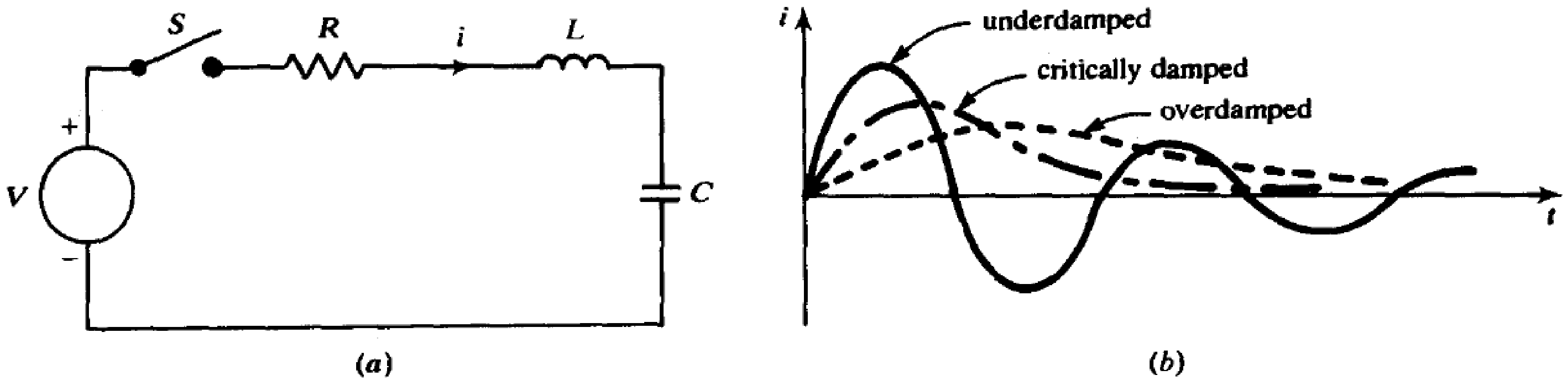


Fig. 4-4

The voltage equation is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = V$$

or, differentiating with respect to time,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0 \qquad (4.15)$$

$$i = i_n = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{4.16}$$

where s_1 and s_2 are the *characteristic roots* of (*4.15*):

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \equiv -\alpha + \beta$$
$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \equiv -\alpha - \beta \tag{4.17}$$

and where the constants of integration, A_1 and A_2, may be determined from the initial conditions

$$i(0) = 0 \qquad L\frac{di}{dt}\bigg|_0 = V$$

(See Problems 4.7 and 4.8.)

Because α is always a positive real number, the transient current (*4.16*) eventually decays in magnitude like an exponential function. The more exact features of this decay depend on the circuit parameters R, L, and C as they enter in the constant β. We define $\omega_0 \equiv 1/\sqrt{LC}$, the *resonant frequency* (in rad/s) of the circuit, so that

$$\beta = \sqrt{\alpha^2 - \omega_0^2}$$

$$\beta = \sqrt{\alpha^2 - \omega_0^2}$$

***Case 1*:** $\alpha > \omega_0$.

Here, β is real and positive, and $\beta < \alpha$. The solution takes the form

$$i = A_1 e^{-(\alpha-\beta)t} + A_2 e^{-(\alpha+\beta)t} \tag{4.18}$$

i.e. the sum of two decaying exponentials. In this case the circuit is said to be *overdamped.*

***Case 2*:** $\alpha = \omega_0$.

It can be shown that as $\beta \to 0$, (*4.16*) goes over into

$$i = (A_1 + A_2 t)e^{-\alpha t} \tag{4.19}$$

The circuit is said to be *critically damped.*

***Case 3*:** $\alpha < \omega_0$.

Now β is a pure imaginary, $\beta = j|\beta|$, and (*4.16*) becomes

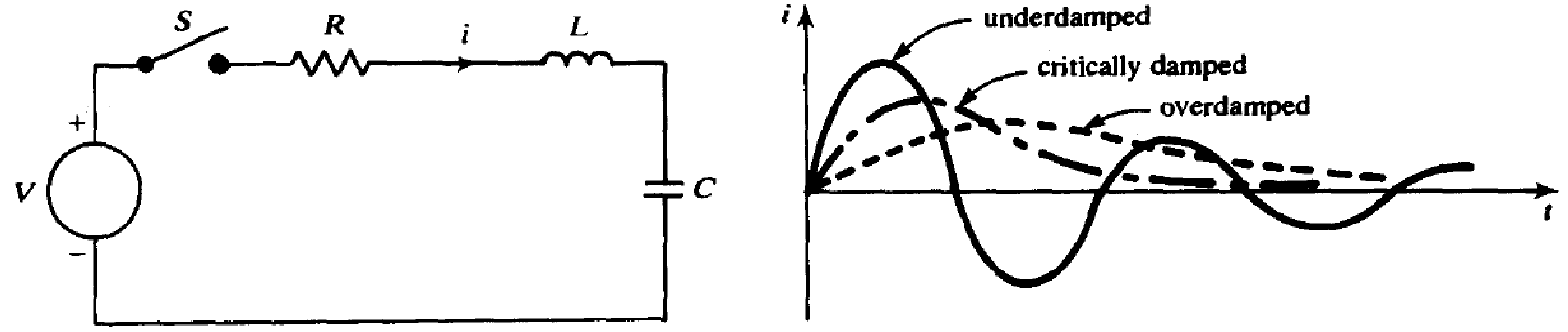
$$i = e^{-\alpha t}(A_1 e^{j|\beta|t} + A_2 e^{-j|\beta|t}) \tag{4.20}$$

or, equivalently (see Problem 4.21),

$$i = Ae^{-\alpha t} \sin(|\beta| t + \psi) \tag{4.21}$$

As given by (*4.21*), the response is a damped sine wave, of frequency $|\beta|$ (rad/s); the circuit is *underdamped.*

Figure 4-4(*b*) illustrates the three kinds of damping. In the critically damped and underdamped cases, the response goes to zero essentially as $e^{-\alpha t}$, and α is called the *damping coefficient.*



(*a*) (*b*)

Análise de circuitos com corrente alternada

9.2 SINUSOIDS

Consider the sinusoidal voltage

$$v(t) = V_m \sin \omega t \tag{9.1}$$

where

V_m = the *amplitude* of the sinusoid

ω = the *angular frequency* in radians/s

ωt = the *argument* of the sinusoid

The sinusoid is shown in Fig. 9.1(a) as a function of its argument and in Fig. 9.1(b) as a function of time. It is evident that the sinusoid repeats itself every T seconds; thus, T is called the *period* of the sinusoid. From the two plots in Fig. 9.1, we observe that $\omega T = 2\pi$,

$$\boxed{T = \frac{2\pi}{\omega}} \tag{9.2}$$

The fact that $v(t)$ repeats itself every T seconds is shown by replacing t by $t + T$ in Eq. (9.1). We get

$$\begin{aligned} v(t+T) &= V_m \sin \omega(t+T) = V_m \sin \omega\left(t + \frac{2\pi}{\omega}\right) \\ &= V_m \sin(\omega t + 2\pi) = V_m \sin \omega t = v(t) \end{aligned} \tag{9.3}$$

Hence,

$$\boxed{v(t+T) = v(t)} \tag{9.4}$$

that is, v has the same value at $t + T$ as it does at t and $v(t)$ is said to be *periodic*. In general,

$$f = \frac{1}{T} \tag{9.5}$$

From Eqs. (9.2) and (9.5), it is clear that

$$\omega = 2\pi f \tag{9.6}$$

While ω is in radians per second (rad/s), f is in hertz (Hz).

Let us now consider a more general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi) \tag{9.7}$$

where $(\omega t + \phi)$ is the argument and ϕ is the *phase*. Both argument and phase can be in radians or degrees.

Let us examine the two sinusoids

$$v_1(t) = V_m \sin \omega t \qquad \text{and} \qquad v_2(t) = V_m \sin(\omega t + \phi) \tag{9.8}$$

shown in Fig. 9.2. The starting point of v_2 in Fig. 9.2 occurs first in time. Therefore, we say that v_2 *leads* v_1 by ϕ or that v_1 *lags* v_2 by ϕ. If $\phi \neq 0$, we also say that v_1 and v_2 are *out of phase*. If $\phi = 0$, then v_1 and v_2 are said to be *in phase*; they reach their minima and maxima at exactly the same time. We can compare v_1 and v_2 in this manner because they operate at the same frequency; they do not need to have the same amplitude.

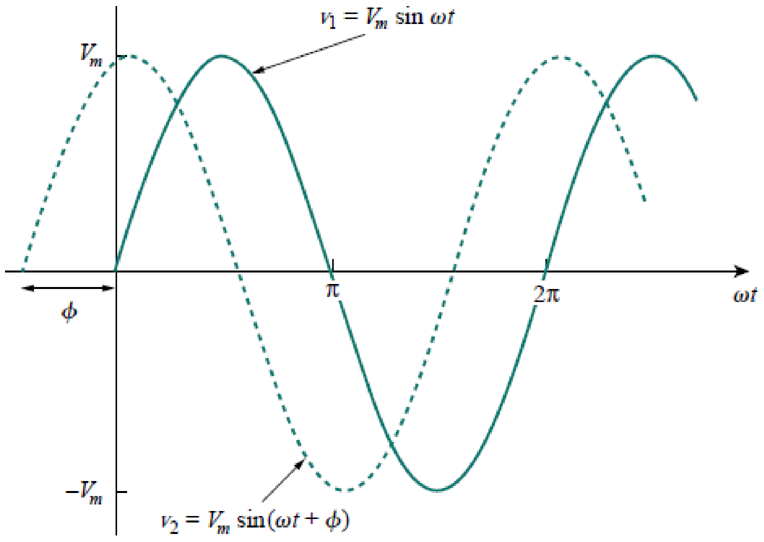


Figure 9.2 Two sinusoids with different phases.

A sinusoid can be expressed in either sine or cosine form. When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes. This is achieved by using the following trigonometric identities:

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned} \tag{9.9}$$

With these identities, it is easy to show that

$$\begin{aligned} \sin(\omega t \pm 180^\circ) &= -\sin \omega t \\ \cos(\omega t \pm 180^\circ) &= -\cos \omega t \\ \sin(\omega t \pm 90^\circ) &= \pm \cos \omega t \\ \cos(\omega t \pm 90^\circ) &= \mp \sin \omega t \end{aligned} \tag{9.10}$$

Using these relationships, we can transform a sinusoid from sine form to cosine form or vice versa.

A graphical approach may be used to relate or compare sinusoids as an alternative to using the trigonometric identities in Eqs. (9.9) and (9.10). Consider the set of axes shown in Fig. 9.3(a). The horizontal axis represents the magnitude of cosine, while the vertical axis (pointing down) denotes the magnitude of sine. Angles are measured positively counterclockwise from the horizontal, as usual in polar coordinates. This graphical technique can be used to relate two sinusoids. For example, we see in Fig. 9.3(a) that subtracting 90° from the argument of $\cos \omega t$ gives $\sin \omega t$, or $\cos(\omega t - 90^\circ) = \sin \omega t$. Similarly, adding 180° to the argument of $\sin \omega t$ gives $-\sin \omega t$, or $\sin(\omega t - 180^\circ) = -\sin \omega t$, as shown in Fig. 9.3(b).

The graphical technique can also be used to add two sinusoids of the same frequency when one is in sine form and the other is in cosine form. To add $A \cos \omega t$ and $B \sin \omega t$, we note that A is the magnitude of $\cos \omega t$ while B is the magnitude of $\sin \omega t$, as shown in Fig. 9.4(a). The magnitude and argument of the resultant sinusoid in cosine form is readily obtained from the triangle. Thus,

$$A \cos \omega t + B \sin \omega t = C \cos(\omega t - \theta) \tag{9.11}$$

where

$$C = \sqrt{A^2 + B^2}, \qquad \theta = \tan^{-1} \frac{B}{A} \tag{9.12}$$

For example, we may add $3 \cos \omega t$ and $-4 \sin \omega t$ as shown in Fig. 9.4(b) and obtain

$$3 \cos \omega t - 4 \sin \omega t = 5 \cos(\omega t + 53.1^\circ) \tag{9.13}$$

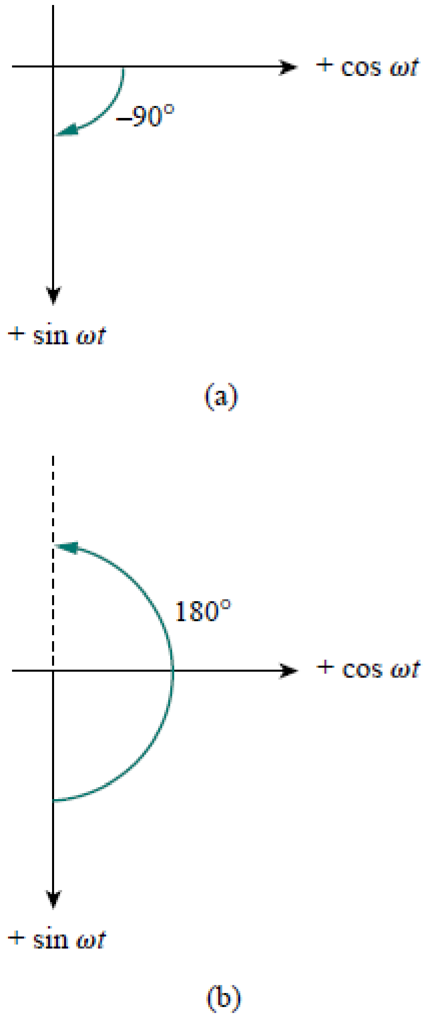


Figure 9.3 A graphical means of relating cosine and sine: (a) $\cos(\omega t - 90^\circ) = \sin \omega t$, (b) $\sin(\omega t + 180^\circ) = -\sin \omega t$.

For example, we may add $3\cos\omega t$ and $-4\sin\omega t$ as shown in Fig. 9.4(b) and obtain

$$3\cos\omega t - 4\sin\omega t = 5\cos(\omega t + 53.1^\circ) \tag{9.13}$$

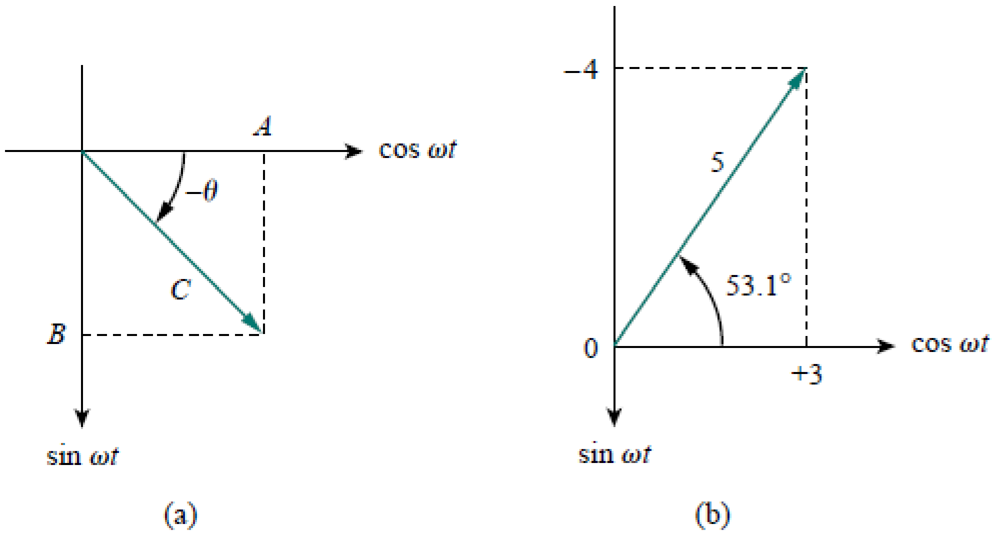


Figure 9.4 (a) Adding $A\cos\omega t$ and $B\sin\omega t$, (b) adding $3\cos\omega t$ and $-4\sin\omega t$.

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\cos(50t + 10^\circ)$$

Solution:

The amplitude is $V_m = 12$ V.

The phase is $\phi = 10^\circ$.

The angular frequency is $\omega = 50$ rad/s.

The period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{50} = 0.1257$ s.

The frequency is $f = \dfrac{1}{T} = 7.958$ Hz.

9.3 PHASORS

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions.

> A **phasor** is a complex number that represents the amplitude and phase of a sinusoid.

Phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources; solutions of such circuits would be intractable otherwise. The notion of solving ac circuits using phasors was first introduced by Charles Steinmetz in 1893. Before we completely define phasors and apply them to circuit analysis, we need to be thoroughly familiar with complex numbers.

A complex number z can be written in rectangular form as

$$z = x + jy \tag{9.14a}$$

where $j = \sqrt{-1}$; x is the real part of z; y is the imaginary part of z. In this context, the variables x and y do not represent a location as in two-dimensional vector analysis but rather the real and imaginary parts of z in the complex plane. Nevertheless, we note that there are some

$$z = r\angle\phi = re^{j\phi} \tag{9.14b}$$

where r is the magnitude of z, and ϕ is the phase of z. We notice that z can be represented in three ways:

$$\begin{aligned} z &= x + jy && \text{Rectangular form} \\ z &= r\angle\phi && \text{Polar form} \\ z &= re^{j\phi} && \text{Exponential form} \end{aligned} \tag{9.15}$$

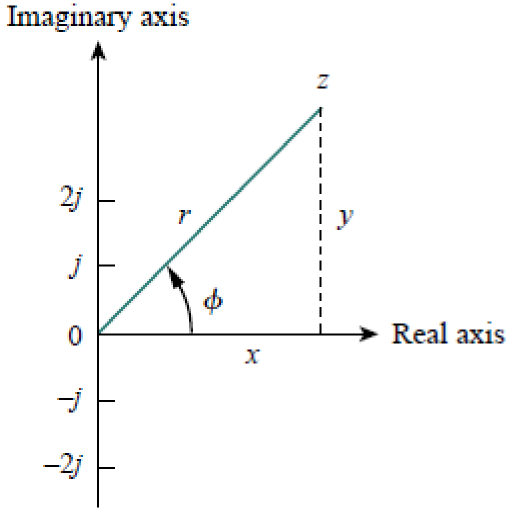


Figure 9.6 Representation of a complex number $z = x + jy = r\angle\phi$.

The relationship between the rectangular form and the polar form is shown in Fig. 9.6, where the x axis represents the real part and the y axis represents the imaginary part of a complex number. Given x and y, we can get r and ϕ as

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x} \tag{9.16a}$$

On the other hand, if we know r and ϕ, we can obtain x and y as

$$x = r\cos\phi, \qquad y = r\sin\phi \tag{9.16b}$$

Thus, z may be written as

$$\boxed{z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)} \tag{9.17}$$

Addition and subtraction of complex numbers are better performed in rectangular form; multiplication and division are better done in polar form. Given the complex numbers

$$z = x + jy = r\underline{/\phi}, \qquad z_1 = x_1 + jy_1 = r_1\underline{/\phi_1}$$
$$z_2 = x_2 + jy_2 = r_2\underline{/\phi_2}$$

the following operations are important.

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{9.18a}$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{9.18b}$$

Multiplication:

$$z_1 z_2 = r_1 r_2\underline{/\phi_1 + \phi_2} \tag{9.18c}$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\underline{/\phi_1 - \phi_2} \tag{9.18d}$$

Reciprocal:

$$\frac{1}{z} = \frac{1}{r}\underline{/-\phi} \tag{9.18e}$$

Square Root:

$$\sqrt{z} = \sqrt{r}\underline{/\phi/2} \tag{9.18f}$$

Complex Conjugate:

$$z^* = x - jy = r\angle{-\phi} = re^{-j\phi} \tag{9.18g}$$

Note that from Eq. (9.18e),

$$\frac{1}{j} = -j \tag{9.18h}$$

The idea of phasor representation is based on Euler's identity. In general,

$$\boxed{e^{\pm j\phi} = \cos\phi \pm j\sin\phi} \tag{9.19}$$

which shows that we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$; we may write

$$\cos\phi = \text{Re}(e^{j\phi}) \tag{9.20a}$$

$$\sin\phi = \text{Im}(e^{j\phi}) \tag{9.20b}$$

where Re and Im stand for the *real part of* and the *imaginary part of*. Given a sinusoid $v(t) = V_m\cos(\omega t + \phi)$, we use Eq. (9.20a) to express $v(t)$ as

$$v(t) = V_m\cos(\omega t + \phi) = \text{Re}(V_m e^{j(\omega t+\phi)}) \tag{9.21}$$

or

$$v(t) = \text{Re}(V_m e^{j\phi} e^{j\omega t}) \tag{9.22}$$

Thus,

$$\boxed{v(t) = \text{Re}(\mathbf{V}e^{j\omega t})} \tag{9.23}$$

where

$$\mathbf{V} = V_m e^{j\phi} = V_m \underline{/\phi} \tag{9.24}$$

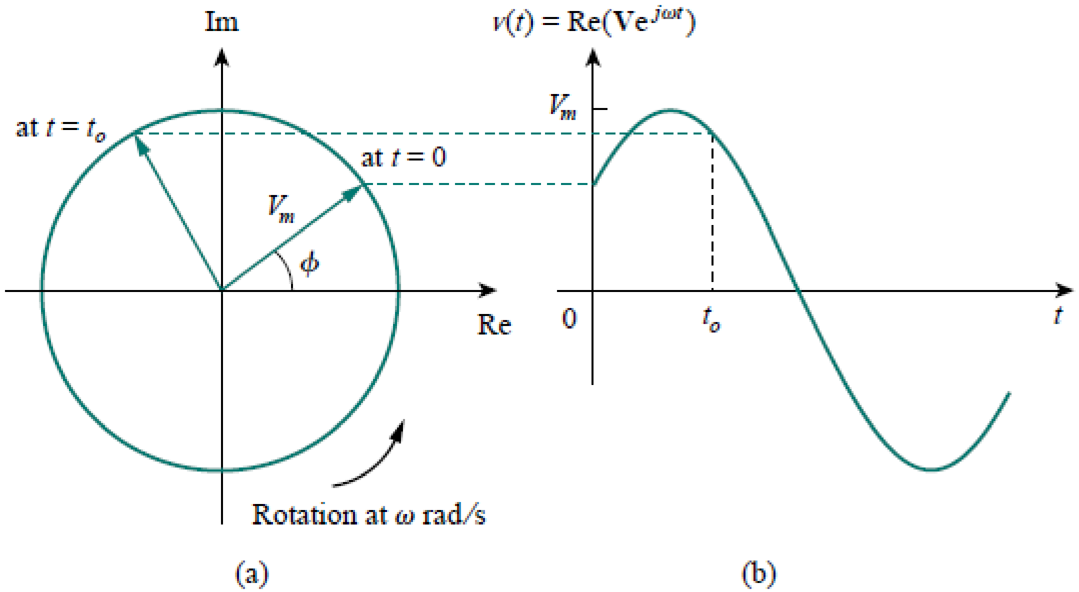


Figure 9.7 Representation of $\mathbf{V}e^{j\omega t}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

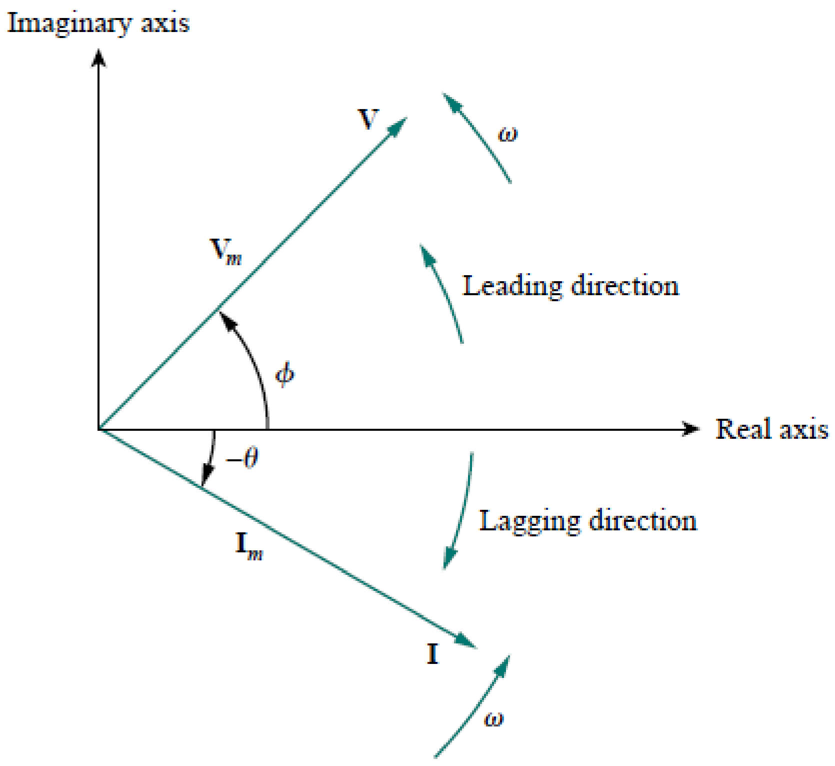


Figure 9.8 A phasor diagram showing $\mathbf{V} = V_m \angle \phi$ and $\mathbf{I} = I_m \angle -\theta$.

$$v(t) = V_m \cos(\omega t + \phi) \quad \Longleftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

TABLE 9.1 Sinusoid-phasor transformation.

Time-domain representation	Phasor-domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$

From Eqs. (9.23) and (9.24), $v(t) = \text{Re}(\mathbf{V}e^{j\omega t}) = V_m \cos(\omega t + \phi)$, so that

$$\begin{aligned}\frac{dv}{dt} &= -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90^\circ) \\ &= \text{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90^\circ}) = \text{Re}(j\omega \mathbf{V} e^{j\omega t})\end{aligned} \tag{9.26}$$

This shows that the derivative $v(t)$ is transformed to the phasor domain as $j\omega\mathbf{V}$

$$\underset{\text{(Time domain)}}{\frac{dv}{dt}} \quad \Longleftrightarrow \quad \underset{\text{(Phasor domain)}}{j\omega\mathbf{V}} \tag{9.27}$$

Similarly, the integral of $v(t)$ is transformed to the phasor domain as $\mathbf{V}/j\omega$

$$\underset{\text{(Time domain)}}{\int v\,dt} \quad \Longleftrightarrow \quad \underset{\text{(Phasor domain)}}{\frac{\mathbf{V}}{j\omega}} \tag{9.28}$$

The differences between $v(t)$ and $\mathbf{V}$ should be emphasized:

1. $v(t)$ is the *instantaneous or time-domain* representation, while $\mathbf{V}$ is the *frequency or phasor-domain* representation.
2. $v(t)$ is time dependent, while $\mathbf{V}$ is not. (This fact is often forgotten by students.)
3. $v(t)$ is always real with no complex term, while $\mathbf{V}$ is generally complex.

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(a) Using polar to rectangular transformation,

$$40\angle 50^\circ = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$$

$$20\angle -30^\circ = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

We begin with the resistor. If the current through a resistor R is $i = I_m \cos(\omega t + \phi)$, the voltage across it is given by Ohm's law as

$$v = iR = RI_m \cos(\omega t + \phi) \tag{9.29}$$

The phasor form of this voltage is

$$\mathbf{V} = RI_m \angle \phi \tag{9.30}$$

But the phasor representation of the current is $\mathbf{I} = I_m \angle \phi$. Hence,

$$\mathbf{V} = R\mathbf{I} \tag{9.31}$$

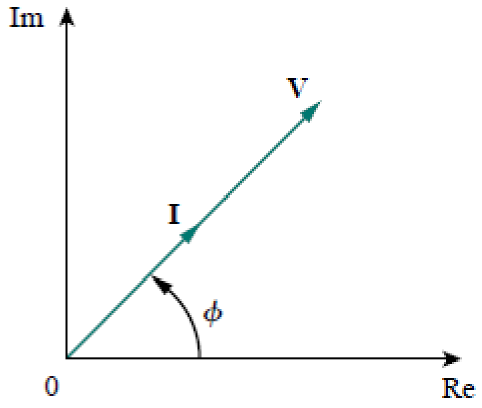


Figure 9.10 Phasor diagram for the resistor.