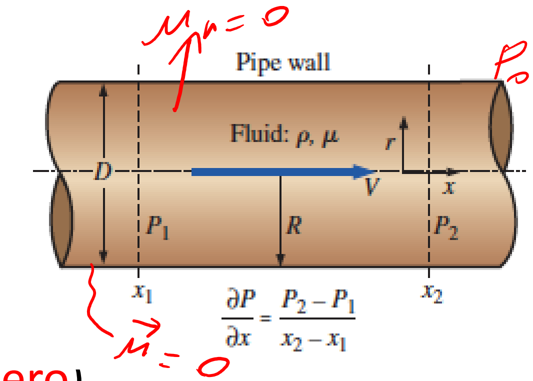
Flow in a round pipe: Poiseuille



1 The pipe is infinitely long in the x-direction.

2 The flow is steady (all partial time derivatives are zero).

3 This is a parallel flow (the r-component of velocity, u_r, is zero).

4 The fluid is incompressible and Newtonian with constant properties, and the flow is laminar. $Re < 2000$

5 A constant pressure gradient is applied in the x-direction such that pressure changes linearly with respect to x.

6 The velocity field is axisymmetric with no swirl, implying that $u_\theta = 0$ and all partial derivatives with respect to θ are zero.

7 We ignore the effects of gravity.

8 The first boundary condition comes from imposing the no slip condition at the pipe wall: (1) at $r = R, \vec{V} = 0$.

9 The second boundary condition comes from the fact that the centerline of the pipe is an axis of symmetry: (2) at $r = 0$, $\frac{\partial u}{\partial x} = 0$. Alternatively: the velocity is finite at the center.

Continuity: $\nabla \cdot \vec{u} = 0$

$u_r = 0$ $\quad$ $u_\theta = 0$

$$\frac{1}{\cancel{r}}\frac{\partial(\cancel{r u_r})}{\partial r} + \frac{1}{\cancel{r}}\frac{\partial(\cancel{u_\theta})}{\partial \theta} + \frac{\partial u}{\partial x} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} = 0$$

Result of continuity: $\quad \underline{u = u(r) \text{ only}}$

NS u:

$0 \quad u_r = 0 \quad u_\theta = 0 \quad \vec{g} = 0$

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + u_r \cancel{\frac{\partial u}{\partial r}} + \frac{u_\theta}{\cancel{r}} \cancel{\frac{\partial u}{\partial \theta}} + u\cancel{\frac{\partial u}{\partial x}}\right) = -\frac{\partial P}{\partial x} + \underbrace{\rho g_x} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{\cancel{r^2}}\cancel{\frac{\partial^2 u}{\partial \theta^2}} + \cancel{\frac{\partial^2 u}{\partial x^2}}\right)$$

$\frac{\partial}{\partial \theta}(\ldots) = 0$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x}$$

NS p:

r-momentum: $\quad \frac{\partial P}{\partial r} = 0$

Result of r-momentum: $\quad P = P(x)$ only $\quad \Rightarrow P = \frac{\partial P}{\partial x} \cdot x + P_0$

Integration of NS for u:

$$r\frac{du}{dr} = \frac{r^2}{2\mu}\frac{dP}{dx} + C_1$$

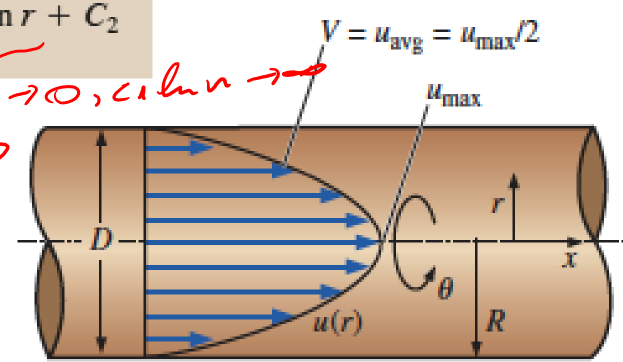
$$u = \frac{r^2}{4\mu}\frac{dP}{dx} + \underbrace{C_1 \ln r} + C_2$$

$u(r = R) = 0$

se $r \to 0$, $C_1 \ln r \to \infty$

$\Rightarrow C_1 = 0$

Axial velocity: $\quad u = \frac{1}{4\mu}\frac{dP}{dx}(r^2 - R^2)$



Poiseuille's law for the flow rate

Maximum axial velocity: $$u_{\max} = -\frac{R^2}{4\mu}\frac{dP}{dx}$$

$$\dot{V} = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} ur\,dr\,d\theta = \frac{2\pi}{4\mu}\frac{dP}{dx}\int_{r=0}^{R}(r^2 - R^2)r\,dr = -\frac{\pi R^4}{8\mu}\frac{dP}{dx}$$

Average axial velocity: $$V = \frac{\dot{V}}{A} = \frac{(-\pi R^4/8\mu)\,(dP/dx)}{\pi R^2} = -\frac{R^2}{8\mu}\frac{dP}{dx}$$

Viscous shear force

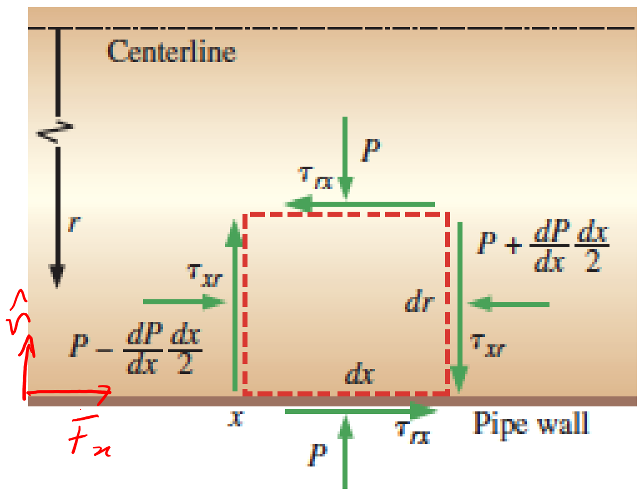
The stress tensor is

$$\tau_{ij} = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rx} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta x} \\ \tau_{xr} & \tau_{x\theta} & \tau_{xx} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mu \frac{\partial u}{\partial r} \\ 0 & 0 & 0 \\ \mu \frac{\partial u}{\partial r} & 0 & 0 \end{pmatrix}$$

Force on $\hat{x}$

Viscous shear stress at the pipe wall: $\tau_{rx} = \mu \dfrac{du}{dr} = \dfrac{R}{2}\dfrac{dP}{dx}$

normal $\hat{n}$



For flow from left to right, dP/dx is negative, so the viscous shear stress on the bottom of the fluid element at the wall is in the direction opposite to that indicated in the figure. (This agrees with our intuition since the pipe wall exerts a retarding force on the fluid.) The shear force per unit area on the wall is equal and opposite to this; hence,

Viscous shear force per unit area acting on the wall: $\dfrac{\vec{F}}{A} = -\dfrac{R}{2}\dfrac{dP}{dx}\vec{i}$

Viscosity and Poiseuille's Law:

https://www.youtube.com/watch?v=wTnI_kfPBhQ