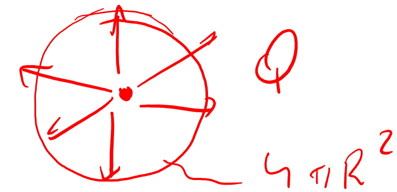
Sources and sinks

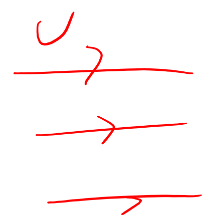
(Faber 4.4)

- The 1/R potential $\boxed{\phi = -\frac{Q}{4\pi R}}$ is a solution of Laplace´s equation in 3D
- It describes isotropic flow with velocity $u_R =$ $Q/4\pi R^2$



- If Q > 0 it is a source and it is a sink otherwise. Q is the discharge rate.

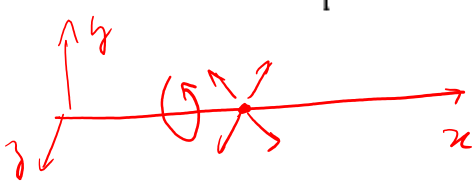
- Free stream potential $\underline{\phi = U x_1.}$



$\vec{u} = U\,\hat{x}$

- Superposition of the two gives

$$u_1 = U + \frac{Q}{4\pi R^2}\cos\theta, \quad (u_2^2 + u_3^2)^{1/2} = \frac{Q}{4\pi R^2}\sin\theta,$$



Sources and sinks

- Or in spherical coordinates,

$$u_R = U\cos\theta + \frac{Q}{4\pi R^2}, \quad u_\theta = -U\sin\theta.$$

$\vec{u} = 0$

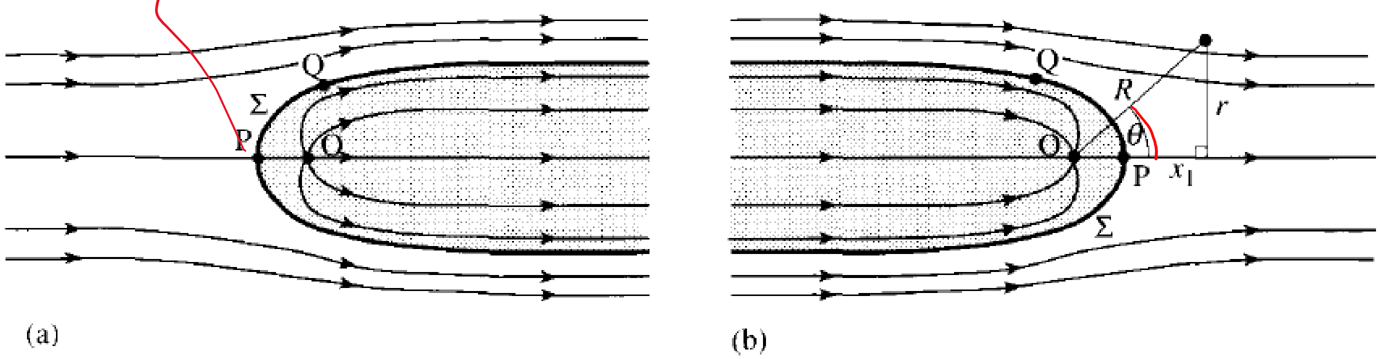
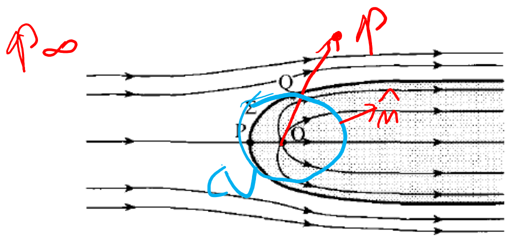


Figure 4.2 Lines of flow past (a) a point source, (b) a point sink. The surface of revolution Σ encloses all the fluid coming from, or destined for, the source or sink respectively.

Excess pressure and force

The excess pressure vanishes at infinity where the velocity is that of the free stream. Then Bernoulli gives for the dynamical pressure:



$$p^* = \frac{1}{2}\rho(U^2 - u_R^2 - u_\theta^2) = \boxed{-\frac{\rho U Q \cos\theta}{4\pi R^2} - \frac{\rho Q^2}{32\pi^2 R^4}}$$

$$H_\infty = H$$

$$\frac{p_\infty}{\rho} + \frac{u_\infty^2}{2} + \cancel{g z_\infty} = \frac{p}{\rho} + \frac{u^2}{2} + \cancel{g z}$$

$$p^* = p - p_\infty = \frac{\rho}{2}(U^2 - u^2) = \frac{\rho}{2}(U^2 - u_R^2 - u_\theta^2)$$

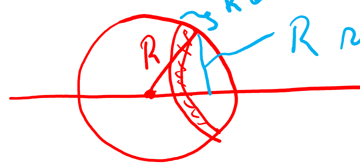
Total force in the direction x, exerted by this excess of pressure on the fluid inside a spherical control surface centered on O, of an arbitrary R.

$$F_x = \frac{1}{2}\rho U Q \int_0^{\pi} \left(\cos^2\theta \sin\theta + \frac{Q \cos\theta \sin\theta}{8\pi R^2 U} \right) d\theta = \boxed{\frac{1}{3}\rho U Q}$$

$F_j = \sigma_{ij} n_j \, dA$; $p = \dfrac{F}{A}$ $\quad dF = p \, dA$

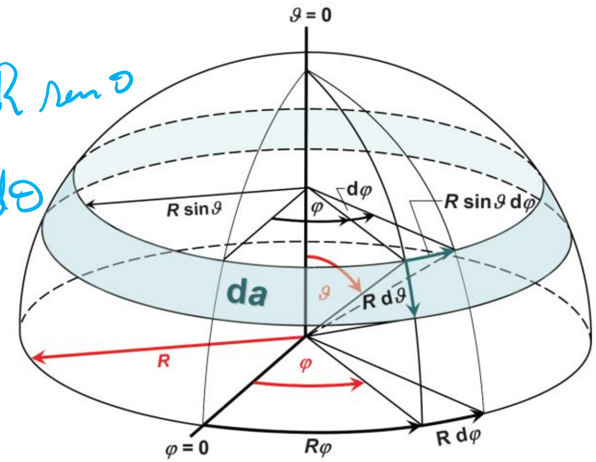
$$F_x = \int_{CS} p^* \, dA \cos\theta$$

$$= \int_0^{\pi} p^* \cos\theta \, R^2 \sin\theta \, d\theta$$



$dA = R\,d\theta \cdot R \sin\theta$

$= R^2 \sin\theta \, d\theta$



Rate of change of momentum

- The total force is equal to the rate of change of momentum in the x direction of the fluid, within the sphere:

$$\sum F_x = \int_0^\pi \overbrace{\rho u_1}^{\rho\vec{V}} \underbrace{u_R 2\pi R^2 \sin\theta \, d\theta}_{\vec{V}\cdot\hat{n}\; dA}$$

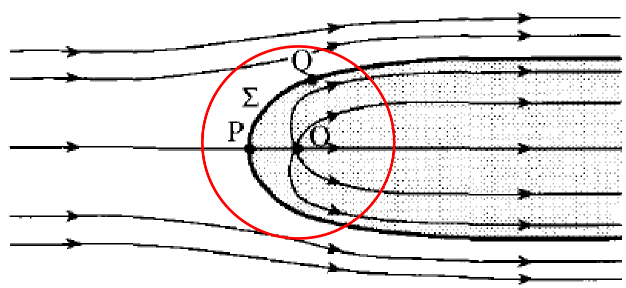
$$= \int_0^\pi \left\{ U^2 \cos\theta + \frac{UQ(1+\cos^2\theta)}{4\pi R^2} + \frac{Q^2\cos\theta}{16\pi^2 R^4} \right\} 2\pi R^2 \sin\theta \, d\theta$$

$$= \boxed{\frac{4}{3}\rho U Q,}$$

Reynolds transport theorem: $\dfrac{dB_{sys}}{dt} = \dfrac{d}{dt}\displaystyle\int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA$ $\quad B = b.m$

$$\sum \vec{F} = \cancelto{0}{\frac{d}{dt}\int_{CV} \rho \vec{V} \, dV} + \int_{CS} \rho \vec{V}(\vec{V}\cdot\vec{n}) \, dA$$

Rate of change of momentum



$p^* \to \frac{1}{3}\rho U Q$

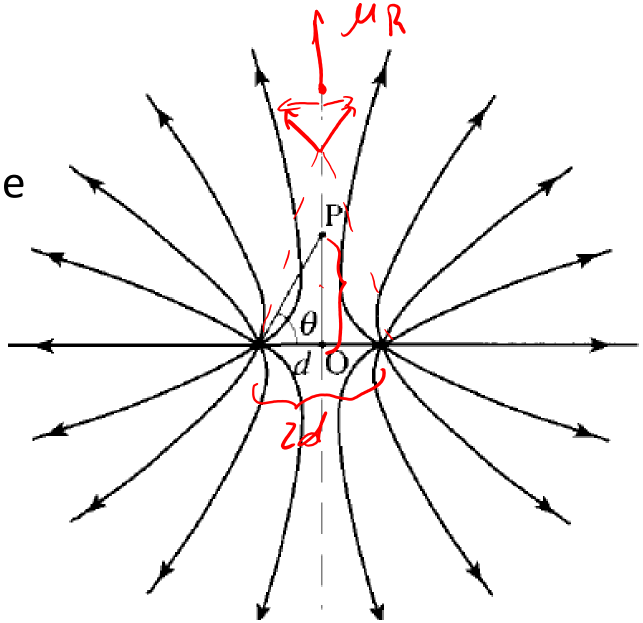
$\sum F_x \to \frac{4}{3}\rho U Q$

- There is then an additional force on the fluid in the x direction of magnitude $\rho U Q$
- This has to be exerted by the source (sink) and thus the source (sink) will experience a reaction force

$$\boxed{\boldsymbol{F} = -\rho U Q.}$$

Two equal sources

Velocity at one source, due to the other:



$OP = r = \dfrac{d}{\cos\theta} = d\ \sec\theta$

$\rightarrow$ $U = Q/4\pi(2d)^2$

$= \dfrac{Q}{16\pi d^2}$

On the plane bissecting the line joining the two sources the normal component of the velocity vanishes. The radial component (in the direction of OP), add and are given by:

$$u_R = \frac{2Q\ \sin\theta}{4\pi(\underbrace{d\ \sec\ \theta}_{r})^2}.$$

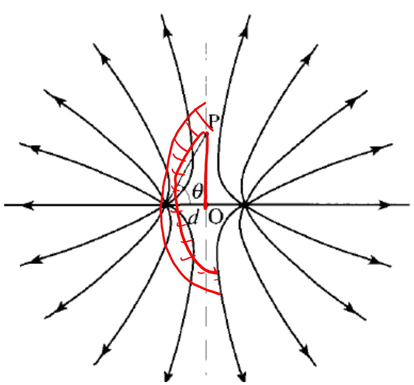
Excess pressure and force

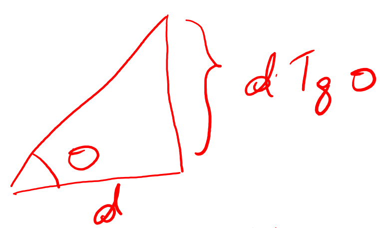
- Assuming that the excess pressure vanishes at infinity, where **u** also vanishes, the excess pressure at P is (Bernoulli),

$$\rightarrow \quad p^*\{\theta\} = -\frac{\rho Q^2 \sin^2\theta \cos^4\theta}{8\pi^2 d^4}. \qquad = p - p_\infty$$

- The fluid to the left of the bissecting plane experiences a force due to this excess pressure, given by

$$-\int_0^\infty p^*\{\theta\} 2\pi \underbrace{d \tan\theta}_{r} \, \mathrm{d}(\underbrace{d \tan\theta}_{r}) = \frac{\rho Q^2}{4\pi d^2}\int_0^{\pi/2} \sin^3\theta \cos\theta \, \mathrm{d}\theta = \boxed{\frac{\rho Q^2}{16\pi d^2}}.$$





$$\boxed{F = \rho U Q.}$$

$$d(\bar{d}\, \mathrm{Tg}\,\theta) = \frac{\bar{d}}{\cos^2\theta}\, d\theta$$

Analytical solutions of Laplace's equation

(Sec. 4.6)

(i) ***Two-dimensional circular polar coordinates (r, θ)***

In this system Laplace's equation becomes

$$\nabla^2 \phi = 0$$

$$\longrightarrow \quad r\frac{\partial}{\partial r}\left\{r\frac{\partial \phi}{\partial r}\right\} + \frac{\partial^2 \phi}{\partial \theta^2} = 0.$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Single-valued solutions in which the variables are separated can readily be found. They are:

$$\phi = \text{constant},$$

$$\phi \propto \phi_0 = \ln r, \tag{4.22}$$

$$\phi \propto \phi_n = \underline{r^n \cos(n\theta)}, \quad or \quad \phi \propto \psi_n = \underline{r^n \sin(n\theta)} \tag{4.23}$$

$$[n = \pm 1, \pm 2, \pm 3 \text{ etc.}].$$

$$\longrightarrow \quad \phi = \text{constant} + A_0\phi_0 + \Sigma(A_n\phi_n + B_n\psi_n)$$

Ex.:

$$\phi_n = r^n \cos(n\theta).$$

$$r \frac{\partial}{\partial r}\left\{ r \frac{\partial \phi}{\partial r} \right\} + \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

2nd term

$$\frac{\partial^2 \phi}{\partial \theta^2} = - r^n n^2 \cos(n\theta)$$

1st term

$$\frac{\partial \phi}{\partial r} = n r^{n-1} \cos(n\theta)$$

$$r \frac{\partial}{\partial r}\left(r^n n \cos(n\theta) \right) = r\, r^{n-1} \cdot n^2 \cos(n\theta)$$

$$= r^n n^2 \cos(n\theta)$$

$$\nabla^2 \phi = 0$$