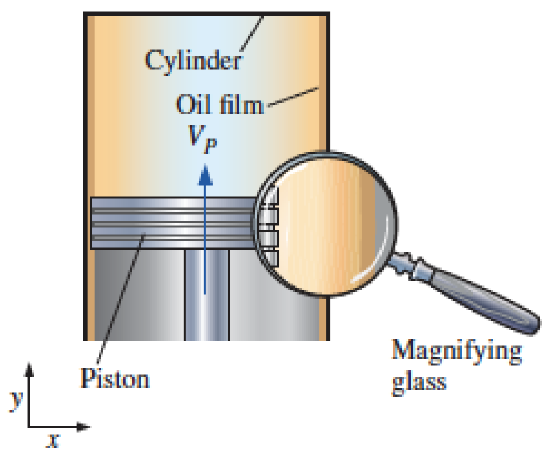
Boundary conditions

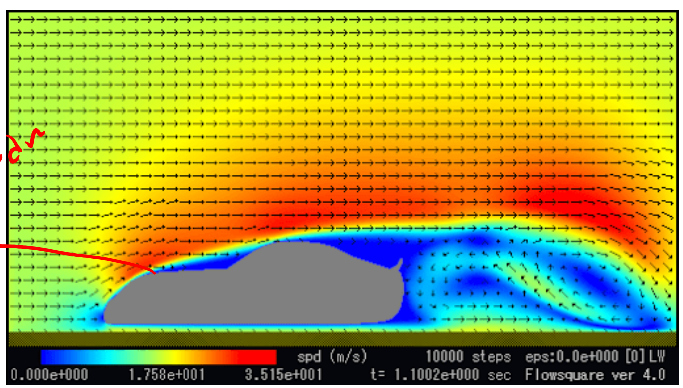
- The most-used boundary condition is the no-slip condition, which states that for a fluid in contact with a solid wall, the velocity of the fluid must equal that of the wall,

No-slip boundary condition:

$$\boxed{\vec{V}_{\text{fluid}} = \vec{V}_{\text{wall}}}$$



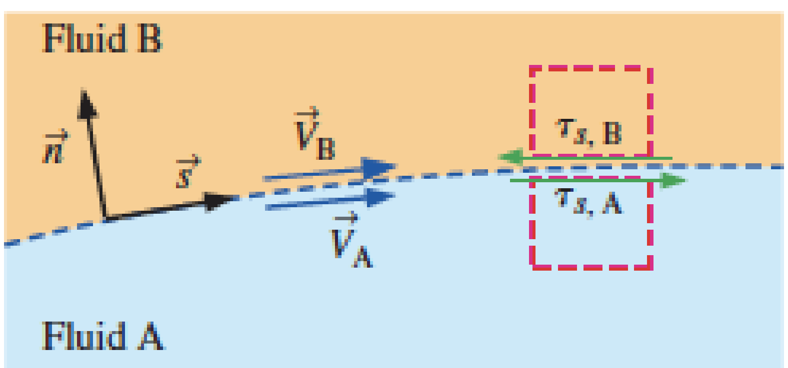
$\vec{u} = \vec{u}_{car}$

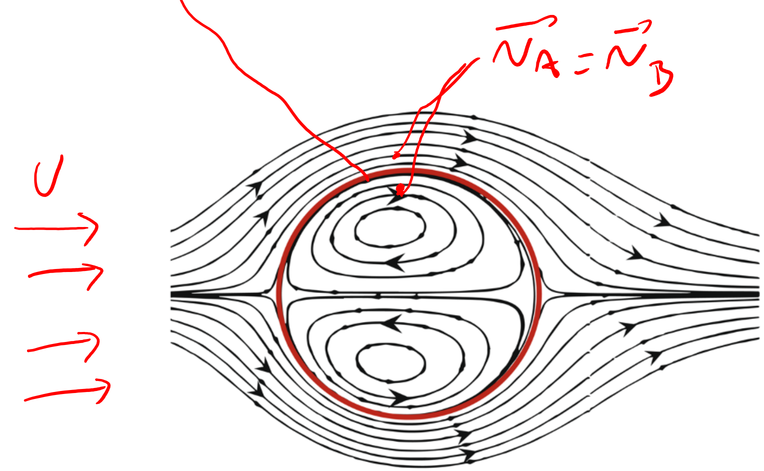


Boundary conditions

- When two fluids (fluid A and fluid B) meet at an interface, the interface boundary conditions are

Interface boundary conditions: $\vec{V}_A = \vec{V}_B$ and $\tau_{s,A} = \tau_{s,B}$

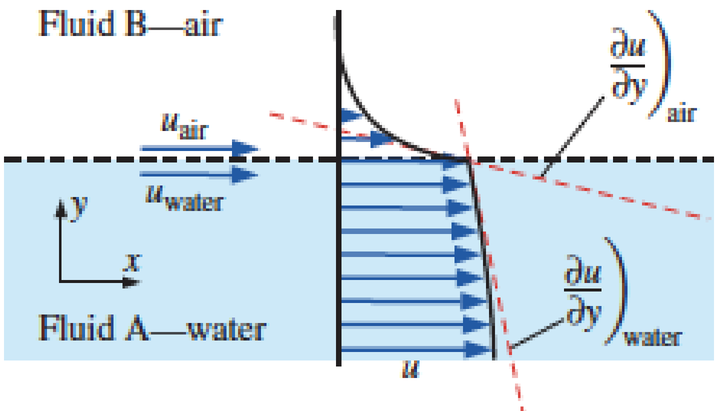




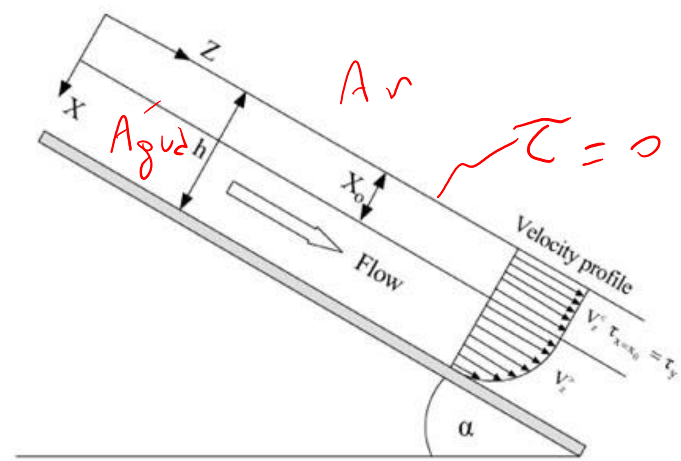
Boundary conditions

- For a liquid in contact with a gas, with negligible surface tension effects, the free- surface boundary conditions are

Free-surface boundary conditions: $P_{\text{liquid}} = P_{\text{gas}}$ and $\tau_{s,\,\text{liquid}} \cong 0$

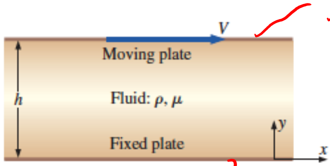


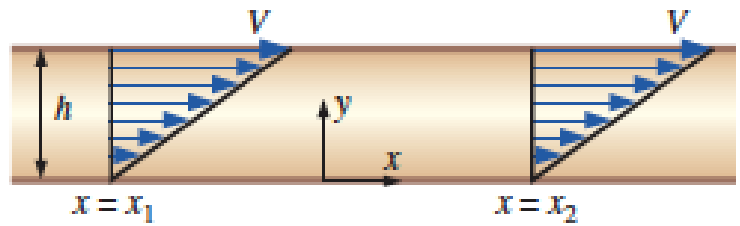
$\mu_{ar} << \mu_{água}$



Fully developed Couette flow

$\vec{u} = V\hat{x}$





$\odot z$

$\vec{u} = 0$

$\frac{\partial \vec{u}}{\partial t} = 0$

- Consider steady, incompressible, laminar flow of a Newtonian fluid in the narrow gap between two infinite parallel plates. The top plate is moving at speed V, and the bottom plate is stationary. The distance between these two plates is h, and gravity acts in the negative z-direction (into the page).

- The boundary conditions come from imposing the no-slip condition: (1) At the bottom plate (y = 0), u = v = w = 0. (2) At the top plate (y = h), u = V, v = 0, and w = 0.

$\nabla \cdot \vec{u} = 0$

- Continuity

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} = 0$$

Invariante por Translação

Result of continuity: $u = u(y)$ only

Navier-Stokes x, y and z componentes:

There is no applied pressure gradient pushing the flow in the x-direction; the flow establishes itself due to viscous stresses caused by the moving upper plate.

$$\cancel{\frac{\partial \vec{u}}{\partial t}}^{\,0} + \underbrace{\vec{u}\cdot\nabla\vec{u}}_{=0} = -\frac{\nabla p}{\rho} + \vec{g} + \eta\nabla^2\vec{u}$$

In the x direction:

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \underbrace{u}_{\neq 0}\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{=0} + \underbrace{\cancel{v}}_{0}\cancel{\frac{\partial u}{\partial y}} + \underbrace{\cancel{w}}_{0}\cancel{\frac{\partial u}{\partial z}}\right) = -\cancel{\frac{\partial P}{\partial x}} + \underbrace{\rho g_x}$$

$$+\mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{=0\text{, pois } \frac{\partial u}{\partial x}=0} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right) \rightarrow \frac{d^2u}{dy^2} = 0$$

In the y direction:

$$\frac{\partial P}{\partial y} = 0$$

Result of y-momentum: $P = P(z)$ only

In the z direction:

$$\frac{\partial P}{\partial z} = -\rho g \rightarrow \frac{dP}{dz} = -\rho g$$

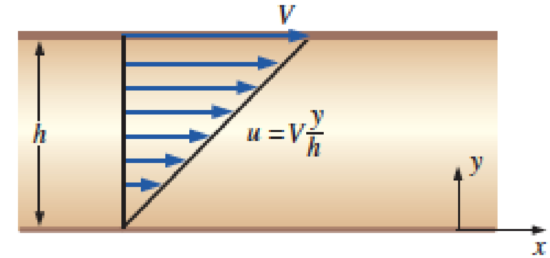
Velocity field

$$u = C_1 y + C_2$$

$$\frac{d^2 u}{\partial y^2} = 0$$

$u(y = h) = V \Rightarrow C_1 = \frac{V}{h}$

$u(y = 0) = 0 \Rightarrow C_2 = 0$

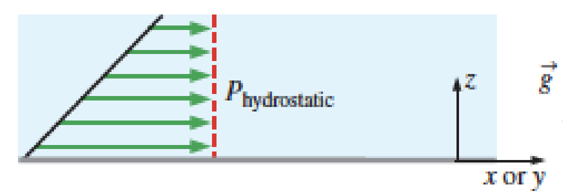


Final result for velocity field: $\boxed{u = V\frac{y}{h}}$

Pressure field

$$P = -\rho g z + C_3$$

Final solution for pressure field: $\boxed{P = P_0 - \rho g z}$

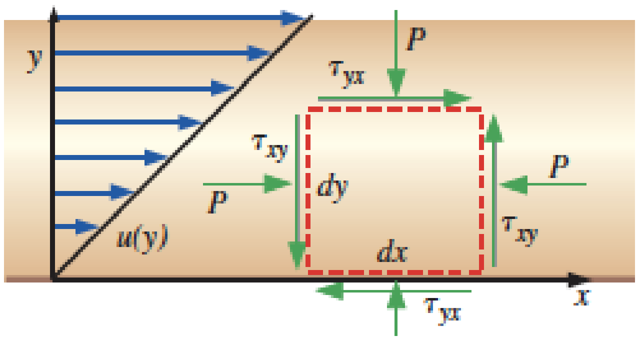


For incompressible flow fields without free surfaces, hydrostatic pressure does not contribute to the dynamics of the flow field.

Shear force on the bottom plate

Deviatoric shear stress tensor

$$\tau_{ij} = \begin{pmatrix} 2\mu\dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & 2\mu\dfrac{\partial v}{\partial y} & \mu\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & 2\mu\dfrac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & \mu\dfrac{V}{h} & 0 \\ \mu\dfrac{V}{h} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Rotational flow

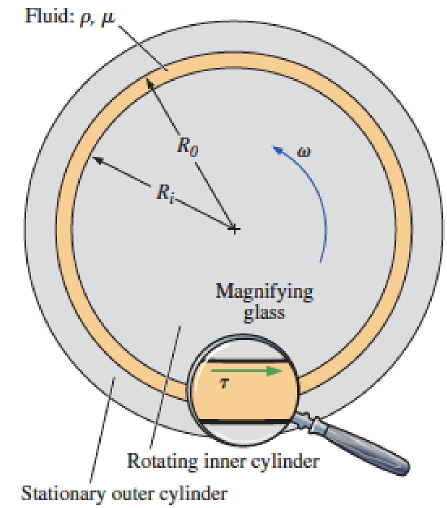
Shear force per unit area acting on the wall: $\dfrac{\vec{F}}{A} = \mu\dfrac{V}{h}\vec{i}$

Discussion The z-component of the linear momentum equation is uncoupled from the rest of the equations; this explains why we get a hydrostatic pressure distribution in the z-direction, even though the fluid is not static, but moving.

The viscous stress tensor is constant everywhere in the flow field, not just at the bottom wall (note that the components of the tensor are not a function of location).

Rotational viscometer

The gap between the two cylinders is very small and contains the fluid.



The magnified region is nearly identical to the parallel plates setup since the gap is small, i.e. $(R_o - R_i) << R_o$.

In a viscosity measurement, the angular velocity of the inner cylinder, ω, is measured, as is the applied torque, $T_{applied}$, required to rotate the cylinder.

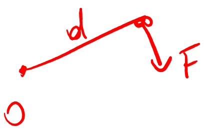
From the previous example, we know that the viscous shear stress acting on a fluid element adjacent to the inner cylinder is approximately equal to

$$\tau = \tau_{yx} \cong \mu \frac{V}{\underbrace{R_o - R_i}_{h}} = \mu \frac{\omega R_i}{R_o - R_i}$$

τ acts to the right on the fluid element adjacent to the inner cylinder wall; hence, the force per unit area acting on the inner cylinder at this location acts to the left with the same magnitude.

The total clockwise torque acting on the inner cylinder wall due to fluid viscosity is thus equal to this shear stress times the wall area times the moment arm,

$T = F \cdot d$



$$\mathrm{T}_{\text{viscous}} = \underbrace{\tau A}_{F} R_i \cong \mu \frac{\omega R_i}{R_o - R_i}\left(2\pi R_i L\right)R_i$$

Under steady conditions, the clockwise torque $\mathrm{T}_{\text{viscous}}$ is balanced by the applied counterclockwise torque $\mathrm{T}_{\text{applied}}$. Equating these we find

Viscosity of the fluid:

$$\boxed{\mu = \underbrace{\mathrm{T}_{\text{applied}}}_{\text{conhecido}} \frac{(R_o - R_i)}{2\pi \underbrace{\omega}_{\text{medido}} R_i^3 L}}$$