Planetary boundary layer

Lecture 3

Homework

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Simplifying the equations

Because the upcoming derivations are sometimes long and involved, it is easy "to lose sight of the forest for the trees". The following summary gives the steps that will be taken in the succeeding sections to develop prognostic equations for mean quantities such as temperature and wind:

Step 1. Identify the basic governing equations that apply to the boundary layer.

Step 2. Expand the total derivatives into the local and advective contributions.

Step 3. Expand dependent variables within those equations into mean and turbulent (perturbation) parts.

Step 4. Apply Reynolds averaging to get the equations for mean variables within a turbulent flow.

Step 5. Add the continuity equation to put the result into flux form.

Additional steps take us further towards understanding the nature of turbulence itself:

Step 6. Subtract the equations of step 5 from the corresponding ones of step 3 to get equations for the turbulent departures from the mean.

Step 7. Multiply the results of step 6 by other turbulent quantities and Reynolds average to yield prognostic equations for turbulence statistics such as kinematic flux or turbulence kinetic energy.

Navier-Stokes (newtonian fluid, uniform rotation, constant gravity)

$$\frac{\partial u_i}{\partial t} = -u_k \frac{\partial u_i}{\partial x_k} - \alpha \frac{\partial P}{\partial x_i} + g_i - \epsilon_{ikm} f_k u_m + \alpha \frac{\partial}{\partial x_k}\left[2\mu\left(e_{i_k} - \frac{1}{3} e_{mm\delta_{ik}}\right)\right]$$

Specific volume:

$$\alpha = \frac{1}{\rho}$$

Deformation rate tensor:

$$e_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

Continuity

$$\left(\frac{d\alpha}{dt} =\right)\frac{\partial \alpha}{\partial t} + u_i \frac{\partial \alpha}{\partial x_i} = \alpha \frac{\partial u_i}{\partial x_i} \ (= \alpha \nabla . \vec{v})$$

Thermodynamics:

$$\frac{\partial \theta}{\partial t} = -u_i \frac{\partial \theta}{\partial x_i} + S_\theta + \frac{\partial}{\partial x_i}\left(\chi \frac{\partial \theta}{\partial x_i}\right)$$

Thermal conductivity χ, diabatic heating source S_θ, potential temperature:

$$\theta = T\left(\frac{P}{P_{00}}\right)^{-R_d/c_p}$$

Reference pressure $P_{00} = 10^5 Pa$.

Scalar conservation (water vapor,…):

$$\frac{\partial q}{\partial t} = -u_i \frac{\partial q}{\partial x_i} + S_q + \frac{\partial}{\partial x_i}\left(\chi_q \frac{\partial q}{\partial x_i}\right)$$

Equation of state:

$$P\alpha = R_d T_v$$

Virtual temperature:

$$T_v = T(1 + 0.61\,q)$$

Simplifying…

Thermodynamic scaling:

$$\alpha = \alpha_0 + \alpha''$$
$$\theta = \theta_0 + \theta''$$
$$\rho = \rho_0 + \rho''$$

Reference state: hydrostatic, adiabatic, barotropic

$\theta_0 = const, \alpha_0(z), \rho_0(z)$

$$\frac{\alpha''}{\alpha_0} \ll 1, etc.$$

Dynamical scaling

$$\hat{u} = \frac{u}{U} \quad ; \quad \hat{v} = \frac{v}{V} \quad ; \quad \hat{w} = \frac{w}{W}$$

$$\hat{x} = \frac{x}{L_x} \quad ; \quad \hat{y} = \frac{y}{L_y} \quad ; \quad \hat{z} = \frac{z}{L_z}$$

$$\hat{t} = \frac{t}{t_\alpha} \quad ; \quad \hat{\alpha}'' = \frac{\alpha''}{A''} \quad ; \quad \hat{\alpha} = \frac{\alpha_0}{A_0}$$

Scales

$$L_x = L_y; \ U = V$$

$$t_\alpha = \frac{L_x}{U}; \ \frac{W}{U} = \frac{L_z}{L_x}$$

Reference scale of height:

$$H = \left(\frac{1}{\alpha_0}\frac{\partial \alpha_0}{\partial z}\right) = \frac{R_d T_0}{g}$$

Simplifying... $\frac{\partial \alpha}{\partial t} + u_i \frac{\partial \alpha}{\partial x_i} = \alpha \frac{\partial u_i}{\partial x_i}$

$$\frac{A"}{L/U}\frac{\partial \hat{\alpha}"}{\partial \hat{t}} = -\frac{UA"}{L_x}\left(\hat{u}\frac{\partial \hat{\alpha}"}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{\alpha}"}{\partial \hat{x}}\right) - \frac{WA"}{L_z}\widehat{w}\frac{\partial \hat{\alpha}"}{\partial \hat{z}} - \frac{WA_0}{L_z}\widehat{w}\frac{\partial \hat{\alpha}_0}{\partial \hat{z}} + \frac{UA_0}{L_x}\hat{\alpha}_0\left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}}\right)\left(1 + \frac{A"}{A_0}\right) + \frac{WA_0}{L_z}\hat{\alpha}_0\frac{\partial \widehat{w}}{\partial \hat{z}}\left(1 + \frac{A"}{A_0}\right)$$

Applying previous relations between scales:

$$\frac{A"}{A_0}\frac{\partial \hat{\alpha}"}{\partial \hat{t}} = -\frac{A"}{A_0}\left(\hat{u}\frac{\partial \hat{\alpha}"}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{\alpha}"}{\partial \hat{x}} + \widehat{w}\frac{\partial \hat{\alpha}"}{\partial \hat{z}}\right) - \frac{L_z}{H}\widehat{w}\hat{\alpha}_0 + \frac{UA_0}{L_x}\hat{\alpha}_0\left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \widehat{w}}{\partial \hat{z}}\right)\left(1 + \frac{A"}{A_0}\right)$$

Imposing $\frac{A"}{A_0} \ll 1$:

$$-\frac{L_z}{H}\widehat{w} + \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \widehat{w}}{\partial \hat{z}}\right) = 0$$

Continuity

$$-\frac{L_z}{H}\widehat{w} + \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \widehat{w}}{\partial \hat{z}}\right) = 0$$

Going back to dimensional variables:

$$-\frac{w}{\alpha_0}\frac{\partial \alpha_0}{\partial z} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \Rightarrow \frac{\partial}{\partial x_i}\left(\frac{u_i}{\alpha_0}\right) = 0$$

This is the anelastic approximation.

If we can accept $L_z \ll H$ (shallow convection approach), we have the solenoidal (or incompressible) approximation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

3^{RD} Navier-Stokes

Define

$$\hat{p} = \frac{pA_0}{fUL_x} + \frac{gz}{fUL_x}$$

where $f = 2\Omega$.

Then:

$$-\alpha\frac{\partial p}{\partial z} = -\frac{\alpha}{A_0}\frac{fUL_x}{L_z}\frac{\partial \hat{p}}{\partial \hat{z}} - g\frac{\alpha}{A_0} = -\frac{fUL_x}{L_z}\hat{\alpha}_0\,\frac{\partial \hat{p}}{\partial \hat{z}} - g\hat{\alpha}_0$$

Replacing in Navier-Stokes for w leads to

$$\frac{UW}{L_x}\left(\frac{\partial \widehat{w}}{\partial \hat{t}} + \hat{u}\frac{\partial \widehat{w}}{\partial \hat{x}} + \hat{v}\frac{\partial \widehat{w}}{\partial \hat{y}}\right) + \frac{W^2}{L_z}\,\widehat{w}\frac{\partial \widehat{w}}{\partial \hat{z}}$$
$$= -\frac{fUL_x}{L_z}\,\hat{\alpha}_0\frac{\partial \hat{p}}{\partial \hat{z}} + fU\cos\phi\,\hat{u} + \nu_0\left[\frac{W}{L_x^2}\left(\frac{\partial^2 \widehat{w}}{\partial \hat{x}^2} + \frac{\partial^2 \widehat{w}}{\partial \hat{y}^2}\right) + \frac{W}{L_z^2}\frac{\partial^2 \widehat{w}}{\partial \hat{z}^2}\right]$$

$$\frac{dw}{dt}$$

After some changes:

$$R_o \delta^2 \frac{d\widehat{w}}{d\hat{t}} = -\hat{\alpha}_0 \frac{\partial \hat{p}}{\partial \hat{z}} + \delta \cos \phi \;\; \hat{u} + \delta^2 E_H \left(\frac{\partial^2 \widehat{w}}{\partial \hat{x}^2} + \frac{\partial^2 \widehat{w}}{\partial \hat{y}^2} + \delta^2 \frac{\partial^2 \widehat{w}}{\partial \hat{z}^2} \right)$$

With the Rossby number:

$$R_o = \frac{U}{f L_x}$$

The horizontal Ekman number:

$$E_H = \frac{\nu_0}{f L_x^2} = \frac{\nu_0}{U L_x} \frac{U}{f L_x} = \frac{R_o}{R_e}$$

the Reynolds number:

$$R_e = \frac{U L_x}{\nu_0}$$

And the aspect ratio

$$\delta = \frac{L_z}{l_X}$$

If $R_o \ll 1, E_H \ll 1, \delta \ll 1$

We get

$$\widehat{\alpha_0}\frac{\partial \hat{p}}{\partial \hat{z}} = 0$$

Or

$$\alpha\frac{\partial p}{\partial z} = -g$$

When vertical acceleration is required

Linearize:

$$g - \alpha \frac{\partial p}{\partial z} = -\alpha_0 \frac{\partial p}{\partial z} + \frac{\alpha''}{\alpha_0} g$$

Also for the state equation:

$$\frac{\alpha''}{\alpha_0} = \frac{\theta''}{\theta_0} - \frac{p''}{\mathrm{p}_0} \approx \frac{\theta''}{\theta_0}$$

Leads to Boussinesq equations ($\beta = \frac{1}{\theta}$ is the isobaric thermal expansion coefficient)

$$\frac{\partial u_i}{\partial t} = -u_k \frac{\partial u_i}{\partial x_k} - \alpha_0 \frac{\partial P}{\partial x_i} - \beta \theta'' g_i - \epsilon_{ikm} f_k u_m + \alpha \frac{\partial}{\partial x_k} \frac{\partial u_i}{\partial x_k}$$

We still need to apply Reynolds decomposition…

$$u_i = U_i + u_i' \ldots$$

Averaging, flux form

$$\frac{\partial U_i}{\partial t} = -U_k \frac{\partial U_i}{\partial x_k} - \alpha_0 \frac{\partial P}{\partial x_i} - \beta\theta" g_i - \epsilon_{ikm} f_k U_m + \nu \frac{\partial}{\partial x_k}\frac{\partial U_i}{\partial x_k} - \frac{\partial}{\partial x_k}\left(\overline{u_i' u_k'}\right)$$

If we take a “first order approximation”:

$$\overline{w'u'} = -K_u \frac{\partial U}{\partial z}$$

The turbulent term looks diffusive.

$$\frac{\partial}{\partial x_k}\left(K \frac{\partial U_i}{\partial x_k}\right)$$

But $K \gg \nu$