(iv) *Three-dimensional spherical polar coordinates* (R, θ, ϕ)

Laplace's equation in spherical polars has separated solutions which form a complete set, like the two-dimensional solutions described by (4.22) and (4.23). We need not list them fully here, because we shall be concerned only with problems in which the flow is axially symmetric, i.e. in which the flow potential does not vary with the azimuthal angle ϕ.[2] In these circumstances Laplace's equation simplifies to

$\nabla^2 \phi = 0$,

$$\frac{\partial}{\partial R}\left(R^2 \frac{\partial \phi}{\partial R}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0,$$

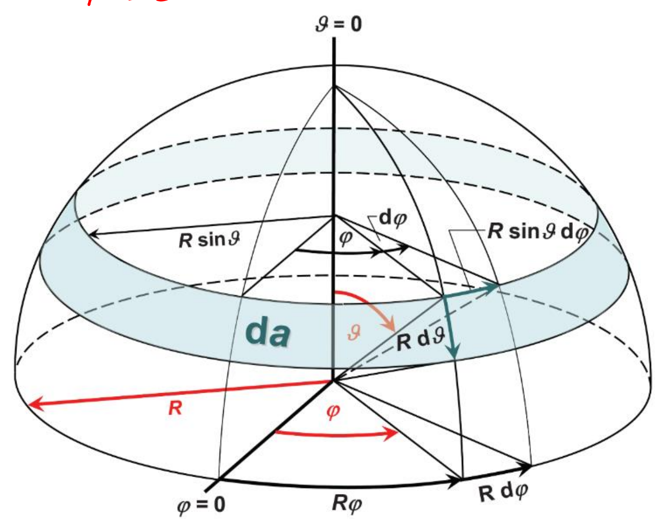
and its separated solutions may be written as

$$\phi = \sum_{m=0} \left(A_m \phi_m^+ + B_m \phi_m^- \right)$$

$$\phi \propto \phi_n^+ = R^n \, \mathrm{P}_n\{\cos\theta\},$$

$$\phi \propto \phi_n^- = R^{-(n+1)} \, \mathrm{P}_n\{\cos\theta\},$$

$$[n = 0, +1, +2, +3 \text{ etc.}].$$



Laplacian in spherical coordinates

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

The Legendre functions $P_n\{\cos\theta\}$ may be expanded as polynomials in their argument, and we shall need the following expressions in particular:

$$P_0\{\cos\theta\} = 1, \tag{4.29}$$

$$P_1\{\cos\theta\} = \cos\theta, \tag{4.30}$$

$$P_2\{\cos\theta\} = \frac{1}{2}(3\cos^2\theta - 1). \tag{4.31}$$

The full functions ϕ_n^+ and ϕ_n^- are properly called *zonal solid harmonics*. They are orthogonal to one another, and all other solutions of Laplace's equation in three dimensions which share their symmetry (or asymmetry) may be expressed as linear combinations of them [cf. (4.24)].

Some of the solutions described by (4.27) and (4.28) are of course trivial. Thus $\phi_0^+ = 1$ for all values of R and θ. As for

$$\phi_1^+ = R\cos\theta = x_1$$

$\phi_1^+ = xU$, $x = R\cos\theta$

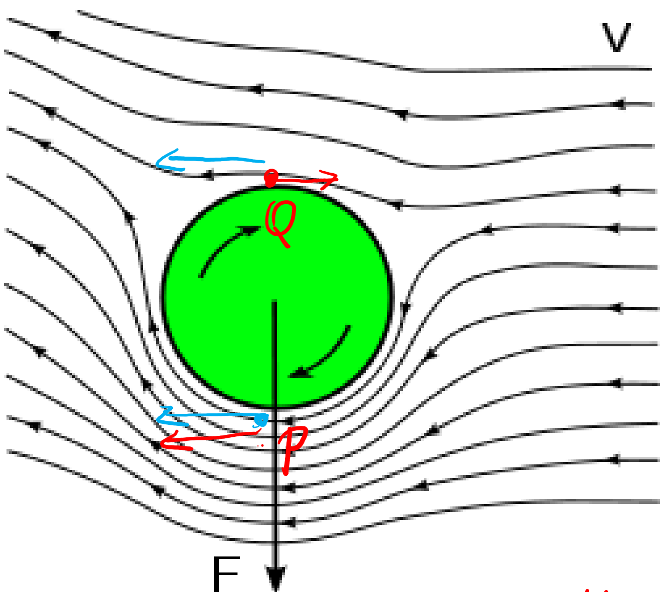
and

$\vec{u} = \dfrac{\partial\phi}{\partial x} = U\hat{x}$

$$\phi_0^- = R^{-1},$$

Potential flow around a sphere and Magnus effect

TP

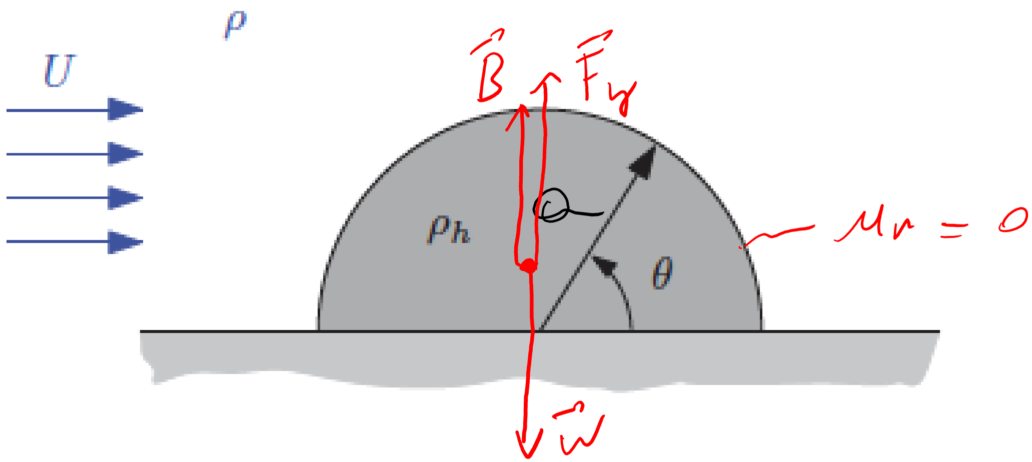


$u_P > u_Q$

$p_P < p_Q$

$, H = \frac{p}{\rho} + \frac{u^2}{2} + gz = C_{te}$

Solid hemisphere on a flat plate



Due to high speed flow at the top of the sphere, we expect a low pressure at the top of the sphere. This pressure results in a lift force on the hemsiphere.

Potential

$$\nabla^2 \phi = 0 \Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0$$

Note: symmetry in φ

Solutions of the Laplace equation

$$\phi_m^+ = r^m P_m(\cos\theta)$$

$$\phi_m^- = r^{-(m+1)} P_m(\cos\theta)$$

where $P_0 = 1$

$P_1 = \cos\theta$

General solution

$$\phi = \sum_m \left(A_m^+ \phi_m^+ + A_m^- \phi_m^- \right)$$

Boundary condition

1) $r \to \infty \Rightarrow \phi = \underbrace{U}_{A_1^+} \underbrace{r\cos\theta}_{\phi_1^+} \Rightarrow \boxed{A_1^+ = U}\ ,\ \boxed{A_{m\neq 1}^+ = 0}$

$\vec{u} = U\hat{x} \Rightarrow \phi = Ux$

$$2)\ r = a\ ,\quad u_r = 0 \Rightarrow \left.\frac{\partial \phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{\partial \phi_n^-}{\partial r} = -(n+1)\, r^{-(n+2)}\, P_n(\cos\theta)$$

We need only the term n=1. It would be complicated (probably impossible) to satisfy this condition for any n.

$$\phi_1^- = A_1^-\, r^{-2} \cos\theta$$

Thus

$$\phi = Ur\cos\theta + \frac{A_1^-}{r^2}\cos\theta = \cos\theta\left(Ur + \frac{A_1^-}{r^2}\right)$$

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=a} = 0 \Rightarrow \boxed{A_1^- = \frac{Ua^3}{2}}$$

$$\Rightarrow \quad \phi = \cos\theta \left[Ur + \frac{Ua^3}{2r^2} \right]$$

Velocity

$$u_r = \frac{\partial \phi}{\partial r} = \cos\theta \left[U - \frac{Ua^3}{r^3} \right]$$

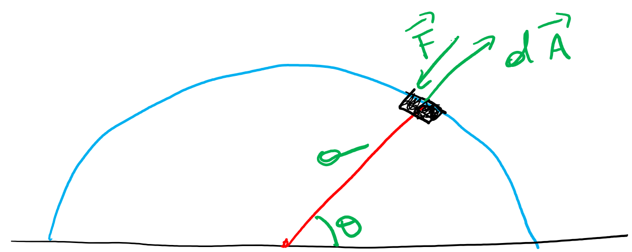
$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\,\mathrm{sen}\,\theta \left[U + \frac{Ua^3}{2r^3} \right]$$

Pressure in r=a

$$u_r = 0 \quad , \quad u_\theta = -\frac{3}{2} U \sin\theta$$

$$p^* = \frac{\rho}{2}\left(U^2 - u^2_{(r=a)}\right) = \frac{\rho}{2}\left[U^2 - \frac{9}{4}U^2 \sin^2\theta\right]$$

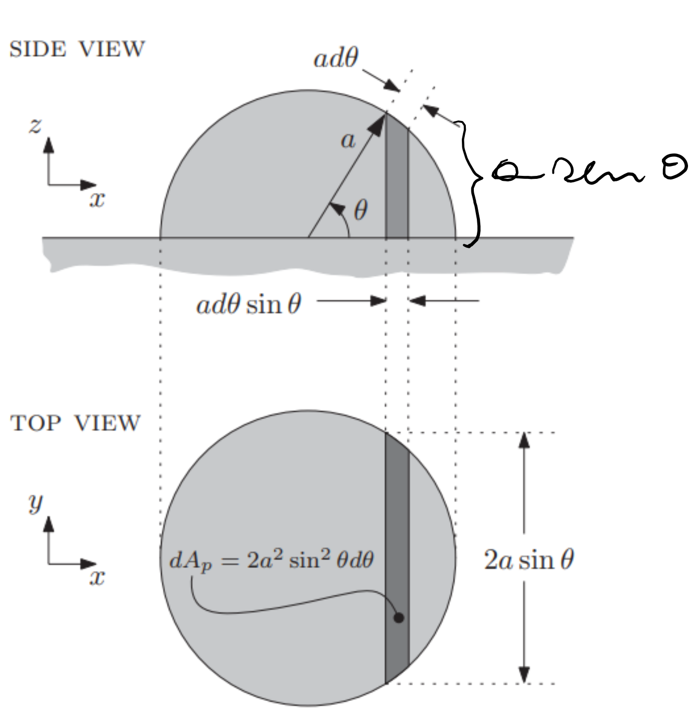
Lift force



$$d\vec{F} = -p^* d\vec{A}$$

Verificar : $F_x = 0$

$$F_y = -\int p^* \, dA_{//}$$



$$dA_{\parallel} = 2a^2 \operatorname{sen}^2\theta \, d\theta$$

$$F_y = -\int_0^{\pi} \frac{\rho}{2} U^2 \left(1 - \frac{9}{4} \operatorname{sen}^2\theta\right) 2a^2 \operatorname{sen}^2\theta \, d\theta$$

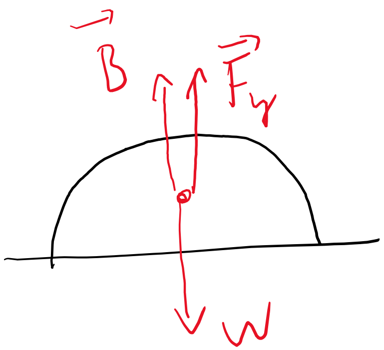
$$= \frac{11}{32} \rho U^2 a^2 \pi$$

Hemisphere weight

$$W = \rho_h \cdot \frac{1}{2} \cdot \frac{4}{3} \pi a^3 g = \frac{2}{3} \rho_h \pi a^3 g$$

Buoyant force

$$B = \rho g V_d = \rho g \frac{1}{2} \frac{4}{3} \pi a^3 = \frac{2}{3} \pi a^3 \rho g$$



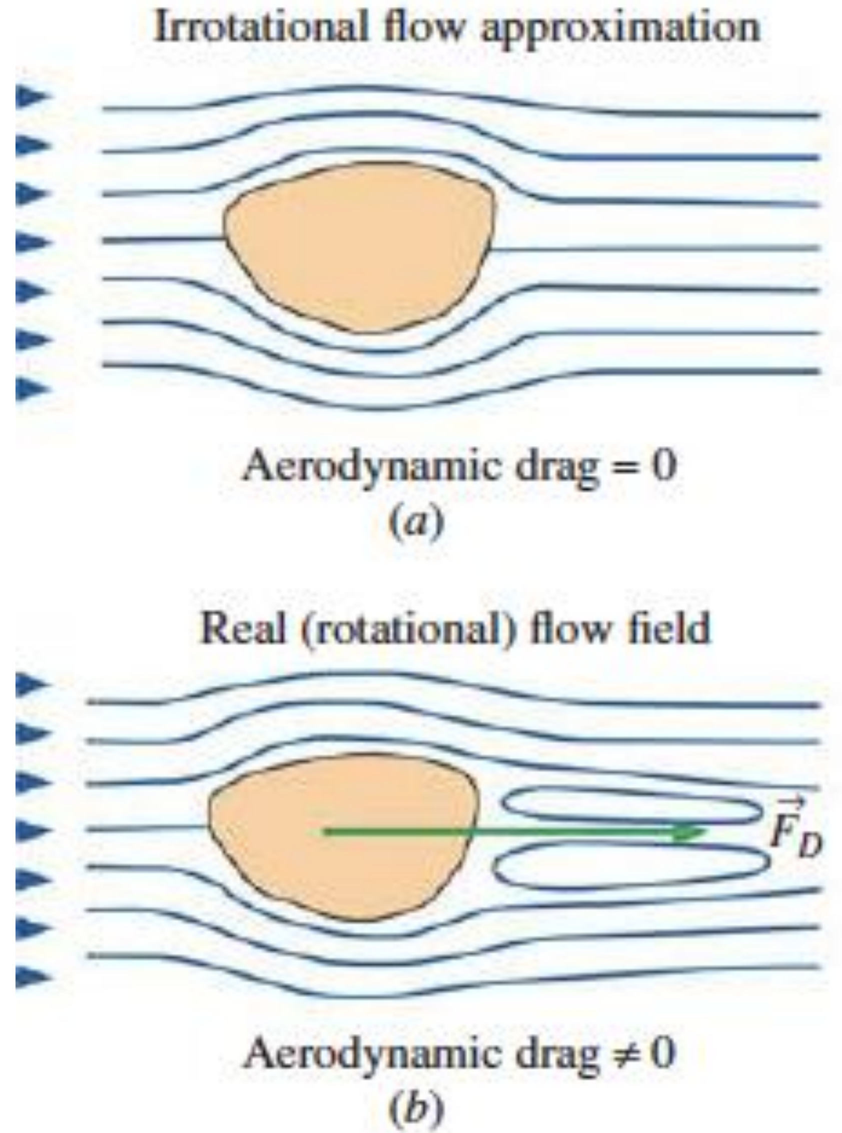
$$W \geq B + F_y$$

$$\Rightarrow \boxed{\rho_h \geq \rho \left(1 + \frac{33}{64} \frac{U^2}{a g} \right)}$$

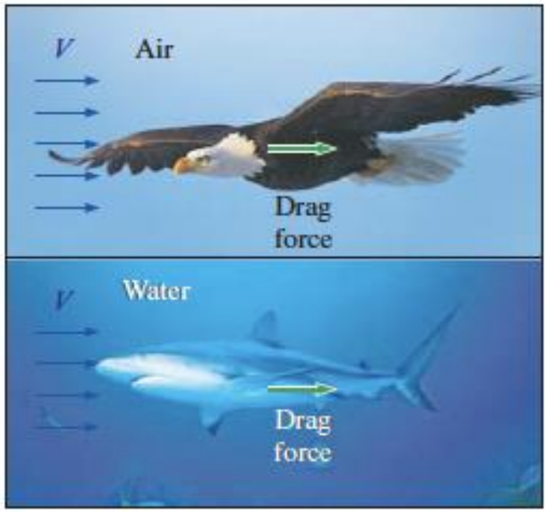
Condition for the hemisphere to remain on the plate

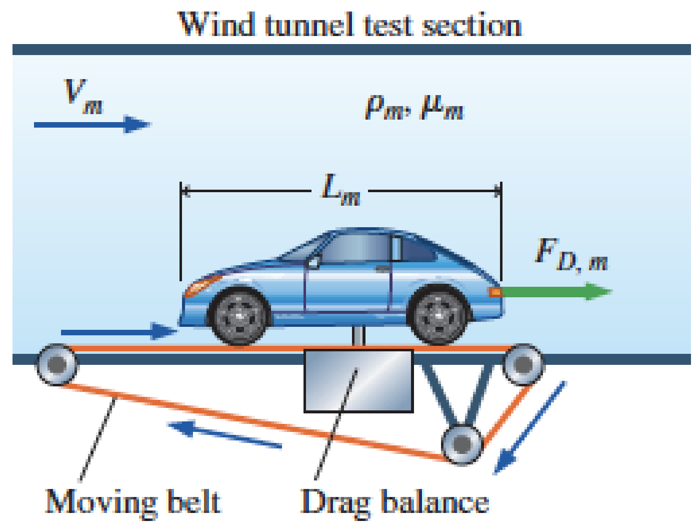
D'Alembert's paradox: In irrotational flow , the aerodynamic drag force on any body of any shape immersed
in a uniform stream is zero.

"*It seems to me that the theory (potential flow), developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future Geometers [i.e. mathematicians - the two terms were used interchangeably at that time] to elucidate*"



Drag force





In a real flow, the pressure on the back surface of the body is significantly less than that on the front surface, leading to a nonzero pressure drag on the body. In addition, the no-slip condition on the body surface leads to a nonzero viscous drag as well.

Thus, the irrotational flow falls short in its prediction of aerodynamic drag for two reasons: it predicts no pressure drag and it predicts no viscous drag.