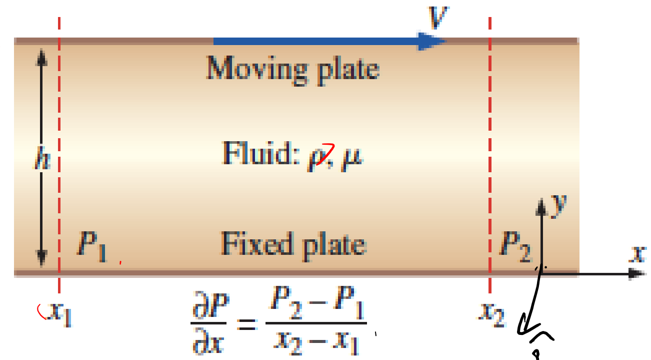
Couette flow with applied pressure gradient

$$\vec{u}\cdot\nabla\vec{u} = \underbrace{u_x}_{\neq 0}\,\cancel{\frac{\partial \vec{u}}{\partial x}}^{\,0} + \underbrace{u_y}_{=0}\frac{\partial \vec{u}}{\partial y} + \underbrace{u_z}_{=0}\frac{\partial \vec{u}}{\partial z} = 0$$

The same as in the Couette flow of the previous slides but the x-component of the momentum equation is now:

P_0



$\mu = \nu \cdot \rho$

$$\cancel{\frac{\partial \vec{u}}{\partial t}}^{\,=0} + \underbrace{\vec{u}\cdot\nabla\vec{u}}_{=0} = -\frac{\nabla p}{\rho} + \vec{g} + \nu\,\nabla^2\vec{u}$$

$$-\frac{\partial P}{\partial x}\frac{1}{\rho} + \nu\,\nabla^2 u = 0$$

Result of x-momentum: $\quad \dfrac{d^2u}{dy^2} = \dfrac{1}{\mu}\dfrac{\partial P}{\partial x}$

Integrating twice yields

$$\frac{\partial p}{\partial x} = \frac{\partial f}{\partial x} = C_{te}$$

$$f = \left(\frac{\partial p}{\partial x}\right)\cdot x + P_0$$

Integration of x-momentum: $\quad u = \dfrac{1}{2\mu}\dfrac{\partial P}{\partial x}y^2 + C_1 y + C_2$

For the pressure

Integration of z-momentum: $\quad P = -\rho g z + f(x)$

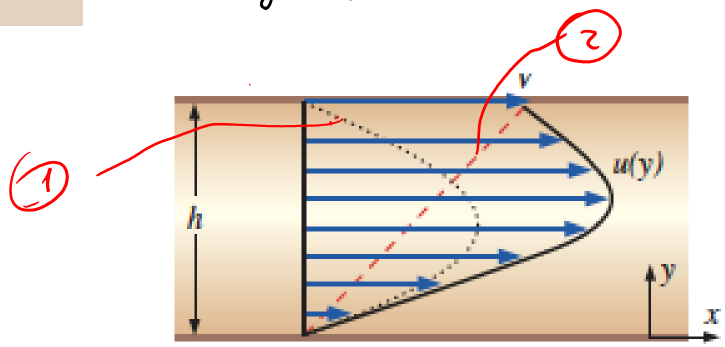
Final result for pressure field: $\quad P = P_0 + \dfrac{\partial P}{\partial x}x - \rho g z$

- Applying the velocity boundary conditions

$$u = \frac{1}{2\mu}\frac{\partial P}{\partial x} \times 0 + C_1 \times 0 + C_2 = 0 \quad \rightarrow \quad \underline{C_2 = 0}$$ $\Leftarrow u(y=0) = 0$

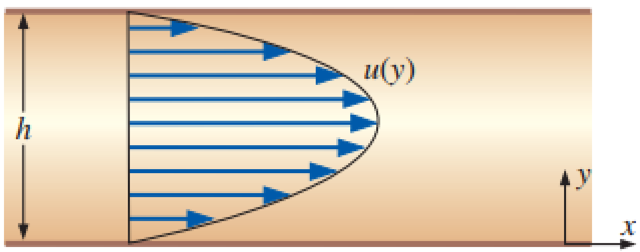
$$u = \frac{1}{2\mu}\frac{\partial P}{\partial x} h^2 + C_1 \times h + 0 = V \quad \rightarrow \quad C_1 = \frac{V}{h} - \frac{1}{2\mu}\frac{\partial P}{\partial x} h$$ $\Leftarrow u(y=h) = V$

$$u = \underbrace{\frac{Vy}{h}}_{2} + \underbrace{\frac{1}{2\mu}\frac{\partial P}{\partial x}(y^2 - hy)}_{1}$$



- $u(y)$ is the velocity profile of Couette flow between parallel plates with an applied negative pressure gradient; the dashed red line indicates the profile for a zero pressure gradient, and the dotted line indicates the profile for a negative pressure gradient with the upper plate stationary ($V = 0$).

If V=0:



Dimensional analysis

- The problem is set in terms of velocity u as a function of y, h, V, m, and dP/dx. There are six variables (including the dependent variable u), and since there are three primary dimensions (mass, length, and time), we expect 6 - 3 dimensionless groups. When we pick h, V, and m as our repeating variables, we get the following result:

Result of dimensional analysis: $\frac{u}{V} = f\left(\frac{y}{h}, \frac{h^2}{\mu V}\frac{\partial P}{\partial x}\right)$

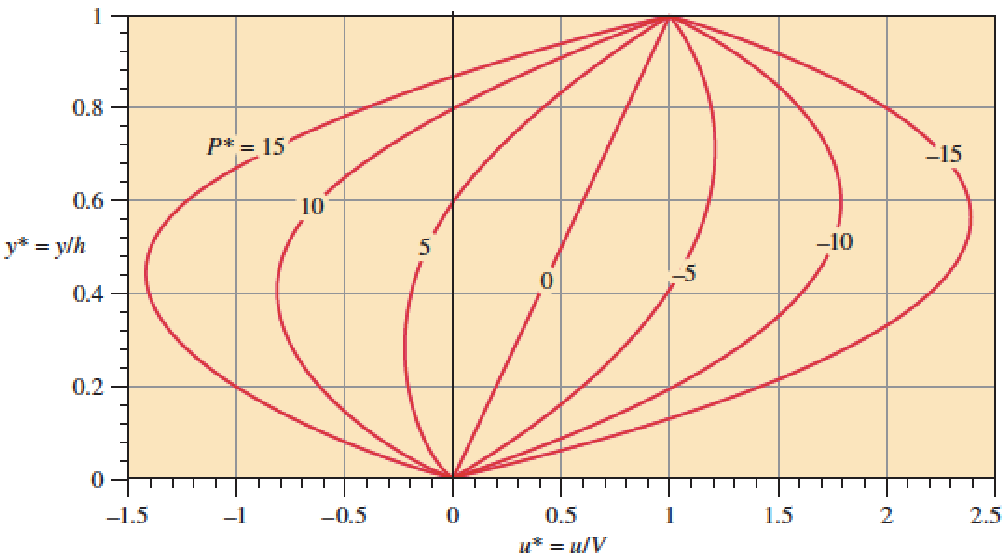
$$u = \frac{Vy}{h} + \frac{1}{2\mu}\frac{\partial P}{\partial x}(y^2 - hy)$$

$$\xrightarrow{\div V} \frac{u}{V} = \frac{y}{h} + \frac{h^2}{2\mu V}\frac{\partial p}{\partial x}\left(\frac{y^2}{h^2} - \frac{y}{h}\right)$$

$$u^* = \frac{u}{V}, \quad y^* = \frac{y}{h}, \quad p^* = \frac{h^2}{\mu V}\frac{\partial p}{\partial x}$$

Dimensional analysis

Dimensionless form of velocity field: $u^* = y^* + \frac{1}{2}P^*y^*(y^* - 1)$



where

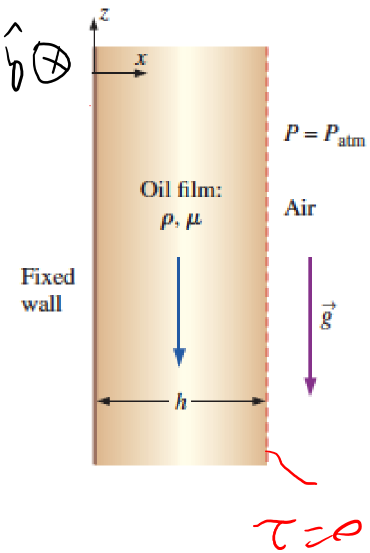
$u^* = \frac{u}{V} \quad y^* = \frac{y}{h} \quad P^* = \frac{h^2}{\mu V}\frac{\partial P}{\partial x}$

SI:

$\nu \to m^2/s$, $P \to N/m^2$

$\mu \to kg/(m \cdot s)$

Oil film falling down a vertical wall



1. The wall is infinite in the yz-plane (y is into the page for a right-handed coordinate system).
2. The flow is steady (all partial derivatives with respect to time are zero).
3. The flow is parallel (the x-component of velocity, u, is zero everywhere).
4. The fluid is incompressible and Newtonian with constant properties, and the flow is laminar.
5. Pressure $P = P_{atm}$ constant at the free surface. In other words, there is no applied pressure gradient pushing the flow; the flow establishes itself due to a balance between gravitational forces and viscous forces. In addition, since there is no gravity force in the horizontal direction, $P = P_{atm}$ everywhere.
6. The velocity field is purely 2D, which implies that derivatives in y are zero.
7. Gravity acts in the negative z direction.
8. The boundary conditions are: no slip at the wall; at x = 0, u = v = w = 0. At the free surface (x = h), there is negligible shear, which for a vertical free surface, in this coordinate system, means $\frac{\partial w}{\partial x} = 0$ at x = h.

Continuity: $\nabla \cdot \vec{u} = 0 \Rightarrow$ $\overset{u=0}{\cancel{\frac{\partial u}{\partial x}}} + \overset{v=0}{\cancel{\frac{\partial v}{\partial y}}} + \frac{\partial w}{\partial z} = 0 \rightarrow \boxed{\frac{\partial w}{\partial z} = 0}$

Result of continuity: $w = w(x)$ only

NS w:

$$\rho\left(\overset{=0}{\cancel{\frac{\partial w}{\partial t}}} + \overset{u=0}{\cancel{u\frac{\partial w}{\partial x}}} + \overset{\frac{\partial}{\partial y}(\dots)=0}{\cancel{v\frac{\partial w}{\partial y}}} + \overset{=0}{\cancel{w\frac{\partial w}{\partial z}}}\right) = \overset{P=cte}{\cancel{-\frac{\partial P}{\partial z}}} + \underbrace{\rho g_z}$$

$$+\mu\left(\frac{\partial^2 w}{\partial x^2} + \underset{=0}{\cancel{\frac{\partial^2 w}{\partial y^2}}} + \underset{=0\ (\nabla\cdot\vec{u}=0)}{\cancel{\frac{\partial^2 w}{\partial z^2}}}\right) \rightarrow \boxed{\frac{d^2 w}{dx^2} = \frac{\rho g}{\mu}}$$

Integration:

$$w = \frac{\rho g}{2\mu}x^2 + C_1 x + C_2$$

$w(x=0)=0$

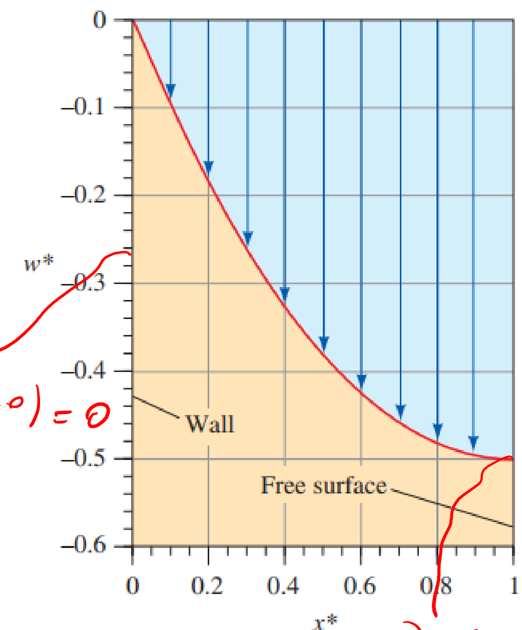
Boundary condition (1): $w = 0 + 0 + C_2 = 0 \quad C_2 = 0$

and

$\frac{\partial w}{\partial x}(x=h)=0$

Boundary condition (2): $\left.\frac{dw}{dx}\right)_{x=h} = \frac{\rho g}{\mu}h + C_1 = 0 \rightarrow C_1 = -\frac{\rho g h}{\mu}$

Velocity field: $w = \frac{\rho g}{2\mu}x^2 - \frac{\rho g}{\mu}hx = \frac{\rho g x}{2\mu}(x - 2h)$



$x^* = x/h$ and $w^* = w\mu/(\rho g h^2)$

$$w^* = \frac{x^*}{2}(x^* - 2)$$

Reading for the next class: Çengel, examples 9-18 and 9-19