Ciências ULisboa

The Navier-Stokes equation

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Overview

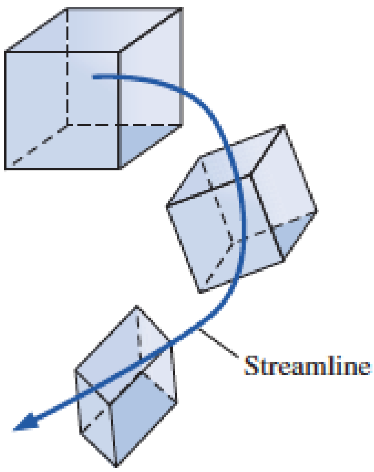
- Understanding the derivation of the differential linear momentum equation for incompressible Newtonian fluids – the Navier-Stokes equation.
- This is a set of partial differential equations that are valid at any point in the flow.
- When solved, together with the continuity equation, these equations yield details about the velocity, density, pressure, etc., at every point throughout the entire flow domain.
- From these fields, by integration, we can find the gross features of the flow such as the net force on the walls or on immersed bodies.
- Obtaining analytical solutions of the equation of motion for simple flow fields.
- Derivation of the Stoke's equation for creeping flow. Obtaining the drag force on a sphere in a uniform stream.
- Other applications of the Stoke's equation.

Differential analysis: mass

We start with the conservation of mass, which through the RTT yields the continuity equation

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0$$



Alternative form of the continuity equation:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla} \cdot \vec{V} = 0$$

Continuity equation in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho u_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho u_\theta)}{\partial \theta} + \frac{\partial (\rho u_z)}{\partial z} = 0$$

Steady continuity equation:

$$\vec{\nabla} \cdot (\rho \vec{V}) = 0$$

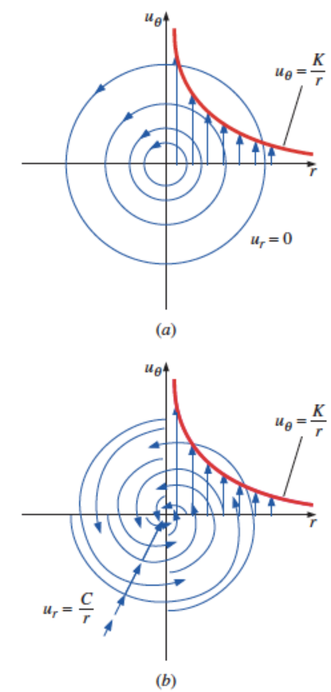
Incompressible continuity equation: $\vec{\nabla}\cdot\vec{V} = 0$

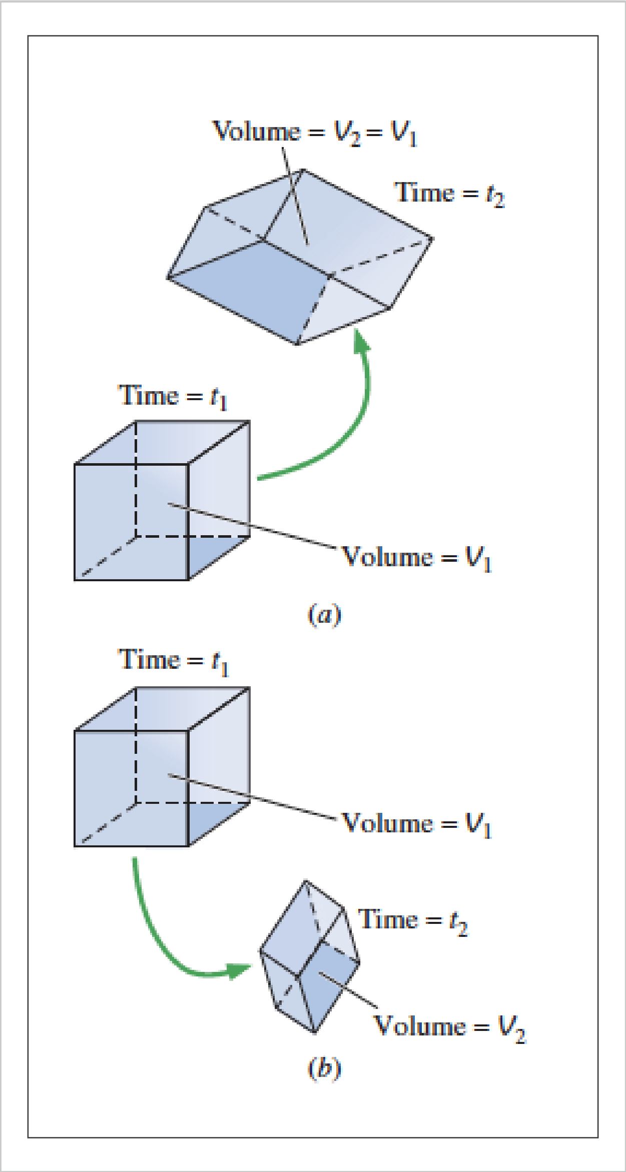
Incompressible continuity equation in Cartesian coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Incompressible continuity equation in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta} + \frac{\partial(u_z)}{\partial z} = 0$$





The volumetric strain rate vanishes for incompressible flows.

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\frac{1}{V}\frac{DV}{Dt} = 0$$

Recall

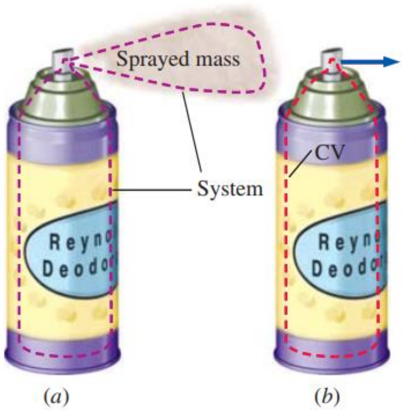


FIGURE 4–53
Two methods of analyzing the spraying of deodorant from a spray can: (*a*) We follow the fluid as it moves and deforms. This is the *system approach*—no mass crosses the boundary, and the total mass of the system remains fixed. (*b*) We consider a fixed interior volume of the can. This is the *control volume approach*—mass crosses the boundary.

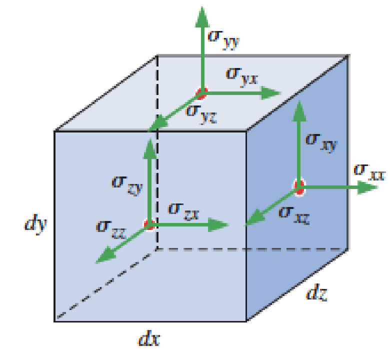
Reynolds transport theorem (RTT)

$$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n}\, dA$$

Surface force acting on a differential surface element: $d\vec{F}_{\text{surface}} = \sigma_{ij} \cdot \vec{n}\, dA$

Differential analysis: momentum

- For a control volume the RTT gives the momentum equation:



$$\sum \vec{F} = \int_{CV} \rho\vec{g}\, dV + \int_{CS} \sigma_{ij}\cdot\vec{n}\, dA = \int_{CV} \frac{\partial}{\partial t}(\rho\vec{V})\, dV + \int_{CS} (\rho\vec{V})\vec{V}\cdot\vec{n}\, dA$$

- The total force acting on the control volume is equal to the rate at which momentum changes within the control volume plus the rate at which momentum flows out of the control volume minus the rate at which momentum flows into the control volume.

- The divergence theorem implies that

$$\int_{CS} (\rho\vec{V})\vec{V}\cdot\vec{n}\, dA = \int_{CV} \vec{\nabla}\cdot(\rho\vec{V}\,\vec{V})\, dV$$

and

$$\int_{CS} \sigma_{ij} \cdot \vec{n}\, dA = \int_{CV} \vec{\nabla} \cdot \sigma_{ij}\, dV$$

- Re-arranging the terms, we find the equation

$$\int_{CV} \left[\frac{\partial}{\partial t}(\rho \vec{V}) + \vec{\nabla} \cdot (\rho \vec{V}\, \vec{V}) - \rho \vec{g} - \vec{\nabla} \cdot \sigma_{ij} \right] dV = 0$$

valid for any CV and thus, we obtain the Cauchy equation of motion

Cauchy's equation:

$$\frac{\partial}{\partial t}(\rho \vec{V}) + \vec{\nabla} \cdot (\rho \vec{V}\, \vec{V}) = \rho \vec{g} + \vec{\nabla} \cdot \sigma_{ij}$$

Other derivations are possible, e.g. by starting from an infinitesimal CV.

Alternative form of Cauchy's equation

- Clearly,

$$\frac{\partial}{\partial t}(\rho\vec{V}) = \rho\,\frac{\partial\vec{V}}{\partial t} + \vec{V}\,\frac{\partial\rho}{\partial t}$$

- The second term of Cauchy's equation can be written as

$$\vec{\nabla}\cdot(\rho\vec{V}\,\vec{V}) = \vec{V}\,\vec{\nabla}\cdot(\rho\vec{V}) + \rho(\vec{V}\cdot\vec{\nabla})\vec{V}$$

- Substituting this into the Cauchy's equation we find

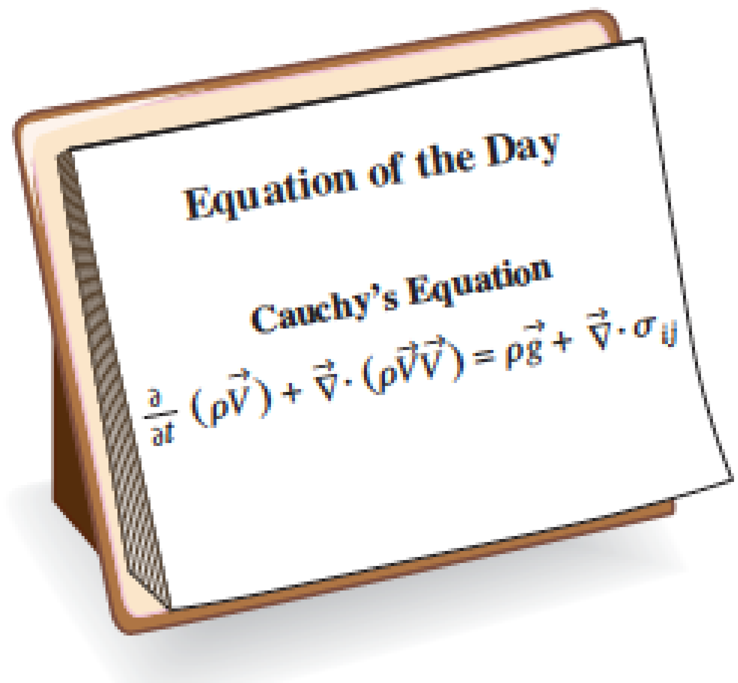
$$\rho\frac{\partial\vec{V}}{\partial t} + \vec{V}\left[\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V})\right] + \rho(\vec{V}\cdot\vec{\nabla})\vec{V} = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij}$$

- The continuity equation implies that the term in brackets vanishes and then

Alternative form of Cauchy's equation:

$$\rho\left[\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij}$$

Cauchy's equation in cartesian components



$\nabla \cdot \vec{u} = 0$

x-component: $$\rho\frac{Du}{Dt} = \rho g_x + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z}$$

y-component: $$\rho\frac{Dv}{Dt} = \rho g_y + \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z}$$

z-component: $$\rho\frac{Dw}{Dt} = \rho g_z + \frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}$$