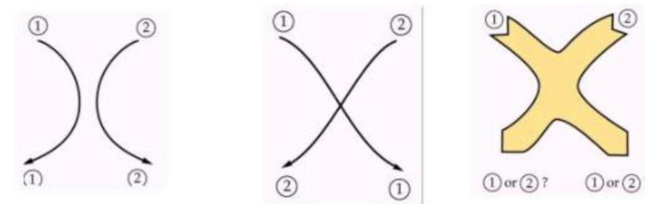
Superfluid

https://www.youtube.com/watch?v=2Z6UJbwxBZI

$\nu \neq 0$ $\nu = 0$



Indistinguishable particles.

Nearly zero viscosity fluids.

Hydrodynamic equations are classical.

Superfluid helium

Suggested reading: https://www.scientificamerican.com/article/superfluid-can-climb-walls/

Flow of living matter

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

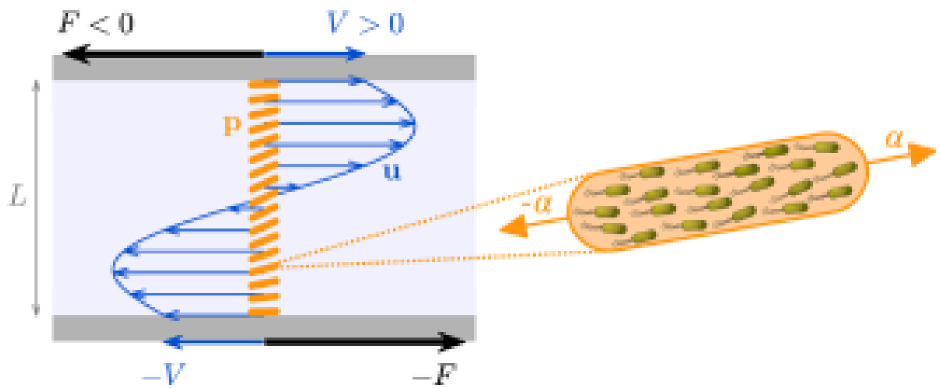


FIG. 1. A suspension containing orientable active units such as bacteria is sheared between two plates of area A separated by a distance L. A force F is exerted on the plates which move at a velocity V. The apparent viscosity, defined as $\eta_{app} = \sigma/\dot{\gamma}$ where $\sigma = F/A$ is the macroscopic shear stress and $\dot{\gamma} = 2V/L$ is the macroscopic shear strain rate, can become negative under certain conditions. When $\eta_{app} < 0$, the velocity profile is nonmonotonic with local velocity gradients at the walls opposing the applied macroscopic velocity gradient.

Lagrange multiplier enforcing $|\boldsymbol{p}| = 1$. The fluid is assumed incompressible ($\nabla \cdot \boldsymbol{u} = 0$), and the flow field obeys, upon neglecting fluid inertia, the Stokes flow equation $\nabla \cdot \boldsymbol{\sigma} = 0$ with

$$\sigma_{ij} = \overbrace{2\eta E_{ij}}^{\text{Fluidos Newtonianos}} - \Pi\delta_{ij} - \frac{\lambda+1}{2}p_i h_j - \frac{\lambda-1}{2}p_j h_i - \alpha p_i p_j, \quad (2)$$

where η is the bulk fluid viscosity ($\eta > 0$), Π is the bulk pressure, and α is the activity coefficient. This coefficient is related to the active stresses in a suspension of particles modeled as force dipoles: the magnitude of α is proportional to the strength of the force pair and the sign of α depends on whether the induced flow is extensile ($\alpha > 0$) or contractile ($\alpha < 0$). Our geometry is a two-dimensional slab of thickness L (Fig. 1) with translational invariance in the direction parallel to the walls. We use no-slip and parallel anchoring as boundary conditions. The fluid is subject to a macroscopic shear rate $\dot{\gamma} = 2V/L$, and the shear stress (simply denoted σ) is uniform across the film.

Using a highly sensitive rheometer, Lopez et al. [22] were able to measure zero and possibly negative values of the apparent viscosity in a suspension of E. Coli at steady-state, thereby demonstrating that microscopic bacterial activity can be converted into macroscopic useful mechanical power.

Loisy et al., PRL, **121**, 018001 (2018)

Non dimensionalized equations of motion

Our goal in this section is to nondimensionalize the equations of motion so that we can properly compare the orders of magnitude of the various terms in the equations. We begin with the incompressible continuity equation,

$$\vec{\nabla}\cdot\vec{V} = 0 \quad \Rightarrow \rho = C^{te}$$

and the vector form of the Navier–Stokes equation, valid for incompressible flow of a Newtonian fluid with constant properties,

$$\rho\frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V}$$

Scaling parameters used to nondimensionalize the continuity and momentum equations, along with their primary dimensions		
Scaling Parameter	Description	Primary Dimensions
L	Characteristic length	$\{L\}$
V	Characteristic speed	$\{Lt^{-1}\}$
f	Characteristic frequency	$\{t^{-1}\}$
$P_0 - P_\infty$	Reference pressure difference	$\{mL^{-1}t^{-2}\}$
g	Gravitational acceleration	$\{Lt^{-2}\}$

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \sim \frac{1}{L}$$

We can define scaled variables:

$$t^* = ft \qquad \vec{x}^* = \frac{\vec{x}}{L} \qquad \vec{V}^* = \frac{\vec{V}}{V} \Rightarrow \vec{V} = V.\vec{V}^*$$

$$P^* = \frac{P - P_\infty}{P_0 - P_\infty} \qquad \vec{g}^* = \frac{\vec{g}}{g} \qquad \vec{\nabla}^* = L\vec{\nabla}$$

In terms of which the continuity and NS equations become

Nondimensionalized continuity: $$\vec{\nabla}^* \cdot \vec{V}^* = 0$$

$$\rho V f \frac{\partial \vec{V}^*}{\partial t^*} + \frac{\rho V^2}{L}\left(\vec{V}^* \cdot \vec{\nabla}^*\right)\vec{V}^* = -\frac{P_0 - P_\infty}{L}\vec{\nabla}^* P^* + \rho g \vec{g}^* + \frac{\mu V}{L^2}\nabla^{*2}\vec{V}^* \quad \times \frac{L}{\rho V^2}$$

$$\underbrace{\left[\frac{fL}{V}\right]}_{St}\frac{\partial \vec{V}^*}{\partial t^*} + \left(\vec{V}^* \cdot \vec{\nabla}^*\right)\vec{V}^* = -\underbrace{\left[\frac{P_0 - P_\infty}{\rho V^2}\right]}_{Eu}\vec{\nabla}^* P^* + \underbrace{\left[\frac{gL}{V^2}\right]}_{Fr^{-2}}\vec{g}^* + \underbrace{\left[\frac{\mu}{\rho V L}\right]}_{Re^{-1}}\nabla^{*2}\vec{V}^*$$

Nondimensionalized Navier–Stokes:

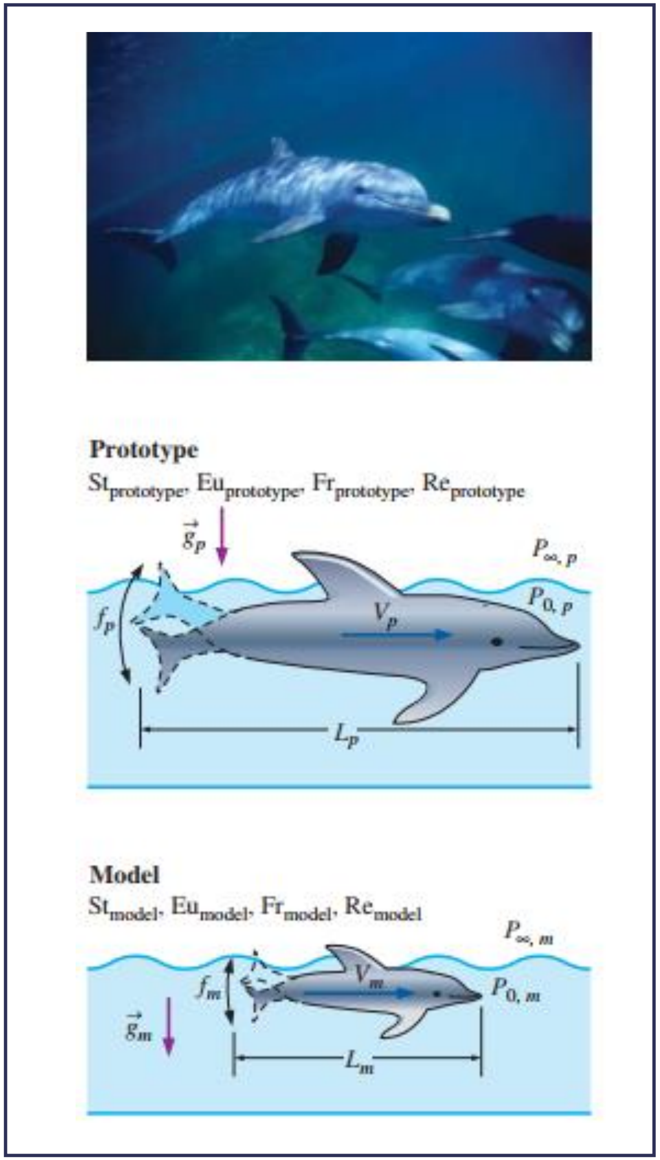
$$[\mathrm{St}]\frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^* \cdot \vec{\nabla}^*)\vec{V}^* = -[Eu]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^*$$

Thus, the relative importance of the terms in the NS equation depends only on the relative magnitudes of the dimensionless parameters in square brackets [ ] known as the Strouhal (St), Euler (Eu), Froude (Fr), and Reynolds (Re) numbers.

Dynamic similarity

- Since there are four dimensionless parameters, dynamic similarity between a model and a prototype requires all four of these to be the same for the model and the prototype ($St_{model} = St_{prototype}$, $Eu_{model} = Eu_{prototype}$, $Fr_{model} = Fr_{prototype}$, and $Re_{model} = Re_{prototype}$).

$$Re = \frac{L \cdot V}{\nu}$$



- If the flow is steady, then f = 0 and the Strouhal number drops out of the list of dimensionless parameters (St = 0). If the characteristic frequency f is very small such that St << 1 the flow is called quasi-steady. This means that at any instant in time (or at any phase of a slow periodic cycle), we can solve the problem as if the flow were steady, and the unsteady term again drops out.

- The effect of gravity is usually important only in flows with free-surface effects (e.g., waves, ship motion, spillways from hydroelectric dams, flow of rivers). For many engineering problems there is no free surface (pipe flow, fully submerged flow around a submarine or torpedo, automobile motion, flight of airplanes, birds, insects, etc.). In such cases, the only effect of gravity on the flow dynamics is a hydrostatic pressure distribution in the vertical direction superposed on the pressure field due to the fluid flow.

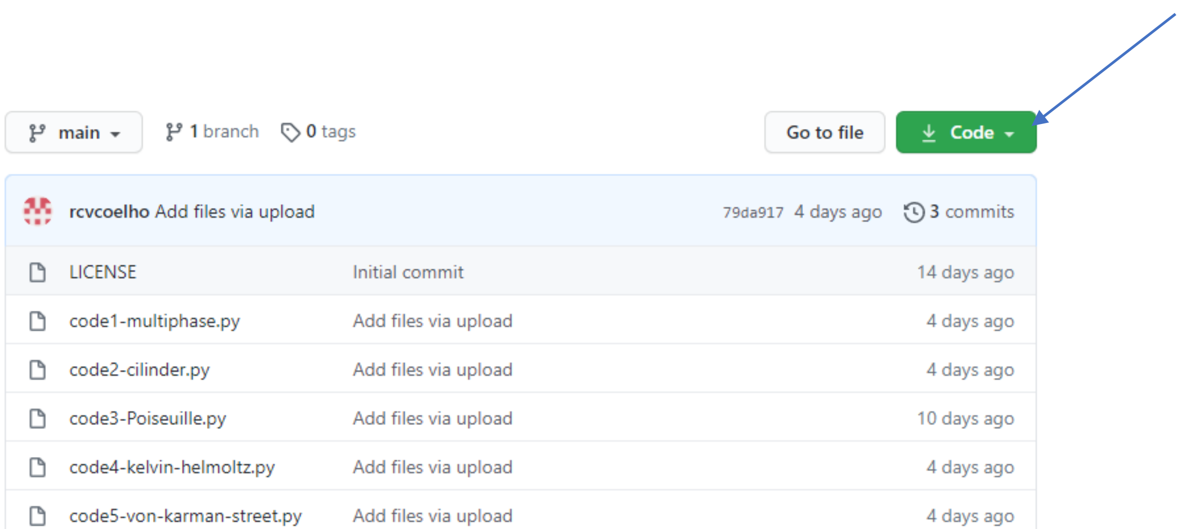
Modified pressure: $$P' = P + \rho g z$$

- In terms of which the NS equation becomes

$$\rho \frac{D\vec{V}}{Dt} = \rho \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{V} \right] = -\vec{\nabla} P' + \mu \nabla^2 \vec{V}$$

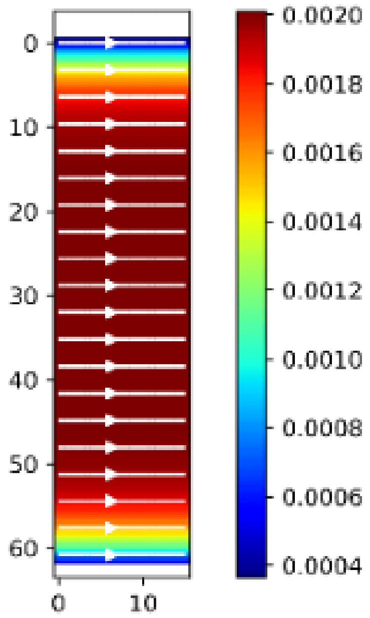
Simple fluid simulations (exercises)

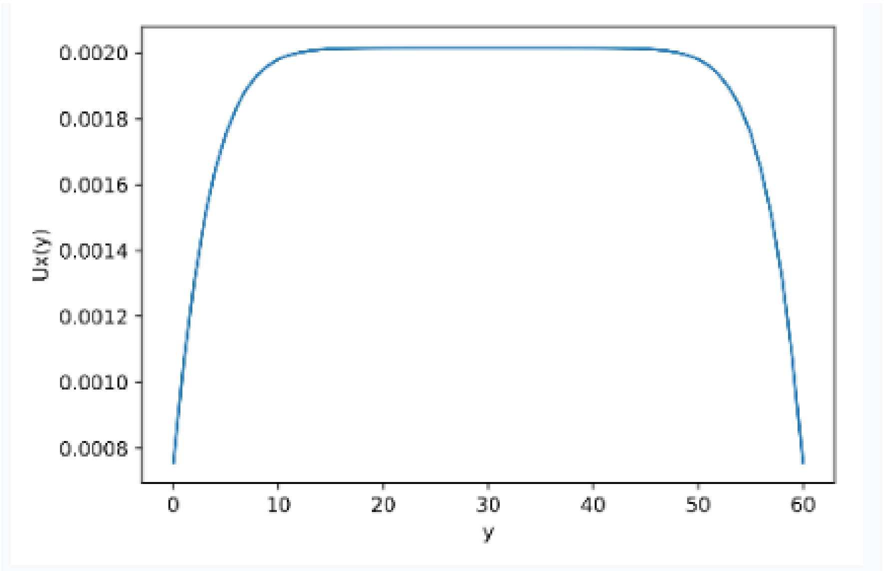
- Numerical solution of the Navier-Stokes equation;
- Lattice Boltzmann method (LBM): implements the Boltzmann equation and recovers the Navier-Stokes equation in the macroscopic limit;
- Use of python: not efficient, but practical and more didactic;
- Available at: https://github.com/rcvcoelho/lbm-python.git



Poiseuille 2D

T=200 (transient state). It becomes a parabola for longer times.





Cylinder

T=76700