Force balance

Navier-Stokes equation

$$\underbrace{\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}}_{=0} = -\frac{\nabla p}{\rho} + \vec{g} + \nu\nabla^2\vec{u}$$

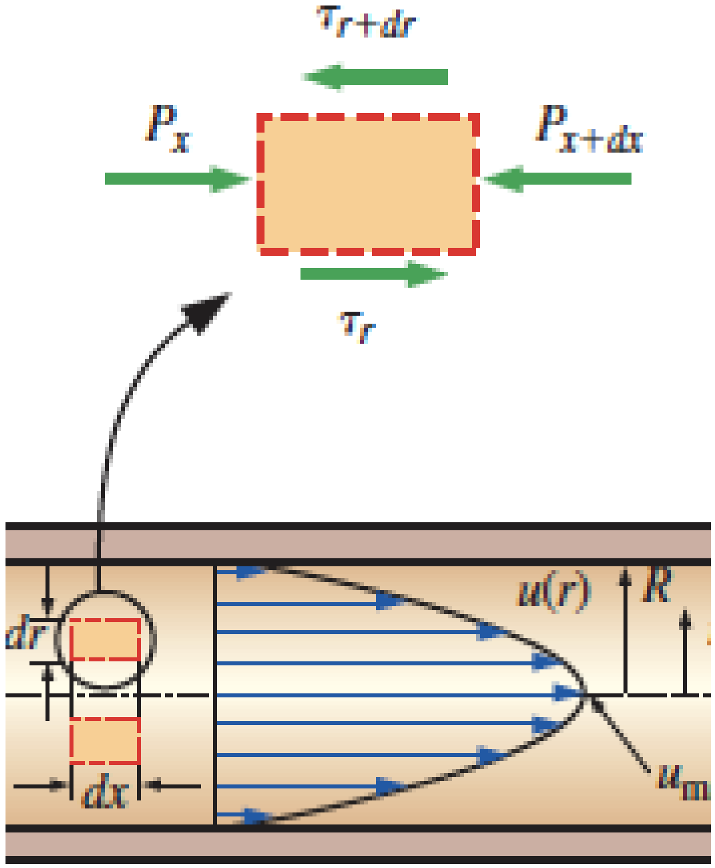
In most of the previous examples, the acceleration of the fluid elements is zero. It means that the viscous force balance the external force (e.g., gravity) or pressure gradients in such a way that the sum of forces acting on a fluid element is zero.

Alternative derivation for flow in a circular pipe

Obtain the momentum equation by applying a momentum balance to a differential volume element, and we obtain the velocity profile by solving it.

Free-body diagram of a ring-shaped differential fluid element of radius r, thickness dr, and length dx oriented coaxially with a horizontal pipe in fully developed laminar flow.

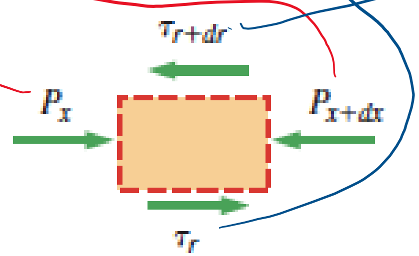
Sec. 8.4, Çengel

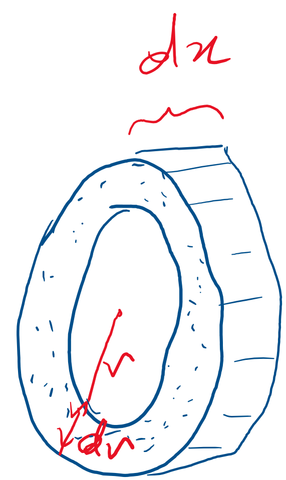


In fully developed laminar flow the axial velocity is, u= u(r). There is no motion in the radial direction. There is no acceleration (check: calculate the acceleration and verify that it is zero).

- Consider a ring-shaped differential volume element of radius r, thickness dr, and length dx oriented coaxially with the pipe.
- The volume element involves only pressure and viscous effects and thus the pressure and shear forces must balance each other. The pressure force acting on a submerged plane surface is the product of the pressure at the centroid of the surface and the surface area. A force balance on the volume element in the flow direction (x) gives

$$(\underbrace{2\pi r\,dr}_{dA}\,P)_x - (2\pi r\,dr\,P)_{x+dx} + (2\pi r\,dx\,\tau)_r - (2\pi r\,dx\,\tau)_{r+dr} = 0$$





Force balance implies

$$(2\pi r\, dr\, P)_x - (2\pi r\, dr\, P)_{x+dx} + (2\pi r\, dx\, \tau)_r - (2\pi r\, dx\, \tau)_{r+dr} = 0 \quad \div\ (2\pi\, dx\, dr)$$

$$r\,\frac{\overbrace{P_{x+dx} - P_x}^{dP}}{dx} + \frac{\overbrace{(r\tau)_{r+dr} - (r\tau)_r}^{d(r\tau)}}{dr} = 0$$

$$r\,\frac{dP}{dx} + \frac{d(r\tau)}{dr} = 0$$

and substituting the stress (component rz): $\tau_{rz} = -\mu(du/dr)$ we find

$$\boxed{\frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{dP}{dx}}$$

Same equation obtained with NS (see slide 39): $\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x}$

Recall

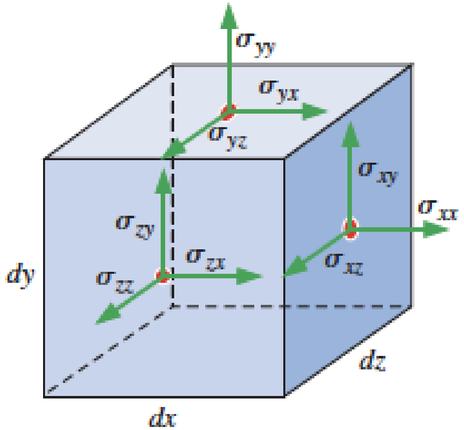
Deviatoric stress tensor

$$\tau_{ij} = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}$$

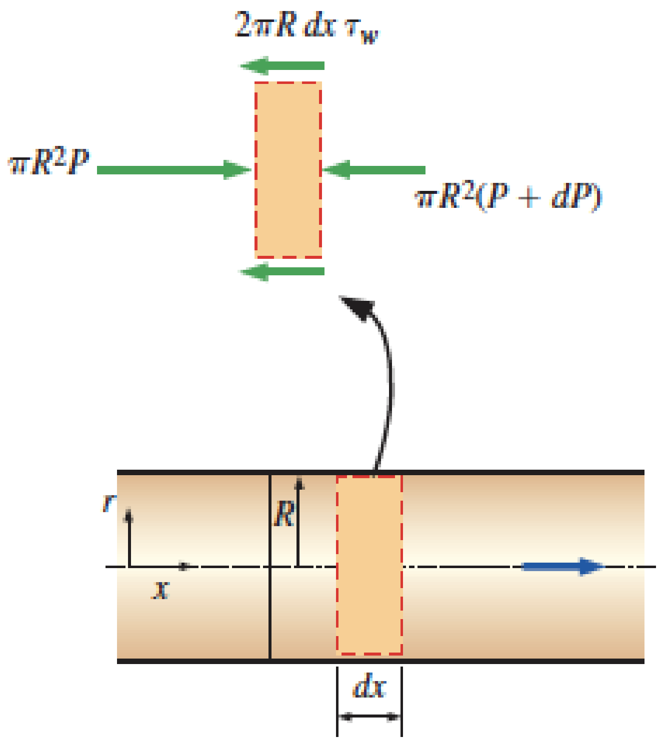
$$= \begin{pmatrix} 2\mu \frac{\partial u_r}{\partial r} & \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] & \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] & 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\ \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) & 2\mu \frac{\partial u_z}{\partial z} \end{pmatrix}$$

Stress tensor

$$\sigma_{ij} = -P\,\delta_{ij} + \tau_{ij}$$



Different fluid element (r from 0 to R)



Force balance:

$$\pi R^2 P - \pi R^2(P + dP) - 2\pi R\,dx\,\tau_w = 0$$

Simplifying:

$$\frac{dP}{dx} = -\frac{2\tau_w}{R}$$

Separation of variables implies that the pressure gradient is constant $\frac{dP}{dx} = -\frac{2\tau_w}{R}$

The velocity profile is obtained by integration and use of the boundary conditions:

$$u(r) = \frac{r^2}{4\mu}\left(\frac{dP}{dx}\right) + \cancel{C_1} \ln r + \cancel{C_2}$$

(...) $u(r=R) = 0$

$= 0$ ($u(r=0)$ e' finito)

$$\boxed{u(r) = -\frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1 - \frac{r^2}{R^2}\right)}$$

The average velocity is

$$V_{avg} = \frac{2}{R^2}\int_0^R u(r) r\, dr = \frac{-2}{R^2}\int_0^R \frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1 - \frac{r^2}{R^2}\right) r\, dr = -\frac{R^2}{8\mu}\left(\frac{dP}{dx}\right)$$

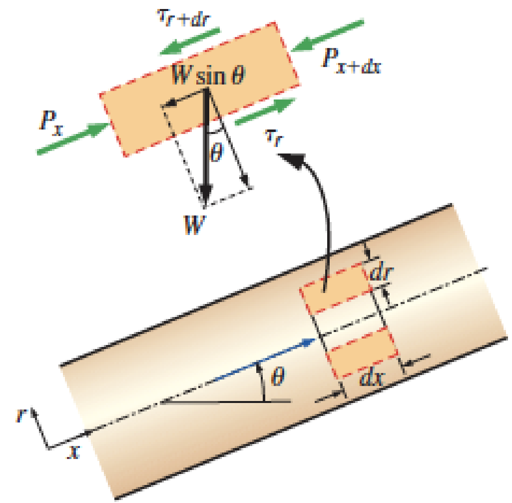
In terms of which the profile becomes

$$u(r) = 2V_{avg}\left(1 - \frac{r^2}{R^2}\right)$$

$r = 0 \Rightarrow u(r) = u_{MAX}$
$= 2V_{avg}$

Effect of gravity

- Gravity has no effect on flow in horizontal pipes, but it has a significant effect on both the velocity and the flow rate in uphill or downhill pipes.
- Relations for inclined pipes can be obtained in a similar manner from a force balance in the direction of flow. The only additional force in this case is the component of the fluid weight in the flow direction, which is



$$W_x = W\sin\theta = \rho g V_{\text{element}} \sin\theta = \rho g(2\pi r\,dr\,dx)\sin\theta$$

Then

$$(2\pi r\,dr\,P)_x - (2\pi r\,dr\,P)_{x+dx} + (2\pi r\,dx\,\tau)_r - (2\pi r\,dx\,\tau)_{r+dr} - \underbrace{\rho g(2\pi r\,dr\,dx)\sin\theta}_{\text{força volumétrica}} = 0 \quad \Leftarrow \frac{D\vec{u}}{Dt} = 0$$

and

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{dP}{dx} + \rho g\sin\theta$$

Effect of gravity

The velocity profile, average velocity and flow rate are:

$$u(r) = -\frac{R^2}{4\mu}\left(\frac{dP}{dx} + \rho g \sin\theta\right)\left(1 - \frac{r^2}{R^2}\right)$$

$$V_{\text{avg}} = \frac{(\Delta P - \rho g L \sin\theta)D^2}{32\mu L} \quad \text{and} \quad \dot{V} = \frac{(\Delta P - \rho g L \sin\theta)\pi D^4}{128\mu L}$$

- As expected, gravity opposes uphill flow, enhances downhill flow, and has no effect on horizontal flow.
- Downhill flow can occur even in the absence of a pressure difference applied by a pump. For the case of P1 = P2 (i.e., no applied pressure difference), the pressure throughout the entire pipe would remain constant, and the fluid would flow through the pipe under the influence of gravity at a rate that depends on the angle of inclination, reaching its maximum value when the pipe is vertical.