Planetary boundary layer

Lecture 4

Homework

4.1 – Florian

4.3 – Jason

4.5 – Maria

4.6 – Mariana

4.7 – Sara

4.8 – Cátia

4.11 – Diogo

Reynolds decomposition and average

$$\frac{\partial \overline{U}_i}{\partial t} + \frac{\partial u_i'}{\partial t} + \overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j} + \overline{U}_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \overline{U}_i}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j} =$$

$$-\delta_{i3} g + \delta_{i3} \left(\frac{\theta_v'}{\overline{\theta}_v} \right) g + f_c \varepsilon_{ij3} \overline{U}_j + f_c \varepsilon_{ij3} u_j' - \left(\frac{1}{\overline{\rho}} \right) \frac{\partial \overline{P}}{\partial x_i} - \left(\frac{1}{\overline{\rho}} \right) \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 \overline{U}_i}{\partial x_j^2} + \nu \frac{\partial^2 u_i'}{\partial x_j^2}$$

$$\frac{\partial \overline{U}_i}{\partial t} + \overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} \overline{U}_j - \left(\frac{1}{\overline{\rho}} \right) \frac{\partial \overline{P}}{\partial x_i} + \nu \frac{\partial^2 \overline{U}_i}{\partial x_j^2} - \frac{\partial \left(\overline{u_i' u_j'} \right)}{\partial x_j}$$

Reynolds decomposition:

$$\frac{\partial u'}{\partial t}$$

$$\frac{\partial \overline{U_i}}{\partial t} + \frac{\partial u_i'}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} + \overline{U_j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \overline{U_i}}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} =$$

$$-\delta_{i3}g + \delta_{i3}\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g + f_c\varepsilon_{ij3}\overline{U_j} + f_c\varepsilon_{ij3}u_j' - \left(\frac{1}{\overline{\rho}}\right)\frac{\partial \overline{P}}{\partial x_i} - \left(\frac{1}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 \overline{U_i}}{\partial x_j^2} + \nu\frac{\partial^2 u_i'}{\partial x_j^2}$$

$$\frac{\partial u_i'}{\partial t} + \overline{U_j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \overline{U_i}}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} =$$

$$+\delta_{i3}\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g + f_c\varepsilon_{ij3}u_j' - \left(\frac{1}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 u_i'}{\partial x_j^2} + \frac{\partial\left(\overline{u_i'u_j'}\right)}{\partial x_j}$$

Scaling

4.2 Free Convection Scaling Variables

Before deriving equations for variances and fluxes, we must detour a bit to learn how experimental data is scaled for presentation. We can then show case study examples of data that correspond to the equations we develop.

Length Scale: Thermals rise until they hit the stable layer capping the ML. As a result, the thermal size scales to z_i. Thermals are the dominant eddy in the convective boundary layer, and all smaller eddies feed on the thermals for energy. Thus, we would expect many turbulent processes to scale to z_i in convective situations.

Velocity Scale: The strong diurnal cycle in solar heating creates a strong heat flux into the air from the earth's surface. The buoyancy associated with this flux fuels the thermals. We can define a ***buoyancy flux*** as $(g/\overline{\theta_v})\,\overline{w'\theta_v'}$.

Scaling

Although the surface buoyancy flux could be used directly as a scaling variable, it is usually more convenient to generate a velocity scale instead, using the two variables we know to be important in free convection: buoyancy flux at the surface, and z_i. Combining these yields a velocity scale known as the ***free convection scaling velocity***, w_*, also sometimes called the ***convective velocity scale*** for short:

$$w_* = \left[\frac{g\, z_i}{\overline{\theta_v}} \left(\overline{w'\theta_v'} \right)_s \right]^{1/3} \qquad (4.2a)$$

This scale appears to work quite well; for example, the magnitude of the vertical velocity fluctuations in thermals is on the same order as w_*. For deep MLs with vigorous heating at the ground, w_* can be on the order of 1 to 2 m/s. Fig 4.1 shows examples of the diurnal variation of w_*.

Scaling

Time Scale: The velocity and length scales can be combined to give the following free convection time scale, t_*:

$$t_* = \frac{z_i}{w_*} \quad (4.2b)$$

This time scale is on the order of 5 to 15 minutes for many MLs. Observations suggest that this is roughly the time it takes for air in a thermal to cycle once between the bottom and the top of the ML.

Temperature Scale: Using surface heat flux with w_*, we can define a temperature scale for the mixed layer, θ_*^{ML}, by:

$$\theta_*^{ML} = \frac{\left(\overline{w'\theta'}\right)_s}{w_*} \quad (4.2c)$$

This scale is on the order of 0.01 to 0.3 K, which is roughly how much warmer thermals are than their environment.

Humidity Scale: Surface moisture flux and w_* can be combined to define a mixed layer humidity scale, q_*^{ML}:

$$q_*^{ML} = \frac{\left(\overline{w'q'}\right)_s}{w_*} \quad (4.2d)$$

Magnitudes are on the order of 0.01 to 0.5 $g_{water}\,(kg_{air})^{-1}$ and scale well to moisture excesses within thermals.

With these convective scales in mind, we can return to the equation derivations.

Average would be 0...

instead:

$$\frac{\partial u_i'}{\partial t} + \overline{U}_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \overline{U}_i}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j} =$$

$$+ \delta_{i3}\left(\frac{\theta_v'}{\overline{\overline{\theta_v}}}\right) g + f_c \varepsilon_{ij3} u_j' - \left(\frac{1}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_j^2} + \frac{\partial (\overline{u_i' u_j'})}{\partial x_j}$$

Basic Derivation. Start with (4.1.1) and multiply by $2u_i'$:

$$2u_i' \frac{\partial u_i'}{\partial t} + 2\overline{U}_j u_i' \frac{\partial u_i'}{\partial x_j} + 2u_i' u_j' \frac{\partial \overline{U}_i}{\partial x_j} + 2u_i' u_j' \frac{\partial u_i'}{\partial x_j} =$$

$$+ 2\delta_{i3} u_i' \left(\frac{\theta_v'}{\overline{\overline{\theta_v}}}\right) g + 2f_c \varepsilon_{ij3} u_i' u_j' - 2\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i} + 2\nu u_i' \frac{\partial^2 u_i'}{\partial x_j^2} + 2u_i' \frac{\partial (\overline{u_i' u_j'})}{\partial x_j}$$

Basic Derivation. Start with (4.1.1) and multiply by $2u_i'$:

$$2u_i' \frac{\partial u_i'}{\partial t} + 2\overline{U}_j u_i' \frac{\partial u_i'}{\partial x_j} + 2u_i'u_j' \frac{\partial \overline{U}_i}{\partial x_j} + 2u_i'u_j' \frac{\partial u_i'}{\partial x_j} =$$

$$+ 2\delta_{i3} u_i' \left(\frac{\theta_v'}{\overline{\theta_v}}\right) g + 2f_c \varepsilon_{ij3} u_i' u_j' - 2\left(\frac{u_i'}{\overline{\rho}}\right) \frac{\partial p'}{\partial x_i} + 2\nu\, u_i' \frac{\partial^2 u_i'}{\partial x_j^2} + 2u_i' \frac{\partial (\overline{u_i'u_j'})}{\partial x_j}$$

Next, use the product rule of calculus to convert terms like $2u_i'\partial u_i'/\partial t$ into $\partial(u_i')^2/\partial t$:

$$\frac{\partial u_i'^2}{\partial t} + \overline{U}_j \frac{\partial u_i'^2}{\partial x_j} + 2u_i'u_j' \frac{\partial \overline{U}_i}{\partial x_j} + u_j' \frac{\partial u_i'^2}{\partial x_j} =$$

$$+ 2\delta_{i3} u_i' \left(\frac{\theta_v'}{\overline{\theta_v}}\right) g + 2f_c \varepsilon_{ij3} u_i' u_j' - 2\left(\frac{u_i'}{\overline{\rho}}\right) \frac{\partial p'}{\partial x_i} + 2\nu u_i' \frac{\partial^2 u_i'}{\partial x_j^2} + 2u_i' \frac{\partial (\overline{u_i'u_j'})}{\partial x_j}$$

For step three, average the whole equation and apply Reynolds averaging rules:

$$\overline{\frac{\partial u_i'^2}{\partial t}} + \overline{\overline{U_j}\frac{\partial u_i'^2}{\partial x_j}} + \overline{2u_i'u_j'\frac{\partial \overline{U_i}}{\partial x_j}} + \overline{u_j'\frac{\partial u_i'^2}{\partial x_j}} =$$

$$+ \overline{2\,\delta_{i3}u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g} + \overline{2f_c\varepsilon_{ij3}u_i'u_j'} - \overline{2\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}} + \overline{2\nu u_i'\frac{\partial^2 u_i'}{\partial x_j^2}} + \overline{2u_i'\frac{\partial\left(\overline{u_i'u_j'}\right)}{\partial x_j}}$$

$$\overline{\frac{\partial u_i'^2}{\partial t}} + \overline{\overline{U}_j \frac{\partial u_i'^2}{\partial x_j}} + \overline{2u_i'u_j'\frac{\partial \overline{U}_i}{\partial x_j}} + \overline{u_j'\frac{\partial u_i'^2}{\partial x_j}} =$$

$$+ 2\,\overline{\delta_{i3}u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g} + \overline{2f_c\varepsilon_{ij3}u_i'u_j'} - \overline{2\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}} + \overline{2\nu u_i'\frac{\partial^2 u_i'}{\partial x_j^2}} + \overline{2u_i'\frac{\partial(\overline{u_i'u_j'})}{\partial x_j}}$$

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \overline{U}_j\frac{\partial \overline{u_i'^2}}{\partial x_j} + 2\overline{u_i'u_j'}\frac{\partial\overline{U}_i}{\partial x_j} + \frac{\partial(\overline{u_j'u_i'^2})}{\partial x_j} =$$

$$+ 2\,\overline{\delta_{i3}u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)}g + 2f_c\varepsilon_{ij3}\overline{u_i'u_j'} - 2\overline{\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}} + 2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}$$

Dissipation

$$+ 2\nu \overline{u_i' \frac{\partial^2 u_i'}{\partial x_j^2}}$$

Dissipation. Consider a term of the form $\partial^2(\overline{u_i'^2})/\partial x_j^2$. Using simple rules of calculus, we can rewrite it as:

$$\frac{\partial^2(\overline{u_i'^2})}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left[\frac{\partial(\overline{u_i'^2})}{\partial x_j}\right] = \frac{\partial}{\partial x_j}\left[\overline{2u_i' \frac{\partial u_i'}{\partial x_j}}\right] = 2\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} + \overline{2u_i' \frac{\partial^2 u_i'}{\partial x_j^2}} =$$

$$2\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{2u_i' \frac{\partial^2 u_i'}{\partial x_j^2}}$$

Dissipation

$$2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}} = \nu\frac{\partial^2(\overline{u_i'^2})}{\partial x_j^2} - 2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}$$

$$2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}} \cong -2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}$$

$$\varepsilon = +\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}$$

Pressure correlation

$$-2\overline{\left(\frac{u_i'}{\bar{\rho}}\right)\frac{\partial p'}{\partial x_i}} \cong -\left(\frac{2}{\bar{\rho}}\right)\frac{\partial\,(\overline{u_i'p'})}{\partial x_i}$$

Coriolis (does no work)

$$2f_c \varepsilon_{ij3} \overline{u_i'u_j'} = 2f_c \varepsilon_{213} \overline{u_2'u_1'} + 2f_c \varepsilon_{123} \overline{u_1'u_2'}$$

$$= -2f_c \overline{u_2'u_1'} + 2f_c \overline{u_1'u_2'}$$

$$= 0$$

simplified

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \overline{U}_j \frac{\partial \overline{u_i'^2}}{\partial x_j} = +2\,\delta_{i3} \frac{g\,\overline{(u_i'\theta_v')}}{\overline{\theta}_v} - 2\overline{u_i'u_j'} \frac{\partial \overline{U}_i}{\partial x_j} - \frac{\partial\,\overline{(u_j'u_i'^2)}}{\partial x_j} - \frac{2}{\overline{\rho}} \frac{\partial\,\overline{(u_i'p')}}{\partial x_i} - 2\varepsilon \qquad (4.3.1\text{g})$$

I II III IV V VI VII

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \overline{U}_j \frac{\partial \overline{u_i'^2}}{\partial x_j} = +2\,\delta_{i3} \frac{g\,\overline{(u_i'\theta_v')}}{\overline{\theta}_v} - 2\overline{u_i'u_j'} \frac{\partial \overline{U}_i}{\partial x_j} - \frac{\partial \overline{(u_j'u_i'^2)}}{\partial x_j} - \frac{2}{\overline{\rho}} \frac{\partial \overline{(u_i'p')}}{\partial x_i} - 2\varepsilon \qquad (4.3.1g)$$

I II III IV V VI VII

Term I represents local storage of variance.

Term II describes the advection of variance by the mean wind.

Term III is a production or loss term, depending on whether the buoyancy flux $\overline{w'\theta_v'}$ is positive (e.g., daytime over land) or negative (e.g., night over land).

Term IV is a production term. The momentum flux $\overline{u_i'u_j'}$ is usually negative in the boundary layer because the momentum of the wind is lost downward to the ground; thus, it results in a positive contribution to variance when multiplied by a negative sign.

Term V is a turbulent transport term. It describes how variance $\overline{u_i'^2}$ is moved around by the turbulent eddies u_j'.

Term VI describes how variance is redistributed by pressure perturbations. It is often associated with oscillations in the air (i.e., ***buoyancy or gravity waves***).

Term VII represents the viscous dissipation of velocity variance.

Turbulent kinetic energy TKE

$$\bar{e} = \frac{1}{2}\left(u'^2 + v'^2 + w'^2\right) = \frac{u_i' u_i'}{2}$$

$$\frac{\partial \bar{e}}{\partial t} + \overline{U}_j \frac{\partial \bar{e}}{\partial x_j} = + \delta_{i3} \frac{g}{\overline{\theta}_v}\left(\overline{u_i'\theta_v'}\right) - \overline{u_i'u_j'} \frac{\partial \overline{U}_i}{\partial x_j} - \frac{\partial\left(\overline{u_j'e}\right)}{\partial x_j} - \frac{1}{\overline{\rho}} \frac{\partial\left(\overline{u_i'p'}\right)}{\partial x_i} - \varepsilon \qquad (5.1a)$$

I II III IV V VI VII

$$\frac{\partial \bar{e}}{\partial t} + \bar{U}_j \frac{\partial \bar{e}}{\partial x_j} = + \delta_{i3} \frac{g}{\bar{\theta}_v}\left(\overline{u_i'\theta_v'}\right) - \overline{u_i'u_j'} \frac{\partial \bar{U}_i}{\partial x_j} - \frac{\partial\left(\overline{u_j'e}\right)}{\partial x_j} - \frac{1}{\bar{\rho}} \frac{\partial\left(\overline{u_i'p'}\right)}{\partial x_i} - \varepsilon \quad (5.1a)$$

I II III IV V VI VII

Term I represents local ***storage*** or tendency of TKE.

Term II describes the ***advection*** of TKE by the mean wind.

Term III is the ***buoyant production or consumption term.*** It is a production or loss term depending on whether the heat flux $\overline{u_i'\theta_v'}$ is positive (during daytime over land) or negative (at night over land).

Term IV is a ***mechanical or shear production/loss term.*** The momentum flux $\overline{u_i'u_j'}$ is usually of opposite sign from the mean wind shear, because the momentum of the wind is usually lost downward to the ground. Thus, Term IV results in a positive contribution to TKE when multiplied by a negative sign.

Term V represents the ***turbulent transport*** of TKE. It describes how TKE is moved around by the turbulent eddies u_j'.

Term VI is a ***pressure correlation term*** that describes how TKE is redistributed by pressure perturbations. It is often associated with oscillations in the air (***buoyancy or gravity waves***).

Term VII represents the viscous ***dissipation*** of TKE; i.e., the conversion of TKE into heat.