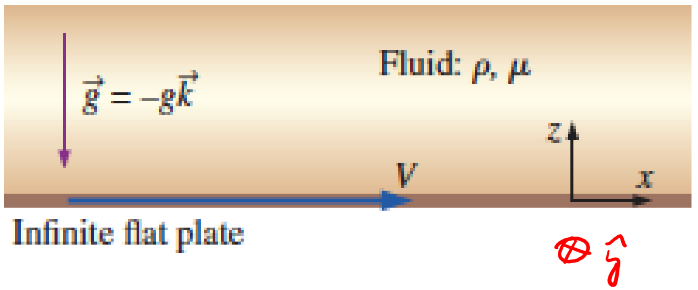
Sudden motion of an infinite flat plate



Consider a Newtonian fluid on top of a flat plate in the xy-plane at z = 0. The fluid is at rest until t = 0, when the plate suddenly starts moving at speed V in the x-direction.

1 The wall is infinite in the x- and y-directions; thus, nothing is special about any particular x- or y-location.
2 The flow is parallel everywhere ($w = 0$).
3 Pressure P = constant with respect to x. In other words, there is no applied pressure gradient pushing the flow in the x-direction; flow occurs due to viscous stresses caused by the moving plate.
4 The fluid is incompressible and Newtonian with constant properties, and the flow is laminar.
5 The velocity field is two-dimensional in the xz-plane; therefore, $v = 0$, and all partial derivatives with respect to y are zero.
6 Gravity acts in the -z-direction

Initial and boundary conditions:

(1) At t = 0, u = 0 everywhere (no flow until the plate starts moving);

(2) at z = 0, u = V for all values of x and y (no-slip condition at the plate);

(3) as z → ∞, u → 0 (far from the plate, the effect of the moving plate is not felt);

(4) at z = 0, P = P_{wall} (the pressure at the wall is constant at any x- or y-location along the plate).

- Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} = 0$$

Result of continuity: $u = u(z, t)$ only

- y – momentum

$$\frac{\partial P}{\partial y} = 0$$

Result of y-momentum: $P = P(z, t)$ only

- z - momentum

$$\frac{\partial P}{\partial z} = -\rho g$$

- x – momentum

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad \rightarrow \quad \rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial z^2}$$

(Diffusion equation for u, with D=ν)

Result of x-momentum:

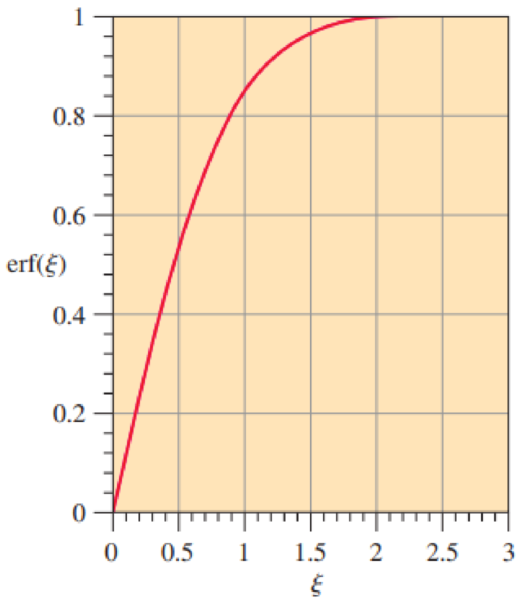
$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial z^2}$$

From the z-component we obtain the pressure

$$P = -\rho g z + f(t)$$

Boundary condition (4): $P = 0 + f(t) = P_{\text{wall}} \rightarrow f(t) = P_{\text{wall}}$

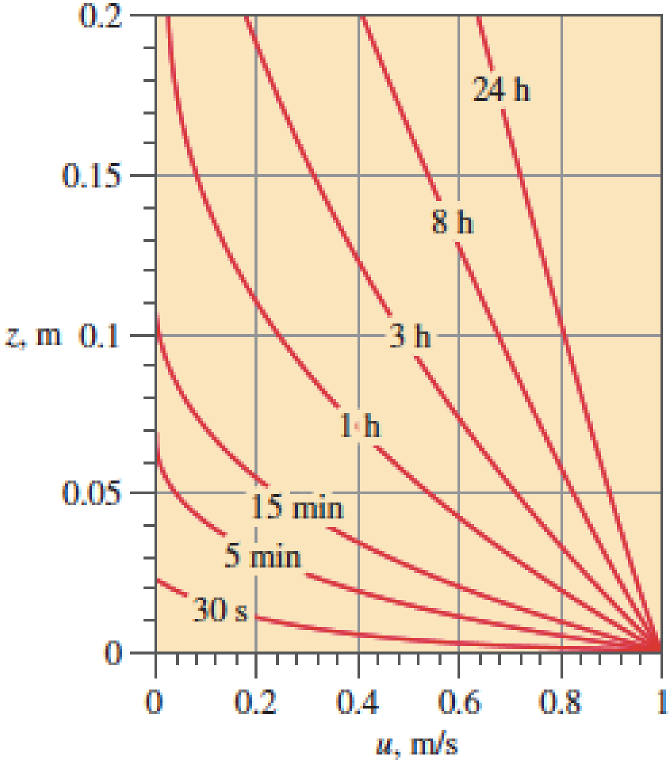
Final result for pressure field: $$P = P_{\text{wall}} - \rho g z$$



Final result for velocity field: $$u = V\left[1 - \operatorname{erf}\left(\frac{z}{2\sqrt{\nu t}}\right)\right]$$

Error function: $$\operatorname{erf}(\xi) = \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-\eta^2} d\eta$$

Verify that this is a solution of the differential equation and that it satisfies the boundary conditions.



After 15 min of flow, the effect of the moving plate is not felt beyond about 10 cm above the plate!

water at room temperature ($\nu = 1.004 \times 10^{-6}$ m^2/s) with $V = 1.0$ m/s

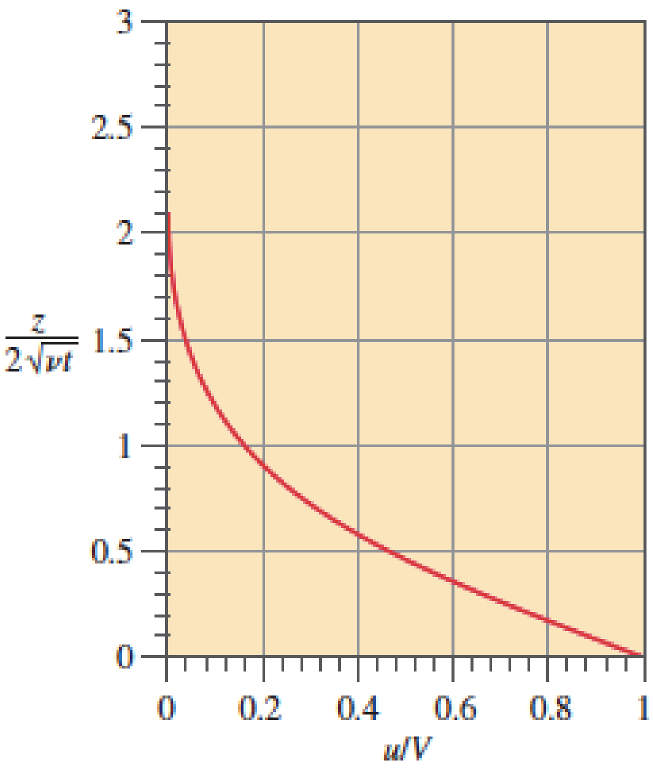
The time required for momentum to diffuse into the fluid seems much longer than we would expect.

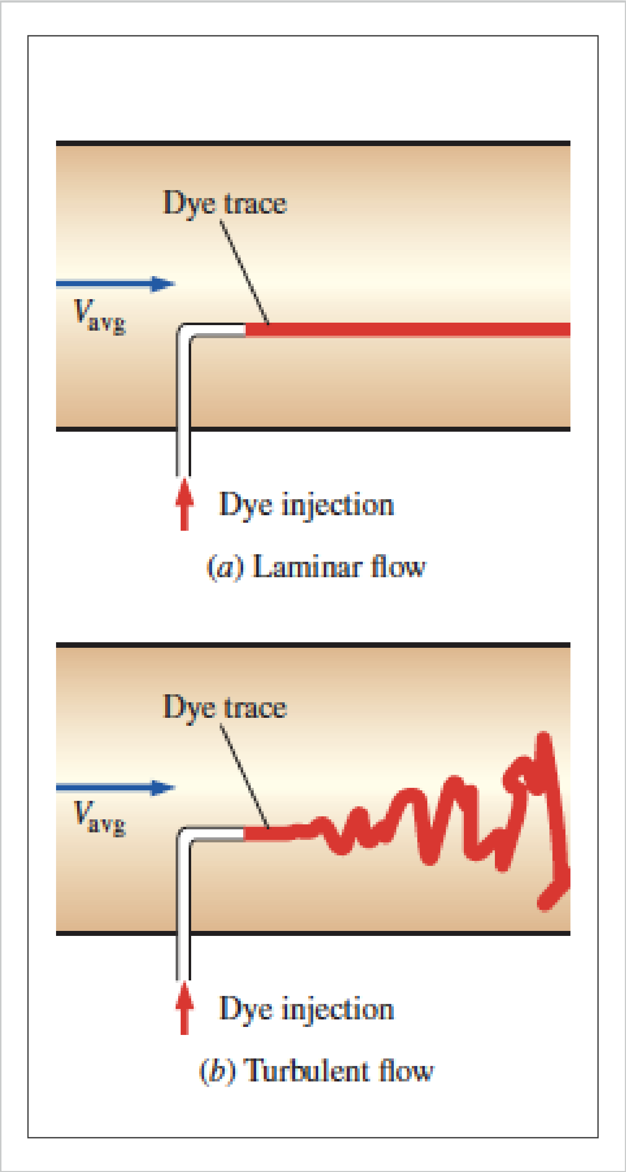
This is because the solution presented here is valid only for laminar flow.

It turns out that if the plate's speed is large enough, or if there are significant vibrations in the plate or disturbances in the fluid, the flow will become turbulent.

In a turbulent flow, large eddies mix rapidly moving fluid near the wall with slowly moving fluid away from the wall.

This mixing process occurs rather quickly, so that turbulent diffusion is usually orders of magnitude faster than laminar diffusion.

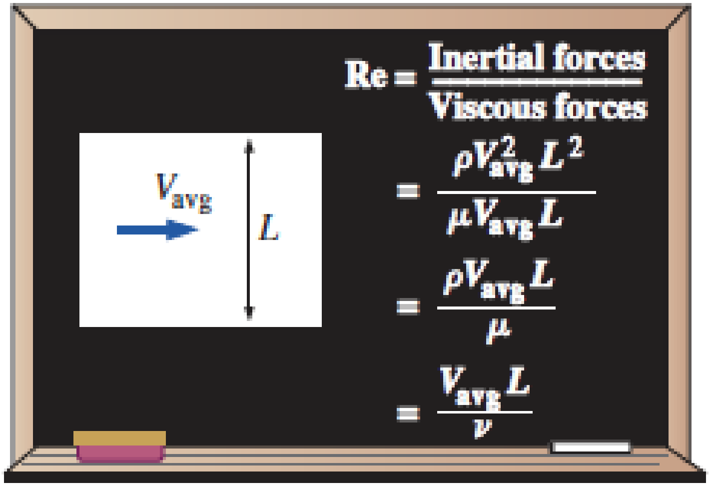




The flow regime in the first case is said to be laminar, characterized by smooth streamlines and highly ordered motion, and turbulent in the second case, where it is characterized by velocity fluctuations and highly disordered motion.

The transition from laminar to turbulent flow does not occur suddenly; rather, it occurs over some region in which the flow fluctuates between laminar and turbulent flows before it becomes fully turbulent.

Most flows encountered in practice are turbulent. Laminar flow is encountered when highly viscous fluids such as oils flow in small pipes or narrow passages.



Reynolds number

$$Re \lesssim 2300 \quad \text{laminar flow}$$
$$2300 \lesssim Re \lesssim 4000 \quad \text{transitional flow}$$
$$Re \gtrsim 4000 \quad \text{turbulent flow}$$