

The Navier-Stokes equation

- To be mathematically solvable, the number of equations must equal the number of unknowns, and thus we need six more equations.
- These equations are called constitutive equations, and they enable us to write the components of the stress tensor in terms of the velocity and pressure fields.
- The first thing we do is to separate the pressure stresses and the viscous stresses.
- For a fluid at rest

Fluid at rest:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

$$\sigma_{ij} = \sigma_{ji}$$

- For moving fluids,

Moving fluids:

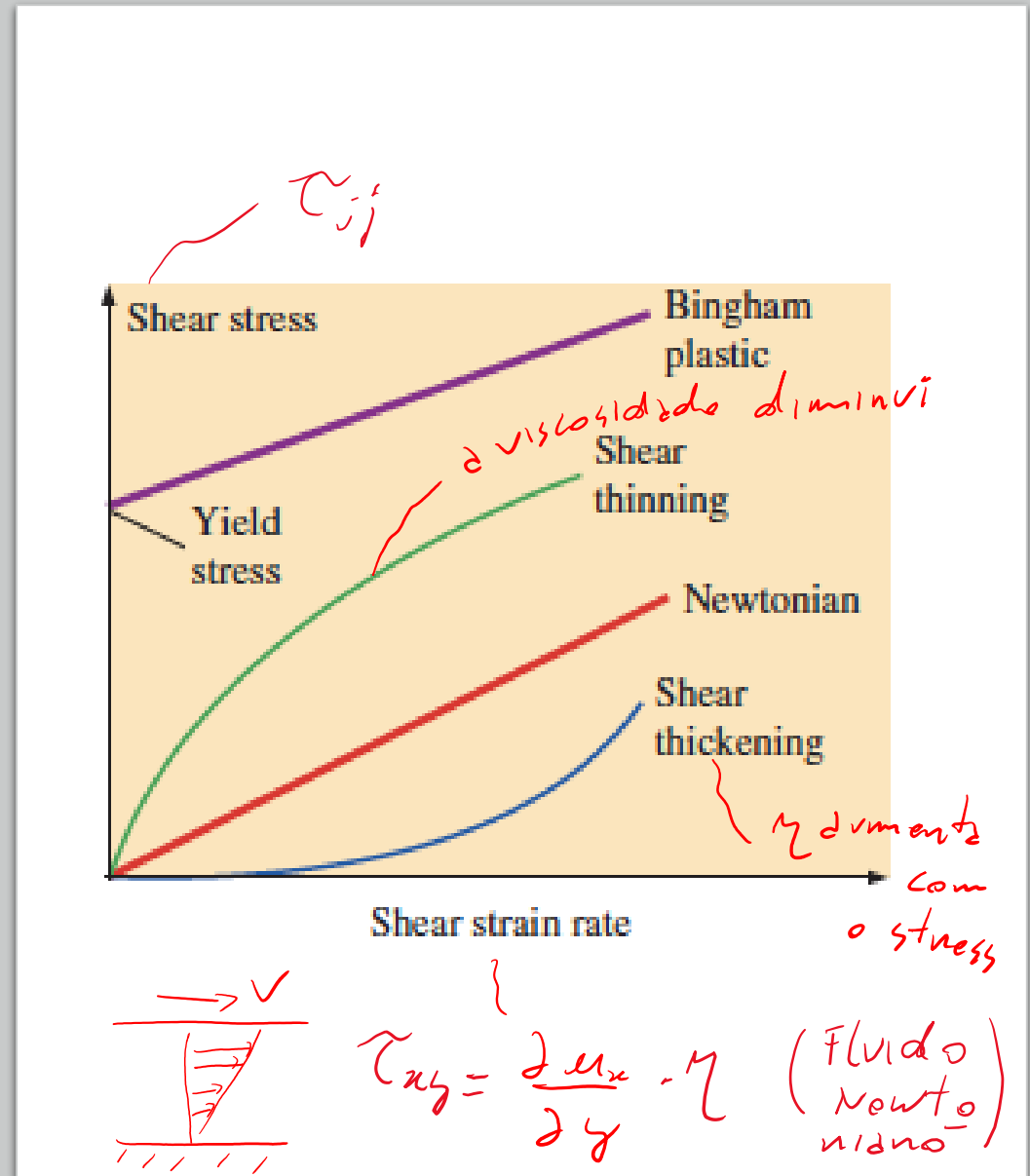
$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

- where we have introduced a new tensor, τ_{ij} , called the viscous stress tensor or the **deviatoric stress tensor**.
- There are constitutive equations that express τ_{ij} in terms of the velocity field and measurable fluid properties such as the viscosity.
- The actual form of the constitutive relations depends on the type of fluid.
- For incompressible fluids P is the mechanical pressure

Mechanical pressure: $P_m = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$

Newtonian and non-Newtonian fluids

- Newtonian fluids, defined as fluids for which the shear stress is linearly proportional to the shear strain rate. Many common fluids, such as air and other gases, water, kerosene, gasoline, and other oil-based liquids, are Newtonian fluids.
- Fluids for which the shear stress is not linearly related to the shear strain rate are called non-Newtonian fluids. Examples include slurries and colloidal suspensions, polymer solutions, blood, paste, and cake batter.



Toothpaste: Bingham plastic



Ketchup: shear thinning



https://www.youtube.com/watch?v=2mYHGn_Pd5M&t=143s

Cornstarch: shear thickening



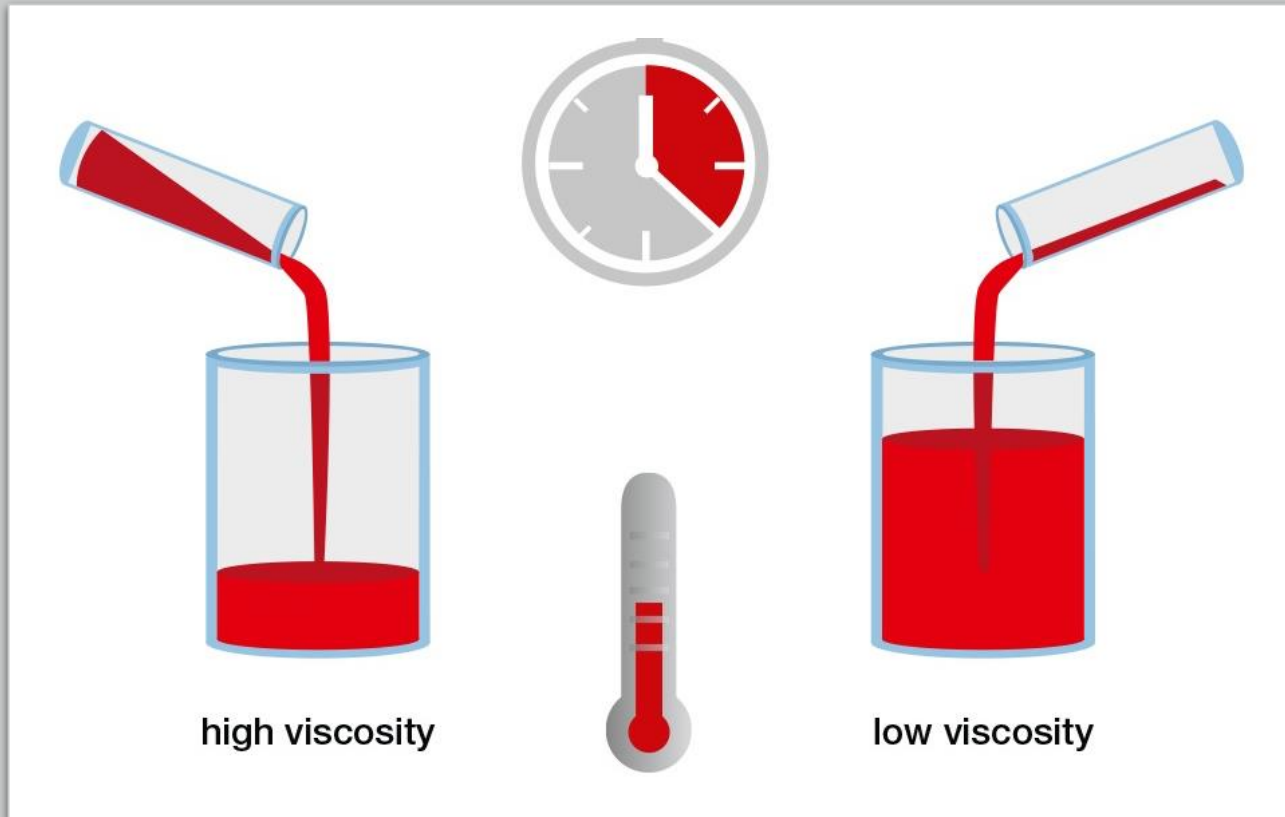
Honey: viscous Newtonian fluid





Viscosity of Newtonian fluids

Characterizes the degree of internal 'friction'



This 'friction', *viscous stress*, is associated with the resistance offered by two adjacent layers of the fluid to their relative motion.

Navier-Stokes equation for incompressible and isothermal flow

viscosidade
cinematica: $\nu = \frac{\mu}{\rho}$

Viscous stress tensor for an incompressible Newtonian fluid with constant properties:

shear "μ"

$$\tau_{ij} = 2\mu\epsilon_{ij}$$

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

where μ is the shear viscosity.

In cartesian coordinates, the deviatoric stress tensor becomes

$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} 2\mu \frac{\partial u}{\partial x} & \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & 2\mu \frac{\partial v}{\partial y} & \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) & \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & 2\mu \frac{\partial w}{\partial z} \end{pmatrix}$$

Stress tensor for Newtonian fluids

$$\sigma_{ij} = \underbrace{\begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}}_{\text{Pressões}} + \underbrace{\begin{pmatrix} 2\mu \frac{\partial u}{\partial x} & \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & 2\mu \frac{\partial v}{\partial y} & \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) & \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & 2\mu \frac{\partial w}{\partial z} \end{pmatrix}}_{\text{Tensões viscosas}}$$

Substituting this into Cauchy's equation we find, in the x direction:

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \mu \frac{\partial}{\partial z} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

We note that as long as the velocity components are smooth functions of x , y , and z , the order of differentiation is irrelevant. For example, the first part of the last term above can be rewritten as

$$\mu \frac{\partial}{\partial z} \left(\frac{\partial w}{\partial x} \right) = \mu \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial z} \right)$$

After some (clever) re-arrangements of the viscous terms we find

$$\begin{aligned} \rho \frac{Du}{Dt} &= -\frac{\partial P}{\partial x} + \rho g_x + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x} \frac{\partial w}{\partial z} + \frac{\partial^2 u}{\partial z^2} \right] \\ &= -\frac{\partial P}{\partial x} + \rho g_x + \mu \left[\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] \end{aligned}$$

$\underbrace{\hspace{10em}}_0$ $\underbrace{\hspace{10em}}_{\nabla^2 \mu_x}$
 $\nabla \cdot \vec{u} = 0$

and thus

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + \mu \nabla^2 u$$

Similarly,

$$\rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \rho g_y + \mu \nabla^2 v$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \rho g_z + \mu \nabla^2 w$$

Incompressible Navier–Stokes equation:

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

Cartesian coordinates

Incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

x-component of the incompressible Navier–Stokes equation:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

y-component of the incompressible Navier–Stokes equation:

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

z-component of the incompressible Navier–Stokes equation:

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

Cylindrical coordinates

Incompressible continuity equation:
$$\frac{1}{r} \frac{\partial(ru_r)}{\partial r} + \frac{1}{r} \frac{\partial(u_\theta)}{\partial \theta} + \frac{\partial(u_z)}{\partial z} = 0$$

r-component of the incompressible Navier–Stokes equation:

$$\begin{aligned} & \rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z \frac{\partial u_r}{\partial z} \right) \\ &= -\frac{\partial P}{\partial r} + \rho g_r + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) - \frac{u_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2} \right] \end{aligned}$$

θ -component of the incompressible Navier–Stokes equation:

$$\begin{aligned} & \rho \left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z} \right) \\ &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \rho g_\theta + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2} \right] \end{aligned}$$

z-component of the incompressible Navier–Stokes equation:

$$\begin{aligned} & \rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) \\ &= -\frac{\partial P}{\partial z} + \rho g_z + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right] \end{aligned}$$

Viscous stress tensor in cylindrical coordinates

$$\tau_{ij} = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} 2\mu \frac{\partial u_r}{\partial r} & \mu \left[r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right] & \mu \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \\ \mu \left[r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right] & 2\mu \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) & \mu \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \\ \mu \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) & \mu \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) & 2\mu \frac{\partial u_z}{\partial z} \end{pmatrix}$$