Universo Primitivo

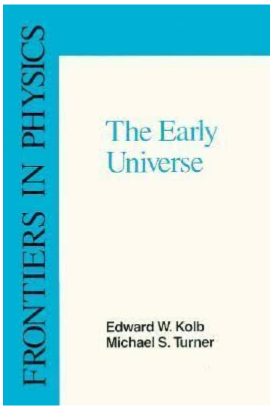
2020-2021 (1º Semestre)

Mestrado em Física - Astronomia

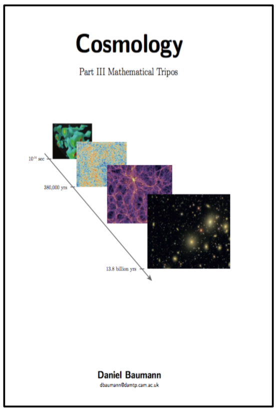
Chapter 3

3. Thermodynamics in an expanding universe
   - Natural Units;
   - Classification and properties of elementary particles;
   - Thermal evolution at equilibrium:
     - Density of states and macroscopic properties
     - Number density, energy density and pressure
       - Ultra-relativistic limit
       - Non-relativistic limit
   - Effective number of degrees of freedom
     - Internal degrees of freedom of particles according to the standard model of particle physics
     - Evolution of relativistic degrees of freedom
   - Entropy at equilibrium
     - Effective number of degrees of freedom in entropy;
     - Entropy conservation an its consequences;
     - Entropy and Temperature – time scaling for relativistic particles
   - Key events in the thermal history of the Universe

References



Ch. 3



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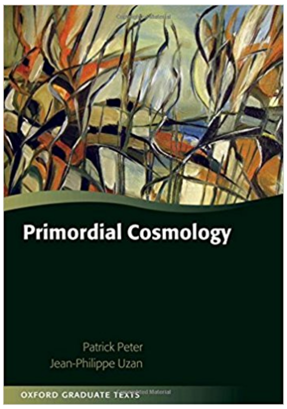
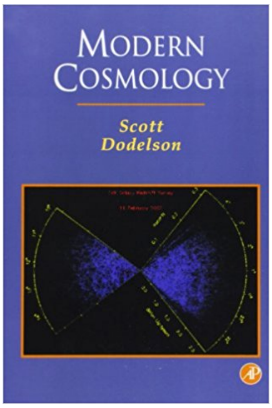
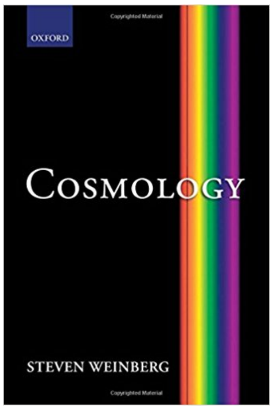


Ch. 4; Sec. 2.3



Ch. 2



Ch. 2

Natural Units

In Particle Physics and Cosmology the expression “**natural units**” usually refers to setting the following fundamental constants equal to unity:

$$c = k_B = \hbar = 1$$

These are the speed of light, the Boltzmann constant and the Planck constant ($\hbar = h/2\pi$).

As a consequence, the following fundamental properties (**time; length, temperature and mass**) can be written in **units of energy** (usually expressed in GeV, MeV, keV):

$$\begin{aligned} 1\ \mathrm{s} &= 1.5 \times 10^{24}\ \mathrm{GeV}^{-1}, \\ 1\ \mathrm{m} &= 5 \times 10^{15}\ \mathrm{GeV}^{-1}, \\ 1\ \mathrm{K} &= 8.6\ \times 10^{-14}\ \mathrm{GeV} = 8.6 \times 10^{-5}\ \mathrm{eV}, \\ 1\ \mathrm{kg} &= 5.6 \times 10^{26}\ \mathrm{GeV}. \end{aligned}$$

Where $1\mathrm{eV} = 1.6 \times 10^{-19}\mathrm{J}$ $\Rightarrow$ $1\mathrm{J} = 6.2 \times 10^{9}\mathrm{GeV}$, $1\ \mathrm{J} = 1\ \mathrm{kg\ m^2 s^{-2}}$

Natural Units

To prove these, use the definitions of the following constants in the IS system and the definition of electron volt in Jules.

$c = 3 \times 10^8$ m s^{-1},	velocidade da luz no vácuo;
$G = 6.67 \times 10^{-11}$ m^3 kg^{-1}s^{-2},	constante gravitacional;
$h = 6.6 \times 10^{-34}$ J s,	constante de Planck;
$e = 1.6 \times 10^{-19}$ C,	carga elementar;
$k_B = 1.38 \times 10^{-23}$ J K^{-1},	constante de Boltzmann.

Example: of the mass of known particles in MeV:

Espécie	Símbolo	Massa (MeV)	Carga (e)
Protão	p	938.3	+1
Neutrão	n	939.6	0
Electrão	e$^-$	0.511	-1
Neutrinos	ν_e , ν_μ , ν_τ	?	0
Fotão	γ	0	0
Matéria Escura	–	?	0?
Energia Escura	–	?	?

Classification of elementary particles

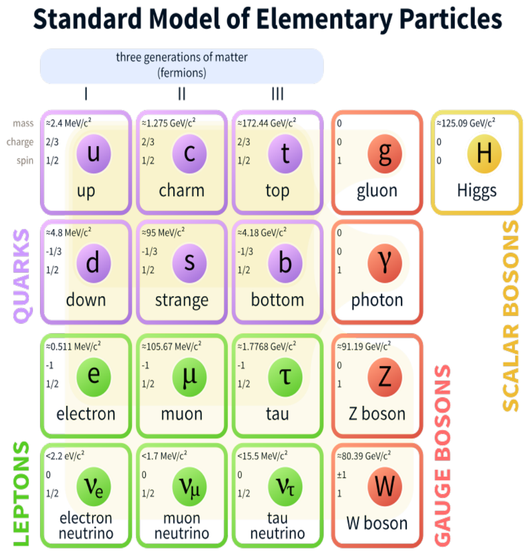
The Standard Model of Particle Physics (SMPF) predicts various families of particles some of them are **fundamental** and other “composite” particles.

Fundamental particles are not know to have internal structure. Composite particles have internal structure (i.e. are made of other particles).

All particles of the SMPF can by classified in the following way:

Name		Spin	Examples
Hadrons	Baryons $= qqq$	$n + \frac{1}{2}$	$p^+, n^0, \Delta, \Lambda, \Sigma, \Omega, \Xi \cdots$
	Mesons $= q\bar{q}$	n	$\pi^{0,\pm}, K^{0,\pm}, J/\psi, D^0, B^0, \eta, \cdots$
Leptons		$\frac{1}{2}$	$e^-, \nu_e, \mu^-, \nu_\mu, \tau^-, \nu_\tau.$
Gauge fields		1	$\gamma, Z^0, W^\pm, g^a.$

Classification of elementary particles



Gauge Fields (exchange Bosons):

Are fundamental particles that mediate interactions:

- Photon γ – electromagnetic;
- 8 gluons g – strong interaction
- Z and $W^{\pm}$ – weak interaction
- *Graviton?* $(h_{\mu\nu})$ – gravitational interaction (quantum gravity)

Leptons:

Are fundamental particles that interact via the electromagnetic and weak forces.

- Come in doublets with respect to the week force
- Only distinguishable by the mass
- Stable doublet: is the electron/electron neutrino

Hadrons:

Have internal structure and interact via all types of forces.

- are made of **quarks**, confined in sets of 2 (Mesons) or 3 (Baryons) particles: up, down; charm, strange; top; bottom (u, d, c, s, t, b)

Scalar Higgs Boson

- Higgs Field: The Higgs mechanism is believed to be the cause the **Electroweak symmetry breaking** and describes the **generation of the mass** of all fermions and massive bosons

Thermal evolution at equilibrium

Fundamental assumptions about the primordial universe:

- All **fluid species** are assumed to **behave as ideal fluids**.

- **Thermal equilibrium** of a fluid species **may be established** whenever the particles' interaction rate, $\Gamma(t)$, (expressed as *the number of interaction events per unit of time*) is larger than the expansion rate of the Universe, $H(t) = \dot{a}/a$:

$$\Gamma(t) \gg H(t)$$

- The best way to describe a fluid component is through its **distribution function** $f(\boldsymbol{x}, \boldsymbol{p}, E, t)$.This gives the mean **number of particle states in the position, $\boldsymbol{x} \pm \boldsymbol{dx}$, with momentum, $\boldsymbol{p} \pm \boldsymbol{dp}$**.

- In classical mechanics f is defined as the number of particles per **phase space** volume: $dN = f(\boldsymbol{x}, \boldsymbol{p}, E, t)\, d^3x\, d^3p$

- If space is **homogeneous,** the distribution function must be *independent of* $\boldsymbol{x}$. Moreover assuming **isotropy**, f must be a function of $p = |\boldsymbol{p}|$, so $f = f(p, E, t)$.

Thermal evolution at equilibrium

The phase-space of a species in Quantum physics:

Uncertainty principle (1927): $\Delta x \, \Delta p \; \gtrsim \; h$

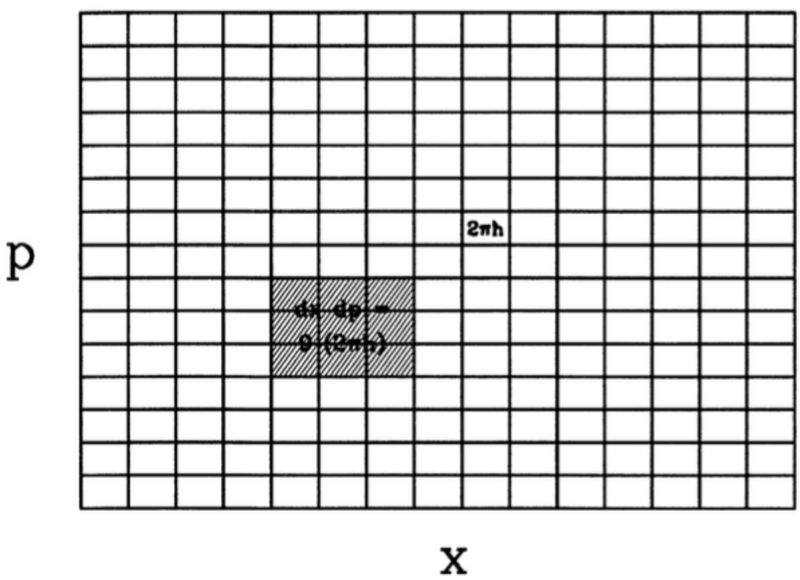


Figure 2.4. Phase space of position and momentum in one dimension. Volume of each cell is $2\pi\hbar$, the smallest region into which a particle can be confined because of Heisenberg's principle. Shaded region has infinitesmal volume $dxdp$. This covers nine cells. To count the appropriate number of cells, therefore, the phase space integral must be $\int dxdp/(2\pi\hbar)$.

Phase space smallest region of confinement:

One-dimension $\{x, p\}$:

$$\Delta x \; \Delta p = 2\pi\hbar$$

Three-dimensions $\{\boldsymbol{x},\boldsymbol{p}\}$:

$$\boldsymbol{\Delta x} \; \boldsymbol{\Delta p} = (2\pi\hbar)^3$$

Number of "cells" in the Phase space:

(natural units $\hbar$=1)

$$\int \frac{\boldsymbol{dxdp}}{(2\pi\hbar)^3} = \frac{\boldsymbol{1}}{(2\pi)^3} \int \boldsymbol{dxdp}$$

Phase space density

Thermal evolution at equilibrium

The phase-space of a species in Quantum physics:

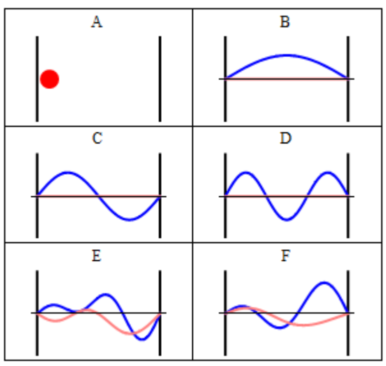
In quantum mechanics the **momentum operator (** $\hat{p} = i\hbar\nabla$ **) eigenstates** of a free particle inside a box of volume, V $= L^3$, has a **discrete spectrum of momentum/energy eigenstates,** described by the (time-independent) Schrödinger equation:

$$\frac{p^2}{2m}\psi = -\frac{\hbar^2\nabla^2}{2m}\psi = E\psi \Leftrightarrow \nabla^2\psi = -k^2\psi$$

where, $k^2 = 2mE/\hbar^2$ and $p = \hbar k$.

The **1D** solution for the boundary condition $\psi(0) = \psi(L) = 0$ is of the form $\psi(x) = A\,\sin(k_n x)$, where:

$$k_n = n\pi/L, \qquad \text{with}\, n > 0$$



The energy of each mode n is:

$$E_n = \frac{p_n^2}{2m} = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2}n$$

In **3D**, the possible momentum and energy states are:

$$\vec{p} = \hbar\,\frac{\pi}{L}\,(n_x, n_y, n_z) \qquad E_{\vec{n}} = \frac{p_{\vec{n}}^2}{2m} = \frac{\hbar^2 k_{\vec{n}}^2}{2m} = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2}\,(n_x^2 + n_y^2 + n_z^2)$$

Thermal evolution at equilibrium

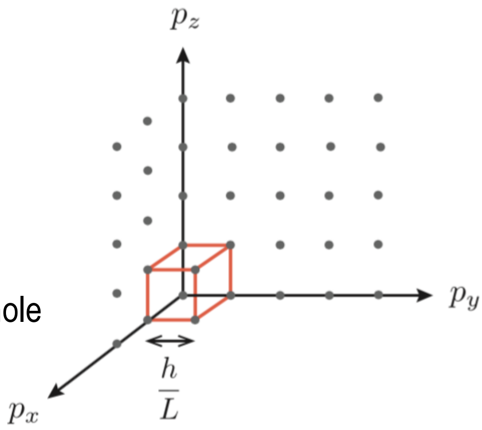
The phase-space of a species in Quantum physics:

Therefore the allowed **momentum eigenstates** in one octant of the $\vec{n} = (n_x, n_y, n_z)$ space is ($\vec{p}_n^2 = 2m\,E$):

$$\vec{p}_n = \frac{\hbar\pi}{L}\vec{n} \qquad \text{or} \qquad \vec{n} = \frac{L}{\hbar\pi}\vec{p}_n$$

So the number of points in this **n-space octant** is:

$$d^3n = \left(\frac{L}{\hbar\pi}\right)^3 d^3p$$



Generally, distribution function integrals are done over the whole $\{\boldsymbol{x}, \boldsymbol{p}\}$-space. That would lead to 8 times larger densities, so:

$$d^3n = \frac{1}{8}\left(\frac{L}{\pi\hbar}\right)^3 d^3p = \left(\frac{L}{h}\right)^3 d^3p = \frac{1}{h^3}\, d^3x\, d^3p$$

Phase space density

If particle species have ***g* internal degrees of freedom** the **density of states in** **natural units** **in** $\{\boldsymbol{x}, \boldsymbol{p}\}$ is:

$$\frac{g}{h^3} = \frac{g}{(2\pi)^3}$$

because $\hbar = h/(2\pi) \equiv 1$ and therefore $h = 2\pi$.

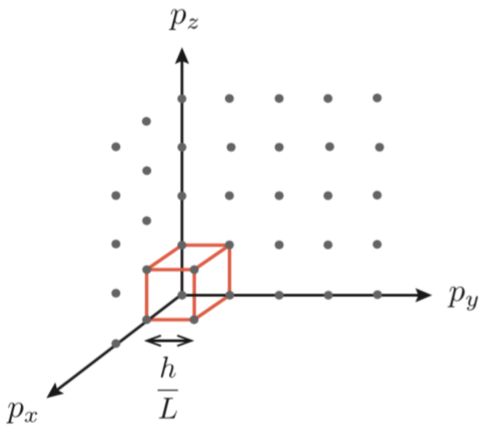
Thermal evolution at equilibrium

From quantum states to microscopic properties:

Under the assumptions of *homogeneity* and *isotropy*, the number of particles $dN = f(\boldsymbol{x}, \boldsymbol{p}, E, t)\, d^3x\, d^3p$ does not depend on $\boldsymbol{x}$ and is only a function of $p = |\boldsymbol{p}|$.

The number density of particle states is therefore:

$$n = \frac{g}{(2\pi)^3}\int \mathrm{d}^3p\, f(p)$$



Likewise, one can obtain the energy density of particles in real space by weighting the each momentum eigen-state by its energy, $E(p) = \sqrt{m^2 + p^2}$, an therefore:

$$\rho = \frac{g}{(2\pi)^3}\int \mathrm{d}^3p\, f(p)\, E(p)$$

The computation of the pressure of particles results in a similar way (This can be derived using statistical mechanics assuming a gas of weakly interacting particles, see next slides).

$$P = \frac{g}{(2\pi)^3}\int \mathrm{d}^3p\, f(p)\, \frac{p^2}{3E}$$

Thermal evolution at equilibrium

Derivation of (done in class),

$$P = \frac{g}{(2\pi)^3} \int \mathrm{d}^3p\, f(p)\, \frac{p^2}{3E}$$

(from Baumann Chap. 3.2)

Lets assume a gas of weakly interacting particles in statistical mechanics.

Consider the area element dA, in the figure on the left. Particles move with $E(|v|)$.

The number of particles in the shaded volume $dV = |v|dt\; dA_s = |v|dt\; d\Omega R^2$ is:

$$\mathrm{d}N = \frac{g}{(2\pi)^3} f(E) \times R^2 |\boldsymbol{v}| \mathrm{d}t\, \mathrm{d}\Omega$$

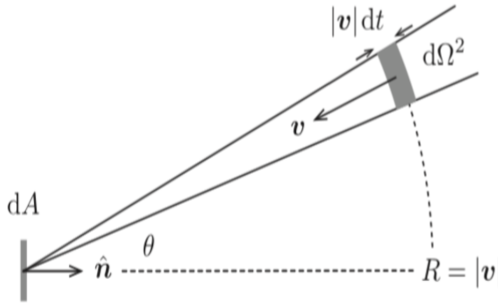


Figure 3.3: Pressure in a weakly interacting gas of particles

Not all particles in dV will hit dA.
*Only a fraction of this particles, with $\hat{v}.\hat{\boldsymbol{n}} = cos(\theta)$, i.e. with the direction, $\boldsymbol{v}$, will hit dA. So, **assuming isotropy**, the number of particles arriving on dA is:*

$$\mathrm{d}N_A = \frac{|\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}}|\, \mathrm{d}A}{4\pi R^2} \times \mathrm{d}N = \frac{g}{(2\pi)^3} f(E) \times \frac{|\boldsymbol{v} \cdot \hat{\boldsymbol{n}}|}{4\pi}\, \mathrm{d}A\, \mathrm{d}t\, \mathrm{d}\Omega$$

Thermal evolution at equilibrium

(Derivation continuation…)

$$\mathrm{d}N_A = \frac{|\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}}|\, \mathrm{d}A}{4\pi R^2} \times \mathrm{d}N = \frac{g}{(2\pi)^3} f(E) \times \frac{|\boldsymbol{v} \cdot \hat{\boldsymbol{n}}|}{4\pi}\, \mathrm{d}A\, \mathrm{d}t\, \mathrm{d}\Omega$$

*If these dN_A particles **collide elastically** at dA, each particle transfers a momentum $2|\boldsymbol{p}.\hat{n}|$ (because the particle is assumed to collide elastically, and is reflected with the same angle of impact).*

So the pressure dP (defined as force / area = momentum / time / area) by these particles at dA is:

$$\begin{aligned} \mathrm{d}P(|\boldsymbol{v}|) &= \int \frac{2|\boldsymbol{p} \cdot \hat{\boldsymbol{n}}|}{\mathrm{d}A\, \mathrm{d}t}\, \mathrm{d}N_A \\ &= \frac{g}{(2\pi)^3} f(E) \times \frac{p^2}{2\pi E} \int \cos^2\theta\, \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi \\ &= \frac{g}{(2\pi)^3} \times f(E)\, \frac{p^2}{3E} \end{aligned}$$

where $|\boldsymbol{v}| = |\boldsymbol{p}|/E$ and the integration is made over the hemisphere of particles moving towards dA (i.e. with $\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}} \equiv -\cos\theta < 0$)

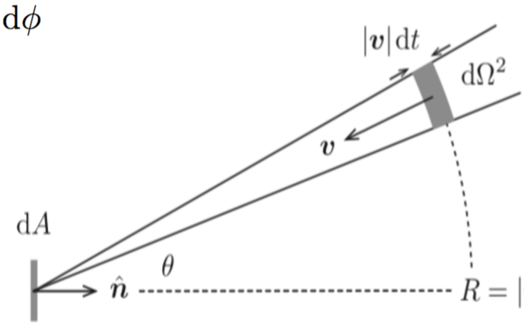


Figure 3.3: Pressure in a weakly interacting gas of particles

Thermal evolution at equilibrium

Kinetic equilibrium

If particles **exchange momentum and energy** in an **efficient way**, the system is said to be in **kinetic equilibrium**. If the system achieves a **maximum entropy state**, then particles are distributed according to the **Fermi-Dirac** or **Bose-Einstein** distribution functions:

$$f(p) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1}$$

+ Fermions
− Bosons

Where T is the temperature of the system and μ is the chemical potential defined as the change of energy with respect of the number of particles, at constant entropy, volume, and number other particle species.

$$\mu_i = \left(\frac{\partial U}{\partial N_i}\right)_{S,V,N_{j\neq i}} \quad \text{or} \quad \mu_i = -T\left(\frac{\partial S}{\partial N}\right)_{U,V,\,N_{j\neq i}}$$

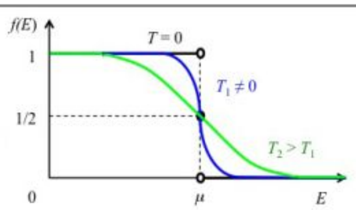
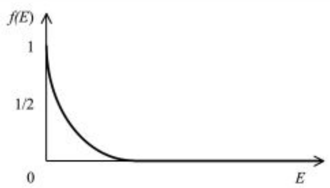
At low temperature $T \ll E - \mu$ both distributions reduce to the **Maxwell-Boltzmann** distribution:

$$f(p) \approx e^{-(E(p)-\mu)/T}$$

Thermal evolution at equilibrium

Particle distribution functions

Quantum Statistics Summary

	Fermi-Dirac distribution	**Bose-Einstein distribution**
Function	$f(E) = \frac{1}{\exp[(E-\mu)/k_B T] + 1}$	$f(E) = \frac{1}{\exp[(E-\mu)/k_B T] - 1}$
Energy Dependence		
Quantum Particles	Undistinguishable particles obeying to the Pauli's Principle: **only one particle per state**	Undistinguishable particles not subject to the Pauli's Principle: **many particles can occupy one state**
Spins	semi-integer spins	integer spins
Properties	At temperature of 0 K, each energy level is occupied by two Fermi particles with opposite spins. Examples: electron, proton, neutron…	At very low temperature, large numbers of Bosons fall into lowest energy state. Examples: photon, gluon, mesons…

Thermal evolution at equilibrium

Chemical equilibrium

- If a particle species, i, is in **chemical equilibrium**, then μ_i is related to the other species chemical potential. For example if one has the following interaction (reaction) among species:

$$1 + 2 \leftrightarrow 3 + 4 \quad \text{then} \quad \mu_1 + \mu_2 = \mu_3 + \mu_4$$

- Photons have chemical potential equal to zero, i.e. $\boldsymbol{\mu_\gamma = 0}$, because the number of photons is not conserved. For example: double scattering interaction $e^- + \gamma \leftrightarrow e^- + \gamma + \gamma$

- This implies that a particle, $\boldsymbol{X}$, and its antiparticle, $\boldsymbol{\bar{X}}$, ($X + \bar{X} \leftrightarrow \gamma + \gamma$) have symmetric chemical potentials $\boldsymbol{\mu_X = -\mu_{\bar{X}}}$.

Thermal equilibrium

- **Thermal equilibrium** is achieved for **species which are both in kinetic and chemical equilibrium**. These species then share the same temperature, $T_i = T$.

Thermal evolution at equilibrium:

Using the distribution functions one can compute the number and energy densities, and pressure from their expressions in slides 11, 12, with, $E(p) = \sqrt{m^2 + p^2}$:

$$n = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p \, f(p)$$

$$\rho = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p \, f(p) \, E(p)$$

$$P = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p \, f(p) \, \frac{p^2}{3E}$$

- In general these expressions are solved numerically.

- However, for some cases of interest it is possible to derive analytical solutions.

- These are the cases of ultra-relativistic particles ($\mathrm{m} \ll T$) and non-relativistic ($m \gg T$) with vanishing chemical potential ($\mu = 0$)

Thermal evolution at equilibrium:

Whenever the **chemical potential is zero** (photons) or **negligible** (e.g. electrons and protons) the number and energy densities can be written as:

$$n = \frac{g}{2\pi^2}\int_0^\infty \mathrm{d}p\, \frac{p^2}{\exp\left[\sqrt{p^2+m^2}/T\right]\pm 1}$$

$$\rho = \frac{g}{2\pi^2}\int_0^\infty \mathrm{d}p\, \frac{p^2\sqrt{p^2+m^2}}{\exp\left[\sqrt{p^2+m^2}/T\right]\pm 1}$$

Defining $\boldsymbol{x} \equiv \boldsymbol{m/T}$ and $\boldsymbol{\xi} \equiv \boldsymbol{p/T}$ these integrals can be written as

$$n = \frac{g}{2\pi^2}T^3\, I_\pm(x)\ , \qquad I_\pm(x) \equiv \int_0^\infty \mathrm{d}\xi\, \frac{\xi^2}{\exp\left[\sqrt{\xi^2+x^2}\right]\pm 1}$$

$$\rho = \frac{g}{2\pi^2}T^4\, J_\pm(x)\ , \qquad J_\pm(x) \equiv \int_0^\infty \mathrm{d}\xi\, \frac{\xi^2\sqrt{\xi^2+x^2}}{\exp\left[\sqrt{\xi^2+x^2}\right]\pm 1}$$

Which in some cases can be evaluated analytically using the Riemann-Zeta and Gama functions. In particular one has:

$$\int_0^\infty \mathrm{d}\xi\, \frac{\xi^n}{e^\xi - 1} = \zeta(n+1)\,\Gamma(n+1)\ ,$$

$$\int_0^\infty \mathrm{d}\xi\, \xi^n e^{-\xi^2} = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}(n+1)\right)\ ,$$

Thermal evolution at equilibrium:

Ultra-relativistic limit: $x \to 0$ ($m \ll T$ and $\mu = 0$)

For $x \to 0$ ($m \ll T$) on has for the integral part of the number density:

$$I_\pm(0) = \int_0^\infty \mathrm{d}\xi\, \frac{\xi^2}{e^\xi \pm 1}$$

Bosons:

$$I_-(0) = \zeta(2+1)\Gamma(2+1) = 2\zeta(3) \simeq 2.4$$

Fermions:

$$I_+(0) = I_-(0) - 2\left(\tfrac{1}{2}\right)^3 I_-(0) = \tfrac{3}{4} I_-(0) = \tfrac{3}{2}\zeta(3)$$

For **Fermions** the integral is not directly related with the Riemann integrals. However one can use the mathematical equality,

$$\frac{1}{e^\xi + 1} = \frac{1}{e^\xi - 1} - \frac{2}{e^{2\xi} - 1}$$

and then apply the Riemann integral.

Thermal evolution at equilibrium:

Ultra-relativistic limit: $x \to 0$ ($m \ll T$ and $\mu = 0$)

So one obtains the following expressions for the **number density**:

$$n = \frac{\zeta(3)}{\pi^2} g T^3 \begin{cases} 1 & \text{bosons} \\ \frac{3}{4} & \text{fermions} \end{cases}$$

Doing a similar computation for the $J_{\pm}(0)$, it is possible to derive the following expression for the **energy density**:

$$\rho = \frac{\pi^2}{30} g T^4 \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases}$$

To compute the pressure for ultra-relativistic particles, $x \to 0$, with μ=0, it is straightforward to show that:

$$P = \frac{1}{3}\rho$$

Thermal evolution at equilibrium:

Non-relativistic limit: $x \gg 1$ ($m \gg T$ and $\mu = 0$)

For $x \gg 1$ ($m \gg T$) the number density integral gives the **same expression for Fermions and Bosons**:

$$I_{\pm}(x) \approx \int_0^\infty d\xi \, \frac{\xi^2}{e^{\sqrt{\xi^2+x^2}}}$$

Most of the contribution to this integral comes from $\xi \ll x$. We can therefore expend the square root in a Taylor expansion to the lowest order in ξ to obtain:

$$I_{\pm}(x) \approx \int_0^\infty d\xi \, \frac{\xi^2}{e^{x+\xi^2/(2x)}} = e^{-x} \int_0^\infty d\xi \, \xi^2 e^{-\xi^2/(2x)} = (2x)^{3/2} e^{-x} \int_0^\infty d\xi \, \xi^2 e^{-\xi^2}$$

This last integral is related with the Gamma Function integral with $\mathrm{n} = 2$. So one gets:

$$I_{\pm}(x) = \sqrt{\frac{\pi}{2}} \, x^{3/2} e^{-x}$$

Which leads to:

Thermal evolution at equilibrium:

Non-relativistic limit: $x \gg 1$ ($m \gg T$ and $\mu = 0$)

The **number density** of non-relativistic particles

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

This tell us that massive particles are exponentially rare at low temperatures.

For the **energy density**, at low temperature one has

$$E(p) = \sqrt{m^2 + p^2} \approx m + p^2/2m$$

The energy density integral can be obtained using this expression:

$$\rho = mn + \frac{3}{2}nT$$

The **pressure** can be also easily computed, giving

$$P = nT$$

Thermal evolution at equilibrium:

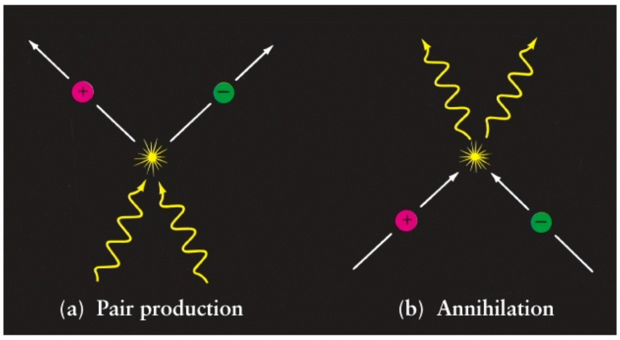
Non-relativistic limit: $x \gg 1$ ($m \gg T$ and $\mu = 0$)

From these expressions one concludes that:

- The densities and pressure of non-relativistic particles are strongly suppressed, by the exponential term $e^{-m/T}$, as temperature, T, drops bellow the particles mass, m. This is known as **Boltzmann suppression** and is due to **particle annihilations**.

- These annihilations occur due to **changes in the interactions** involving the particle species. For example in the case of $X + \bar{X} \leftrightarrow \gamma + \gamma$ (**particle-antiparticle pair production)** at low temperature (typically below $\sim m$), the thermal particle energies are not sufficient for pair production.

- Particle annihilations are typically associated with phase transitions, such as happens to the less massive quarks in the QCD phase transition.



Thermal evolution at equilibrium:

Non-relativistic limit: $x \gg 1$ ($m \gg T$ and $\mu = 0$)

From these expressions one concludes that:

- The transition from relativistic to non-relativistic behaviour is not instantaneous (in fact about **80% of particle-antiparticle annihilations** take place in the temperature range $\boldsymbol{T} \in [\boldsymbol{m/6, m}]$).

- When $m \gg T$ the energy density and pressure of non-relativistic particles,
  - $\rho = n\left(m + \frac{3}{2}T\right) \simeq nm$
  - $P = nT \ll \rho = nm$

- This means that non relativistic particles have in general negligible pressure. They behave as a "pressureless dust", (i.e. as P=0 'matter')

- Note that $P = nT \Leftrightarrow PV = Nk_BT$ (in SI units) is the ideal gas law.

In a nutshell: decoupled non-relativistic particles behave as a gas of pressureless matter

Thermal evolution at equilibrium:

Effective number of degrees of freedom of relativistic species

For a plasma of relativistic species, with bosons (labelled by i) and fermions (labelled by j) we have that:

$$\left.\begin{aligned} \rho_B^{(i)} &= \frac{\pi^2}{30} g_i T_i^4, \\ \rho_F^{(j)} &= \frac{7}{8}\frac{\pi^2}{30} g_j T_j^4 \end{aligned}\right\} \quad \rho_r = \sum_{i\ \text{bosoes}} \frac{\pi^2}{30} g_i T_i^4 + \sum_{i\ \text{fermioes}} \frac{7}{8}\frac{\pi^2}{30} g_i T_i^4$$

The total energy density of relativistic species can therefore be written as:

$$\boxed{\rho_r = \sum_i \rho_i = \frac{\pi^2}{30} g_\star(T) T^4}$$

where $T = T_\gamma$ is the photons temperature and $g_*$is the ***effective number of degrees of freedom*** of the fluid at temperature T:

$$\boxed{g_* = \sum_{i\ \text{bosões}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{j\ \text{fermiões}} g_j \left(\frac{T_j}{T}\right)^4}$$

Thermal evolution at equilibrium:

Effective number of degrees of freedom of relativistic species

This expression allows that different species may not be in thermal equilibrium with the photon component. In fact we can distinguish two situations:

- For **relativistic particles in thermal equilibrium with the photons** we have:

$$g_\star^{th}(T) = \sum_{i=b} g_i + \frac{7}{8} \sum_{i=f} g_i$$

when a species become non-relativistic, it is removed from the sums in g_*^{th}.
So, when T is away from the "mass thresholds" of particles g_*^{th} is independent of temperature

- For **relativistic particles that are not in thermal equilibrium (or decoupling)** from the photon fluid, g_* varies with temperature:

$$g_\star^{dec}(T) = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=f} g_i \left(\frac{T_i}{T}\right)^4$$

Thermal evolution at equilibrium:

Inventory of internal degrees of freedom of fundamental particles

type		mass	spin	g
quarks	$t, \bar{t}$	173 GeV	$\frac{1}{2}$	$2 \cdot 2 \cdot 3 = 12$
	$b, \bar{b}$	4 GeV		
	$c, \bar{c}$	1 GeV		
	$s, \bar{s}$	100 MeV		
	$d, \bar{s}$	5 MeV		
	$u, \bar{u}$	2 MeV		
gluons	g_i	0	1	$8 \cdot 2 = 16$
leptons	$\tau^\pm$	1777 MeV	$\frac{1}{2}$	$2 \cdot 2 = 4$
	$\mu^\pm$	106 MeV		
	$e^\pm$	511 keV		
	$\nu_\tau, \bar{\nu}_\tau$	< 0.6 eV	$\frac{1}{2}$	$2 \cdot 1 = 2$
	$\nu_\mu, \bar{\nu}_\mu$	< 0.6 eV		
	$\nu_e, \bar{\nu}_e$	< 0.6 eV		
gauge bosons	W^+	80 GeV	1	3
	W^-	80 GeV		
	Z^0	91 GeV		
	γ	0		2
Higgs boson	H^0	125 GeV	0	1

Internal degrees of freedom of fundamental particles in the Standard Model of Particle Physics:

- Massless spin-1 (photons and gluons): 2 polarizations
- Massive spin-1 ($W^\pm, Z^0$): 3 polarizations
- Massive spin-1/2 leptons ($e^\pm, \mu^\pm, \tau^\pm$): 2 spins
- Massive spin-1/2 quarks: 2 spin and 3 colour states
- Neutrinos/anti-neutrinos: 1 helicity state

So **the internal degrees of freedom for relativistic bosons and fermions in equilibrium** are:

$g_b = 28$ photons (2), $W^\pm$ and Z^0 $(3 \cdot 3)$, gluons $(8 \cdot 2)$, and Higgs (1)

$g_f = 90$ quarks $(6 \cdot 12)$, charged leptons $(3 \cdot 4)$, and neutrinos $(3 \cdot 2)$

This gives:

$$g_\star = g_b + \frac{7}{8} g_f = 106.75$$

Thermal evolution at equilibrium:

Evolution of relativistic degrees of freedom (SMPP)

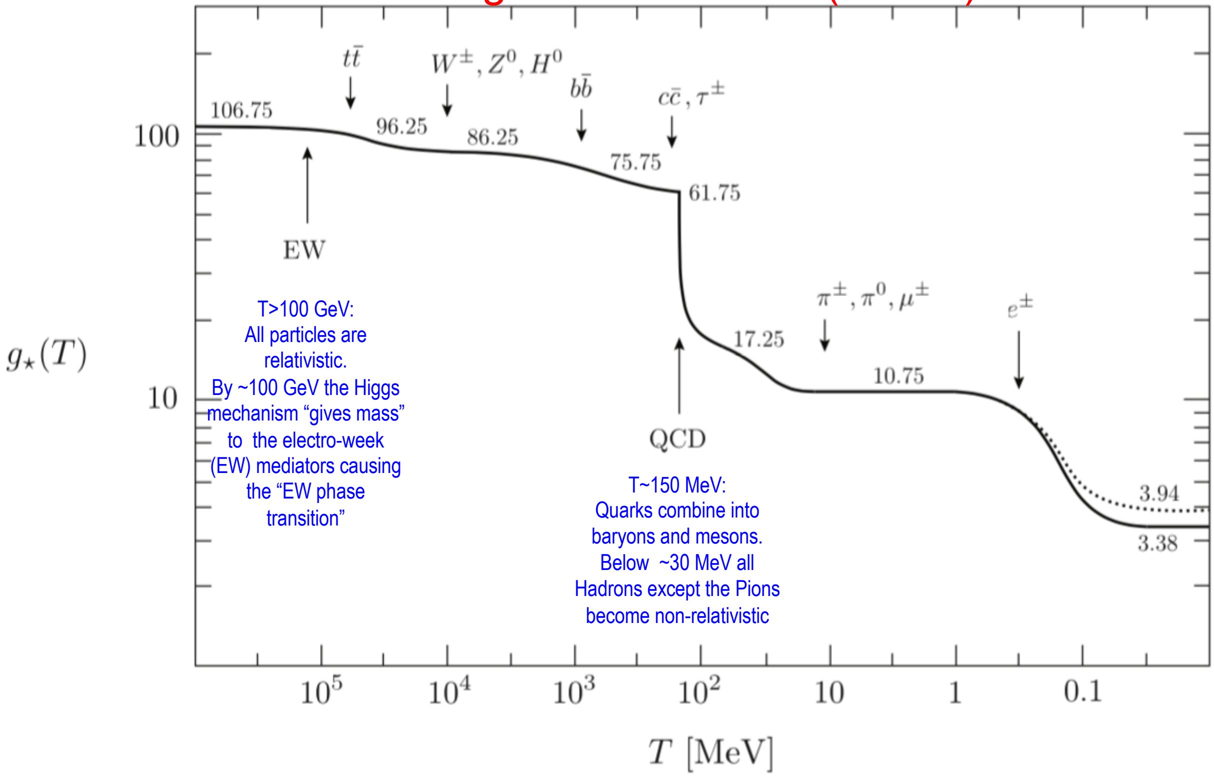


Figure 3.4: *Evolution of relativistic degrees of freedom $g_\star(T)$ assuming the Standard Model particle content. The dotted line stands for the number of effective degrees of freedom in entropy $g_{\star S}(T)$.*

Thermal evolution at equilibrium:

Entropy at equilibrium

From to the first law of thermodynamics ($\mathrm{d}U = TdS - PdV$; with $\mu_i = 0$) one has:

$$TdS \underset{(1)}{=} d(\rho V) + PdV \underset{(2)}{=} d\big[(\rho + P)V\big] - VdP \underset{(3)}{=} Vd\rho + (\rho + P)dV.$$

From (3) one can derive that:

$$\left(\frac{\partial S}{\partial \rho}\right)_V = \frac{V}{T} \qquad\qquad \left(\frac{\partial S}{\partial V}\right)_\rho = \frac{\rho + P}{T}$$

The Schwartz theorem applied to the thermodynamic variable Free Energy: $dF = -SdT - PdV$ allows one to write:

$$\frac{\partial P}{\partial T} = \frac{\partial S}{\partial V} = \frac{\rho + P}{T}$$

From (2) and the above equation on obtains:

$$\begin{aligned} \mathrm{d}S &= \frac{1}{T}\Big(\mathrm{d}[(\rho + P)V] - V\mathrm{d}P\Big) \\ &= \frac{1}{T}\mathrm{d}\big[(\rho + P)V\big] - \frac{V}{T^2}(\rho + P)\,\mathrm{d}T \\ &= \mathrm{d}\left[\frac{\rho + P}{T}V\right], \end{aligned}$$

Thermal evolution at equilibrium:

Entropy at equilibrium

This expression allows defining **entropy** and **entropy density** (or specific entropy), up to a constant, as:

$$S = \frac{\rho + P}{T} V \qquad s \equiv \frac{S}{V} = \frac{\rho + P}{T}$$

The **specific entropy of a relativistic boson species** i can then be computed as (using the expressions of ρ_i, P_i, obtained earlier):

$$s_i = \frac{\pi^2}{30} g_i \left(1 + \frac{1}{3}\right) \frac{T_i^4}{T_i} = \frac{2\pi^2}{45} g_i T_i^3$$

Relativistic Bosons

where the 1/3 term comes from the pressure $P_i = \rho_i/3$.

A similar result holds for **relativistic fermion species**:

$$s_i = \frac{7}{8} \frac{\pi^2}{30} \left(1 + \frac{1}{3}\right) g_i T_i^3 = \frac{7}{8} \frac{2\pi^2}{45} g_i T_i^3$$

Relativistic Fermions

Thermal evolution at equilibrium:

Entropy at equilibrium

For a plasma of relativistic species, with bosons (labelled by i) and fermions (labelled by j) we have that:

$$\left.\begin{aligned} s_B &= \frac{2\pi^2}{45} g_i T_i^3 \\ s_F &= \frac{7}{8} \frac{2\pi^2}{45} g_i T_i^3 \end{aligned}\right\} \quad s = \sum_{i \text{ bosoes}} \frac{2\pi^2}{45} g_i T_i^3 + \sum_{i \text{ fermioes}} \frac{7}{8} \frac{2\pi^2}{45} g_i T_i^3$$

The total energy density of relativistic species can therefore be written as:

$$s = \frac{2\pi^2}{45} g_{*s} T^3$$

where $T = T_\gamma$ is the photons temperature and g_* is the ***effective number of degrees of freedom in entropy*** of the fluid at temperature T:

$$g_{*s} = \sum_{i \text{ bosões}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{j \text{ fermiões}} g_j \left(\frac{T_j}{T}\right)^3$$

Thermal evolution at equilibrium:

Entropy at equilibrium

One should note that g_{*s} is a function of $(T_i/T)^3$ whereas g_*, varies as $(T_i/T)^4$
This means that:

- Relativistic species in thermal equilibrium ($T_i = T$): $g_{*s} = g_*$
- Non-relativistic decoupling species ($T_i \neq T$): $g_{*s} \neq g_*$

In other words, if one writes

$$g_{\star S}(T) = g_{\star S}^{th}(T) + g_{\star S}^{dec}(T)$$

One has that $g_{\star S}^{th}(T) = g_{\star}^{th}(T)$ for relativistic species in thermal equilibrium, and $g_{\star S}^{dec}(T) \neq g_{\star}^{dec}(T)$ for non-relativistic species in the process of decoupling from fluid.

Slide 29 shows both g_{*s} (dotted line) and g_* (solid line).

At high values of the degrees of freedom (i.e. higher temperatures) the curves appear on top of each other because the differences are small and only more visible at low T.

Thermal evolution at equilibrium:

Conservation of Entropy

A most important result about the evolution of the fluid in thermal equilibrium is that its **entropy remains constant with the expansion** (as opposed to its energy density that decreases with time).
This can be proved by taking the **time derivative** of S:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\rho + P}{T} V\right]$$
$$= \frac{V}{T}\left[\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t}(\rho + P)\right] + \frac{V}{T}\left[\frac{\mathrm{d}P}{\mathrm{d}t} - \frac{\rho + P}{T}\frac{\mathrm{d}T}{\mathrm{d}t}\right] = 0$$

- The first term vanishes, because

  $$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$$

  (FLRW continuity equation) and $V = L^3 a^3$.

- The second term also vanishes, because

  $$\frac{\partial P}{\partial T} = \frac{\partial S}{\partial V} = \frac{(\rho + P)}{T}$$

Thermal evolution at equilibrium:

Conservation of Entropy

Entropy conservation has **two important consequences**:

- From $S = sV = const. \Rightarrow s \propto a^{-3}$

  In fact, whenever the number density $n = N/V \propto a^{-3}$ (i.e. away particle mass thresholds) one also has that $n/s = N/S$. Since $S = const.$, one can set it to 1, to conclude that $n/s = N$. The same holds for individual species:

$$N_i \equiv \frac{n_i}{s}$$

- Since $s = \frac{2\pi^2}{45} g_{*s} T_\gamma^3$ and $sV = const.$ one has:

$$g_{*S}(T)T^3a^3 = g_{*S}(T_i)T_i^3a_i^3 = const. \longrightarrow T \propto g_{\star S}^{-1/3} a^{-1}$$

  Away from particle mass thresholds ($g_{*s} = const.$) one concludes that the **temperature of a relativistic fluid scales with the inverse of** $a(t)$**.** More generally ($A_i = g_{*S}^{1/3}\, T_i a_i$):

$$T = T_i \left(\frac{g_{*S}(T_i)}{g_{*S}(T)} \right)^{1/3} \frac{a_i}{a} = A_i\, g_{*S}^{-1/3} a^{-1}$$

Thermal evolution at equilibrium:

Conservation of Entropy: Temperature – time dependence

We can now combine this equation in energy density equation of relativistic particles and in the Friedmann equation to relate temperature with density and time. We have:

$$\rho_r = \frac{\pi^2}{30} g_* T^4 = \frac{\pi^2}{30} g_* \left(A_i\, g_{*S}^{-1/3} a^{-1} \right)^4 = \frac{\pi^2}{30} A_i^4 \left(g_* g_{*S}^{-4/3} \right) a^{-4}$$

Plugging this result in the Friedman Equation (accounting only for relativistic particles) one obtains:

$$H^2 = \frac{8\pi G}{3} \rho_r = \frac{8\pi G}{3} \frac{\pi^2}{30} A_i^4 \left(g_* g_{*S}^{-4/3} \right) a^{-4}$$

These results show that, whenever $g_*(T)$ and $g_{*s}(T)$ are constants (i.e. **away from particle mass thresholds)** one obtains:

- the well know scaling for radiation $\boldsymbol{\rho_r \propto a^{-4}}$
- the solution of the Friedman equation with $\boldsymbol{\rho = \rho_r \propto a^{-4}}$ is: $\boldsymbol{a \propto t^{1/2}}$

At particle mass thresholds $g_*(T)$ and $g_{*s}(T)$ are a function of temperature. The solution of the Friedmann equation is numerical and generally leads to deviations to the $\boldsymbol{a \propto t^{1/2}}$ scaling.

Plugging this in the temperature scaling, one finally obtains $\boldsymbol{T \propto g_{*s} a^{-1} \propto t^{-1/2}}$.

Thermal evolution at equilibrium:

Conservation of Entropy: Temperature – time dependence

Doing the maths, one can obtain the exact time dependence of the temperature of the relativistic fluid. Typically one obtains:

$$\frac{T}{1\,\mathrm{MeV}} \simeq 1.5\, g_\star^{-1/4} \left(\frac{1\,\mathrm{sec}}{t}\right)^{1/2}$$

(which allows to write the rule of thumb: $T \sim 1\,\mathrm{MeV}$ at about 1 second after the Big-Bang)

This expression allows one to establish a direct **correspondence between a given energy scale of the relativistic fluid and time** until the end of the radiation domination period. Beyond the radiation domination phase one needs to account for the other terms in the Friedman equation (see next slide).

Thermal evolution at equilibrium:

Key events in the thermal history of the universe

Event	time t	redshift z	temperature T
Inflation	10^{-34} s (?)	–	–
Baryogenesis	?	?	?
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV
Reionization	100–400 Myr	11–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	13.8 Gyr	0	0.24 meV

The previous sets of equations allows one compute all thermodynamic properties of the primordial relativistic fluid and establish their dependence with time and redshifts.

All we need to know is of what the universe is made of and the physics of each of its components!

Thermal evolution at equilibrium:
Key events in the thermal history of the universe

Baryogenesis:
Quantum field theory requires the existence of anti-particles. This poses a problem: *particle-antiparticle creation and annihilation (allowed by the Heisenberg principle) creates/destroys **equal amounts of particle and anti-particles**.*
However, we do observe an excess of matter (mostly baryons) over anti-matter!
Models of baryogenesis attempt to describe this observational evidence using some dynamical mechanism (instead of assuming this particle-anti-particle asymmetry *ab initio*)

Electroweak phase transition:
At ~100 GeV particles acquire mass through the Higgs mechanism. This leads to a drastic change of the weak interaction. The gauge bosons Z^0, $W^{\pm}$ become massive and soon after decouple from thermal equilibrium.

QCD phase transition:
Above ~150 MeV quarks are asymptotically free (i.e. weakly interacting). Below this energy/mass threshold the strong force (mediated by the gluons) becomes more intense; the more massive quarks start to decouple from the fluid. The less massive become confined (with the gluons) inside the baryons (3 quarks + gluons) and mesons (quarks+anti-quark + gluons)

Thermal evolution at equilibrium:
Key events in the thermal history of the universe

Dark Matter freeze-out:
Present observations indicates that dark matter is very-weakly interacting (or non-interacting). Depending on the mass of the dark matter candidates one should expect that it should decouple from the fluid early on. For example, if dark matter is made of WIMPs (weakly interactive massive particle), one should expect that their abundance should freeze around 1 MeV

Neutrino decoupling:
Neutrinos only interact with the rest of the plasma through the weak force. They are expected to decouple from the fluid at ~0.8 MeV.

Electron-positron annihilation:
Electrons and positron annihilate soon after the neutrinos. Positrons vanish, because electron-positron pair production is strongly suppressed below ~1MeV

Big Bang Nucleosynthesis:
At ~0.1MeV (~3 minutes after the Big-Bang) protons and neutrons combine to form the first light nuclear elements.

Thermal evolution at equilibrium:

Key events in the thermal history of the universe

Recombination:

At ~0.3 eV (260-380 kyr) free electrons combine with nuclei to form atoms. Predominantly Hydrogen: $e^{-}+p^{+}\rightarrow \mathrm{H}+\gamma$. Below this range of energies, this chemical reaction can no longer occur in the reverse order.

Photon decoupling:

By ~0.23 eV (380 kyr) the primordial fluid is reduced to photons, that no longer interact with matter (free electrons). The Cosmic Microwave Background radiation propagates freely in the Universe.

Thermal evolution at equilibrium:

Brief history of the Universe