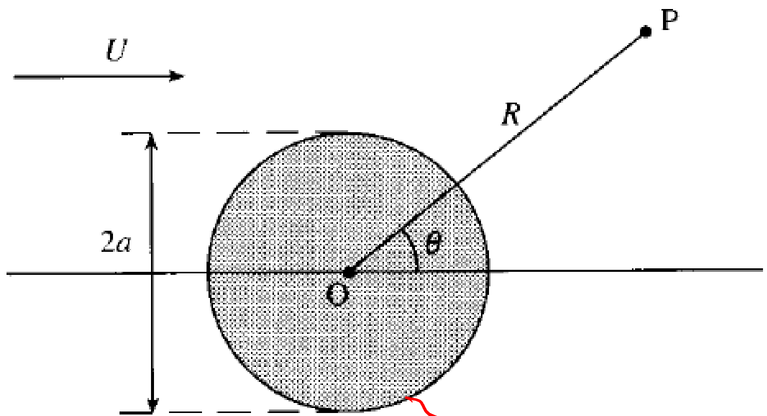
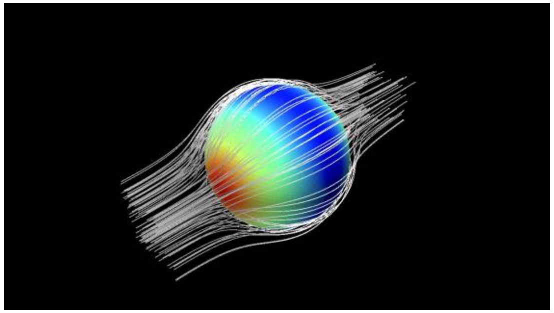
# Potential flow around a sphere




$u_r = 0$

Faber 4.7



Potential

$$\nabla^2 \phi = 0 \Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0$$

Boundary condition 1: at infinity the velocity is the free stream, which implies that

$$r \to \infty \Rightarrow \phi = U r \cos\theta \Rightarrow \underline{A_1^+ = U} \,,\; A_{n\neq 1}^+ = 0$$

$$\vec{u} = U\hat{x} = U\cos\theta$$

Boundary condition 2: at the surface of the sphere *R* = *a,* contact between the sphere and fluid require that the radial component of the fluid velocity is the same as that of the sphere.

$$r = a \,,\quad u_r = 0 \Rightarrow \left.\frac{\partial\phi}{\partial r}\right|_{r=a} = 0$$

Solution:

$$\phi = \underbrace{A_0^+}_{=0} + U r\cos\theta + A_1^- \underline{r^{-2}} \cos\theta$$

$$\phi_n^- = r^{-n-1} P_n(\cos\theta)$$

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=a} = 0 \quad \Rightarrow \quad \boxed{A_1^- = \frac{Ua^3}{2}}$$

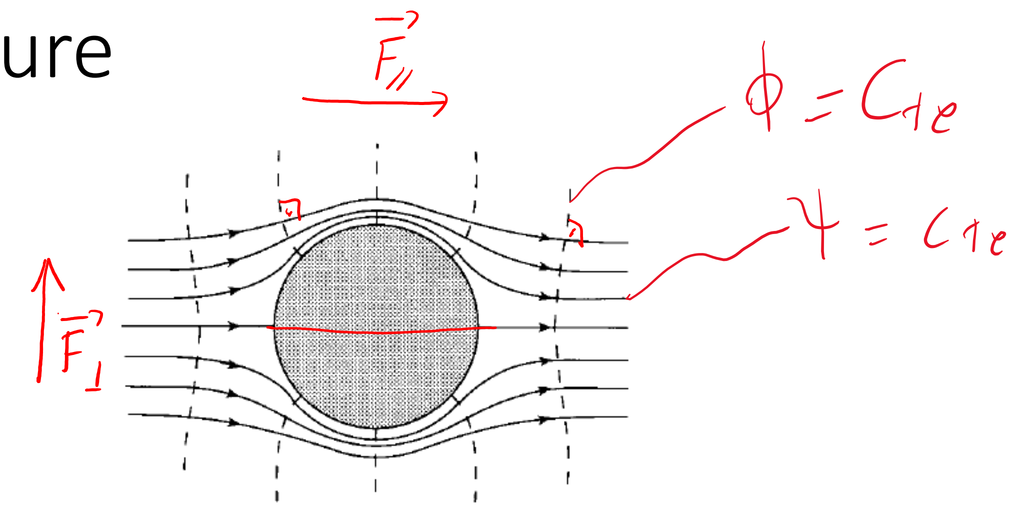
$$\Rightarrow \quad \phi = \cos\theta \left[ Ur + \frac{Ua^3}{2r^2} \right]$$

Velocity

$$u_r = \frac{\partial \phi}{\partial r} = \cos\theta \left[ U - \frac{Ua^3}{r^3} \right]$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\sin\theta \left[ U + \frac{Ua^3}{2r^3} \right]$$

# Excess pressure




With $p^*$ defined to be zero at large distances, we have

$$p^* = \frac{1}{2}\rho(U^2 - u_R^2 - u_\theta^2), \quad = p(R) - p_\infty$$

so that in contact with the sphere

$$p^*_{R=a} = \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right).$$

Because the excess pressure at $R = a$ is completely symmetrical about the equatorial plane, a sphere which is in uniform motion relative to fluid experiences no force, apart from its own weight and the hydrostatic upthrust which we have suppressed. This is an example of *d'Alembert's paradox* [§7.8],

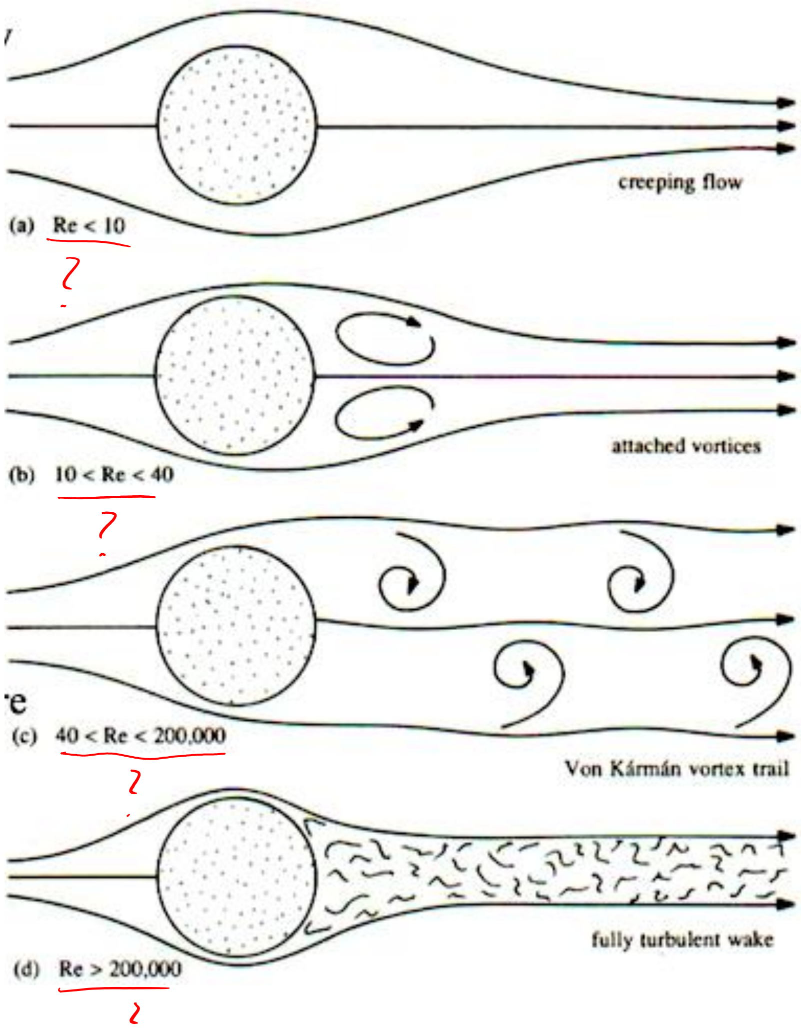
# Lift & drag forces

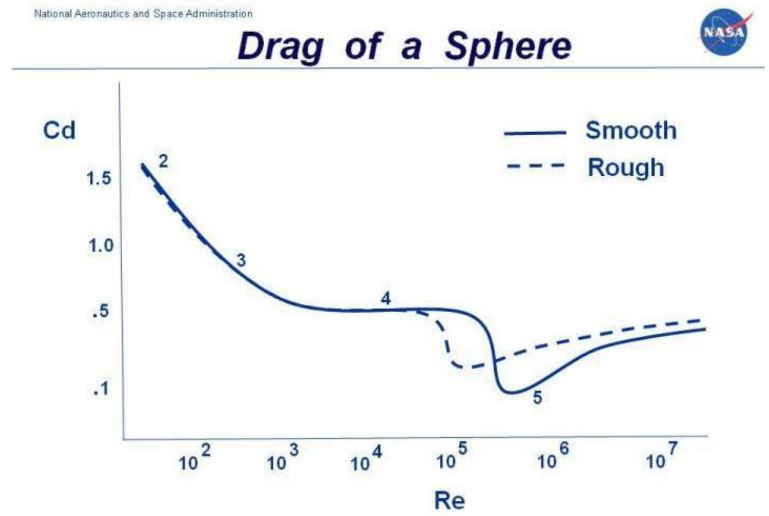
- The component of the resultant pressure and shear forces that acts in the flow direction is called the drag force (or just drag), and the component that acts normal to the flow direction is called the lift force (or just lift).

$$F_{\parallel} = -\int p^* \, dA_{\parallel} = 0$$ drag

$$F_{\perp} = -\int p^* \, dA_{\perp} = 0$$ lift

# Different regimes



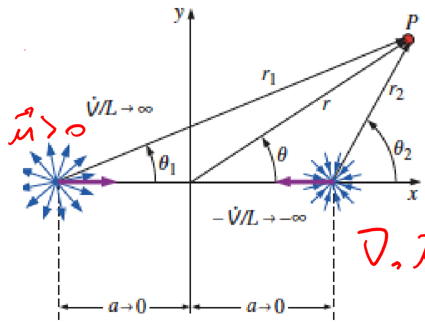




www.youtube.com/watch?v=fcjaxC-e8oY

# Doublet: line source and sink close to origin

We have seen before that

*Composite stream function:* $\psi = \dfrac{-\dot{V}/L}{2\pi} \arctan \left[ \dfrac{2ar \sin\theta}{r^2 - a^2} \right]$




$\nabla \cdot \vec{u} > 0$

$\nabla \cdot \vec{u} < 0$

By Taylor expanding the arctan around zero:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$tg^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

$a << r$

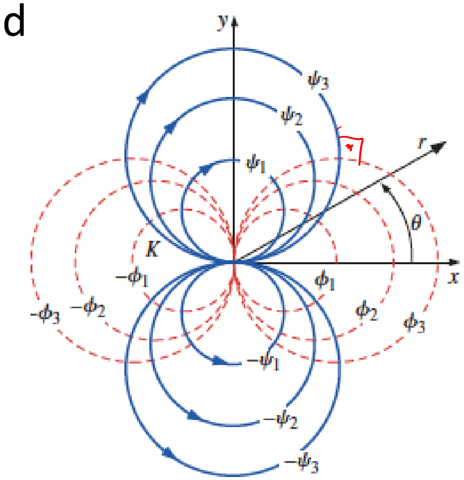
*Stream function as $a \to 0$:* $\boxed{\psi \to \dfrac{-a(\dot{V}/L) r \sin\theta}{\pi (r^2 - a^2)}}$

# Doublet: line source and sink close to origin

Let $a$ tend to zero at constant doublet strength $K$, to find

*Doublet along the x-axis:* $$\psi = \frac{-a(\dot{V}/L)}{\pi}\frac{\sin\theta}{r} = \boxed{-K\frac{\sin\theta}{r}}$$

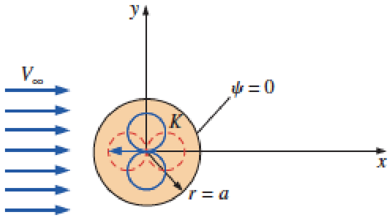
*Doublet along the x-axis:* $$\phi = K\frac{\cos\theta}{r}$$




Streamlines (solid) and equipotential lines (dashed) for a doublet of strength $K$ located at the origin in the $xy$-plane and aligned with the $x$-axis.

# Superposition of a uniform stream and a doublet: Flow over a circular cylinder

*Superposition*: $$\psi = V_\infty r \sin\theta - K\frac{\sin\theta}{r}$$

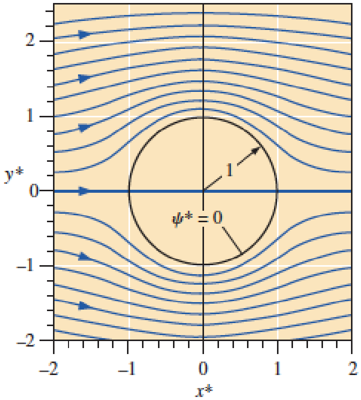



For convenience we set $\psi = 0$ when $r = a$

*Doublet strength*: $$K = V_\infty a^2$$

*Alternate form of stream function*: $$\boxed{\psi = V_\infty \sin\theta\left(r - \frac{a^2}{r}\right)}$$

$$\psi^* = \sin\theta\left(r^* - \frac{1}{r^*}\right)$$




*Nondimensional streamlines*: $$r^* = \frac{\psi^* \pm \sqrt{(\psi^*)^2 + 4\sin^2\theta}}{2\sin\theta}$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \boxed{V_\infty \cos\theta\left(1 - \frac{a^2}{r^2}\right)} \quad u_\theta = -\frac{\partial\psi}{\partial r} = \boxed{-V_\infty \sin\theta\left(1 + \frac{a^2}{r^2}\right)}$$