The Homogeneous Universe

The background evolution

The dynamics of the expansion

$$H^2(a) = H_0^2 \left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_K a^{-2} + \Omega_\Lambda \right]$$

Given a model and a cosmology (**also called a universe, which at this level is determined by the values of the densities Ω and H_0**) we need to integrate the Friedmann equation to get the solution for a(t).

Note that the Friedmann equation is already a solution for H(a).

Solutions a(t) are easily found by solving integrals numerically.

Let us see some cases.

Note: all physical models (the ones that are not only mathematical solutions) need to include radiation, a fundamental species in the Universe. However, the measurement of the CMB radiation shows that Ω_r is very small. In terms of impact to the background dynamics it is only relevant in the early universe. We will not consider it in most of the following models.

Cosmological models with only one species

$$\boxed{\left(\frac{\dot a}{a}\right)^2 = H_0^2\,\frac{\Omega}{a^m} \qquad (m = 3(1+w))}$$

$$\Leftrightarrow \quad \dot a^2\, a^{m-2} = H_0^2\,\Omega \qquad \Leftrightarrow \quad \dot a\, a^{\frac{m}{2}-1} = H_0\sqrt{\Omega}$$

$$\Leftrightarrow \quad \int_0^t a^{\frac{m}{2}-1}\,da = H_0\sqrt{\Omega}\,dt \qquad \Leftrightarrow \quad \left(a^{m/2} \propto t\right)$$

$$\Leftrightarrow \quad \frac{2}{m}\,a^{m/2} = H_0\sqrt{\Omega}\,t$$

Note the integration is made from t=0 (where a=0) and so it does not introduce another free parameter

$$\Leftrightarrow \quad a^{m/2} = H_0\sqrt{\Omega}\,\frac{m}{2}\,t$$

$$\Leftrightarrow \quad \left[\, a = \left(H_0\sqrt{\Omega}\,\frac{m}{2}\,t\right)^{2/m} \right]$$

Note that with only one species, its energy density is necessarily $\Omega = 1$

So, for a one-species dominated fluid we find:

$$\boxed{\Omega(a) \propto a^{-3(1+w)} \quad \text{and} \quad a \propto t^{\frac{2}{3(1+w)}}}$$

Examples are:

Einstein-de Sitter universe (sCDM): only matter, $\Omega_m = 1$, $w = 0$

From the previous result: $$a(t) = \left(\frac{3H_0}{2}t\right)^{2/3}$$ expansion rate: ~2/3

We can also compute the evolution of $H(t) = \dot{a}(t) / a(t) \sim 1/t$ → in the EdS universe the **Hubble radius** grows faster than the scale factor → $r_H \sim t \sim a^{3/2}$

The expansion rate solution a(t) can be inverted to compute the **age of the universe**, which is just the value of t today, when $a(t_0) = 1$. For the EdS universe:

$$t_0 = 2/(3\ H_0)$$

Single-species universes are fully determined by the Hubble constant (they have only one free parameter).

If H_0 is large → the universe is younger (for a given model)

The inverse of the Hubble constant defines the **Hubble time, t_H** $= 1/H_0$

$1\,pc = 3.0857 \times 10^{16}\,m$

$H_0 = 100\,h\,km/s/Mpc$

$1\,yr = 31\,556\,926\,s$

Its value is:

$t_H = 3.08577 \times 10^{17}\,h^{-1}\,s$

$\boxed{t_H = 9.778\,h^{-1}\,Gyr}$ $\Rightarrow$ 13.97 Gyr (h=0.7)

From Friedmann's equation, we see that for any model, the age of universe i.e. the solution for t(a), is an integral times $1/H_{0.}$

So, any age can be given in terms of a Hubble time (that absorbs the uncertainty on the H_0 value).

Radiation-dominated universe: only radiation, $\Omega_r = 1$, $w = 1/3$

$$a^2(t) = 2 H_0 \Omega_r^{1/2} t$$

$$a \propto t^{1/2}$$

Note the expansion is slower than in EdS because, due to pressure, "gravity is stronger"

The age of the universe in this case is $t_0 = 1/2\ t_H$

→ the radiation-dominated slow expansion leads to a universe that is younger than the one with a matter-dominated faster expansion

Milne universe: only curvature, $\Omega_K = 1$, $w = -1/3$

$$a(t) = H_0 \, t \qquad\qquad \text{Fast expansion}$$

The age of the Milne universe is exactly the Hubble time $t_0 = t_H$

Note that we are consistently finding that **models with faster expansion rates lead to older universes.**

Does this seem counter-intuitive?

de Sitter universe: only cosmological constant, $\Omega_\Lambda = 1$, $w = -1$

In this case, the formula of the general solution is undetermined, and we need to go back to the Friedmann equation to find the solution,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} \qquad \left(\frac{\Lambda}{3} = \Omega_\Lambda H_0^2\right)$$

This tells us that if there is only a non-evolving species, then the Hubble function remains constant: $H(a) = H_0$, and the Friedmann equation is: $\dot{a}(t) = a(t) H_0$

The solution is an exponential expansion → $a(t) = C \exp(H_0 t)$

Given the condition $a(t_0) = 1$, the constant is $C = 1 / \exp(H_0 t_0)$ → $\mathbf{a(t) = \exp[H_0(t-t_0)]}$

Inverting this solution, we can find t(a): $H_0 t = \ln[a \exp(H_0 t_0)]$, i.e.

$$\mathbf{t(a) = t_0 + t_H \ln(a)}$$

If we go from a=1 to a=0, ln(a) is negative, and the time decreases from t_0 to $t(a=0) = -\infty$ → **the age of the universe is infinite.**

Cosmological models with two species

Matter and radiation: $\Omega_m + \Omega_r = 1$ $\quad \left(a_{eq} = \frac{\Omega_r}{\Omega_m} \right) = (1 - \Omega_m) / \Omega_m$

Now there is one free density parameter → **different cosmologies are possible from one model**

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2 \left(\frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} \right) = H_0^2 \, \Omega_m a^{-3} \left(1 + \frac{a_{eq}}{a}\right)$$

$$\frac{da}{dt} = H_0 \frac{\sqrt{\Omega_m}}{\sqrt{a}} \sqrt{1 + \frac{a_{eq}}{a}} = H_0 \frac{\sqrt{\Omega_m}}{\sqrt{a}} \sqrt{1 + \frac{1}{y}} \qquad y = a / a_{eq}$$

It is possible to write an integral expression for t(a) and solve it analytically:

$$t(a) = \frac{t_H}{\sqrt{\Omega_m}} \, a_{eq}^{3/2} \int_0^y \frac{y}{\sqrt{1+y}} \, dy$$

$$t(a) = \frac{t_H}{\sqrt{\Omega_m}} a_{eq}^{3/2} \left(2y\sqrt{y+1}\Big|_0^y - \int_0^y 2\sqrt{y+1}\,dy \right)$$

$$= \frac{t_H}{\sqrt{\Omega_m}} a_{eq}^{3/2} \left(2y\sqrt{y+1}\Big|_0^y - \frac{4}{3}(y+1)^{3/2}\Big|_0^y \right)$$

$$t(a) = \frac{t_H}{\sqrt{\Omega_m}} a_{eq}^{3/2} \quad (y+1)^{1/2}\left(2y - \frac{4}{3}(y+1)\right)\Big|_0^y =$$

$$= (y+1)^{1/2}\left(2y - \frac{4}{3}y - 4/3\right)\Big|_0^y = 2(y+1)^{1/2}\left(y - \frac{2}{3}y - \frac{2}{3}\right)\Big|_0^y =$$

$$= 2(y+1)^{1/2}\left(\frac{1}{3}y - \frac{2}{3}\right)\Big|_0^y = \frac{2}{3}(y+1)^{1/2}(y-2)\Big|_0^y =$$

$$= \frac{2}{3}(y+1)^{1/2}(y-2) - \frac{2}{3}(-2) = \frac{2}{3}\left[(y+1)^{1/2}(y-2)+2\right]$$

$$\boxed{t(a) = \frac{t_H}{\sqrt{\Omega_m}} a_{eq}^{3/2} \frac{2}{3}\left[\sqrt{\frac{a}{a_{eq}}+1}\left(\frac{a}{a_{eq}}-2\right)+2\right]}$$

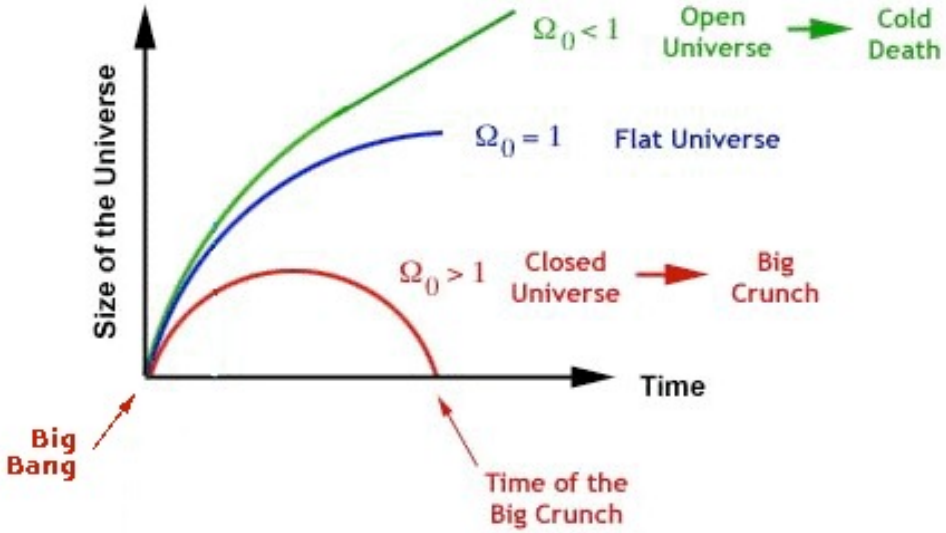
Note the free parameters are t_H (i.e. H_0) and a_{eq} (i.e. Ω)

Matter and curvature: $\Omega_m + \Omega_K = 1$

$$\frac{da}{dt} = H_0\sqrt{|\Omega_K|}\left(\frac{a_K}{a}+1\right)^{1/2} \qquad a_K = \frac{\Omega_m}{\Omega_K}$$

This model can have various cosmologies, grouped in three types:

- $\Omega_K > 0$: **Open CDM (oCDM)**, a(t) expands fast, and the universe is older
- $\Omega_K = 0$: **Standard CDM (sCDM)**, a(t) expands slower
- $\Omega_K < 0$: **Friedmann-Einstein**, a(t) expands slower and contracts



These are the three well-known classical GR cosmologies

Cosmological models with three species

ΛCDM: Matter, curvature and cosmological constant: $\Omega_m + \Omega_K + \Omega_\Lambda = 1$

Note: remember ΛCDM also includes radiation, that we neglect here.

We are left then with two free density parameters, and we can place all the possible ΛCDM cosmologies in **the (Ω_m , Ω_Λ) plane**.

Let us find the possible qualitative behaviours of the various cosmologies:

The goal is to analyse the general evolution behaviour, not the actual a(t) rate.

The general possibilities are expanding or contracting.

From observations, we know the universe is expanding now. So either it has always expanded and will continue to do so in the future, or it had already a contracting phase (or it will have in the future). In this case, $H(t)$ will be (or was) <0

So, we may look for **H(a) = 0 as an indicator of a transition from expansion to collapse (or collapse to expansion).**

This means that, using the Friedmann equation, it is useful to consider the **third-order polynomial f(a)**:

$$\left[\Omega_m + \Omega_\Lambda a^3 + (1-\Omega_m-\Omega_\Lambda)\, a\right] = f(a)$$

Its roots f(a)=0 (for a>0) will correspond to the instants of **transition**

if root a < 1 → transition in the past
if root a > 1 → transition in the future

The flat line $(\Omega_\Lambda = 1 - \Omega_m)$

Consider the particular case of $\Omega_K = 0$.

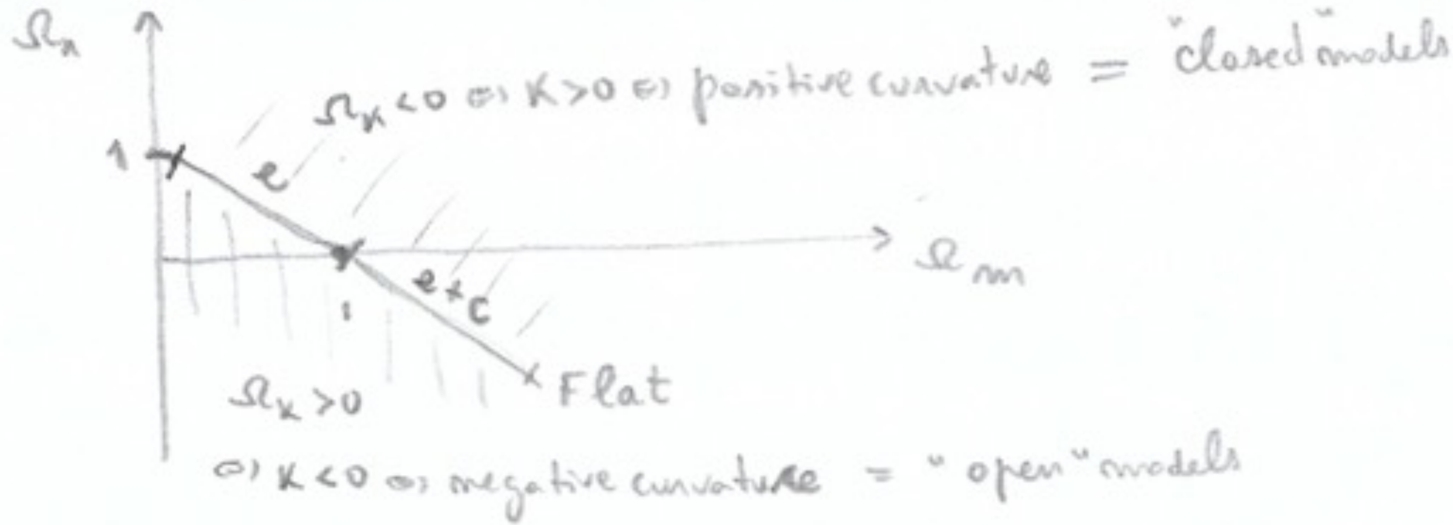
Then all ΛCDM cosmologies lie in the flat line $\Omega_\Lambda = 1 - \Omega_m$

and the transition polynomial simplifies to $f(a) = \Omega_m + (1-\Omega_m)\,a^3$

with roots $$a = \left(\frac{\Omega_m}{\Omega_m - 1}\right)^{1/3}$$

This means that, for cosmologies with $\Omega_m > 1$, there is a transition and the larger is Ω_m the earlier the transition occurs.

For cosmologies with $\Omega_m < 1$ there is no transition



Flat cosmologies lie on this line, and they can be of two types: always expanding (e), or expanding + contracting (e+c).

The line also separates positive curvature and negative curvature cosmologies.

The no-Λ line ($\Omega_\Lambda = 0$)

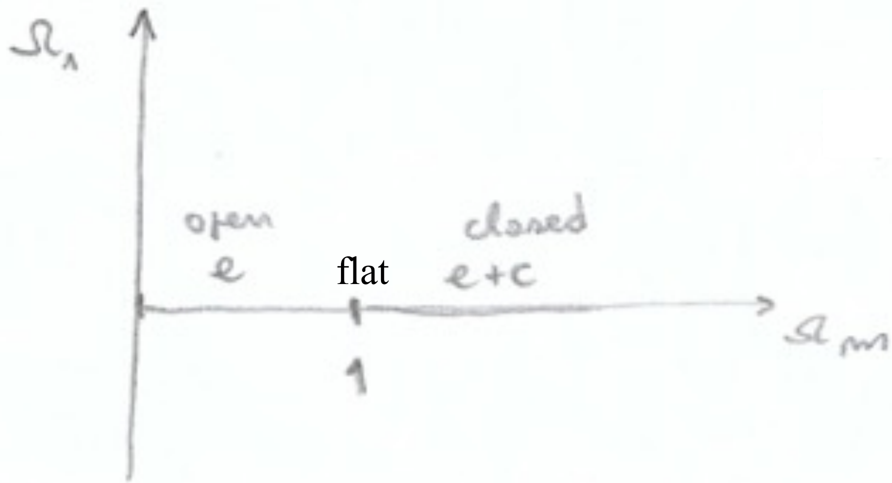
In this particular case of $\Omega_K = 1 - \Omega_m$ we have

$$f(a) = \Omega_m + (1-\Omega_m)\,a$$

$$f(a) = 0 \Rightarrow \left[a = \frac{\Omega_m}{\Omega_m - 1} \right]$$

$\Omega_m < 1 \Rightarrow$ No roots $a > 0$

$\Omega_m > 1 \Rightarrow$ Roots $\quad \uparrow \Omega_m \Rightarrow$ sooner recollapse



We recover the 3 classical cosmologies.

Note however that in general open curvature does not imply (e),

and closed curvature does not necessarily lead to (e+c)

The collapse in the future region (a > 1)

Turning now to the general case, $g(a) = \Omega_m (1-a) + a + \Omega_\Lambda (a^3 - a)$

let us consider examples of collapse in the future:

$a = 2$

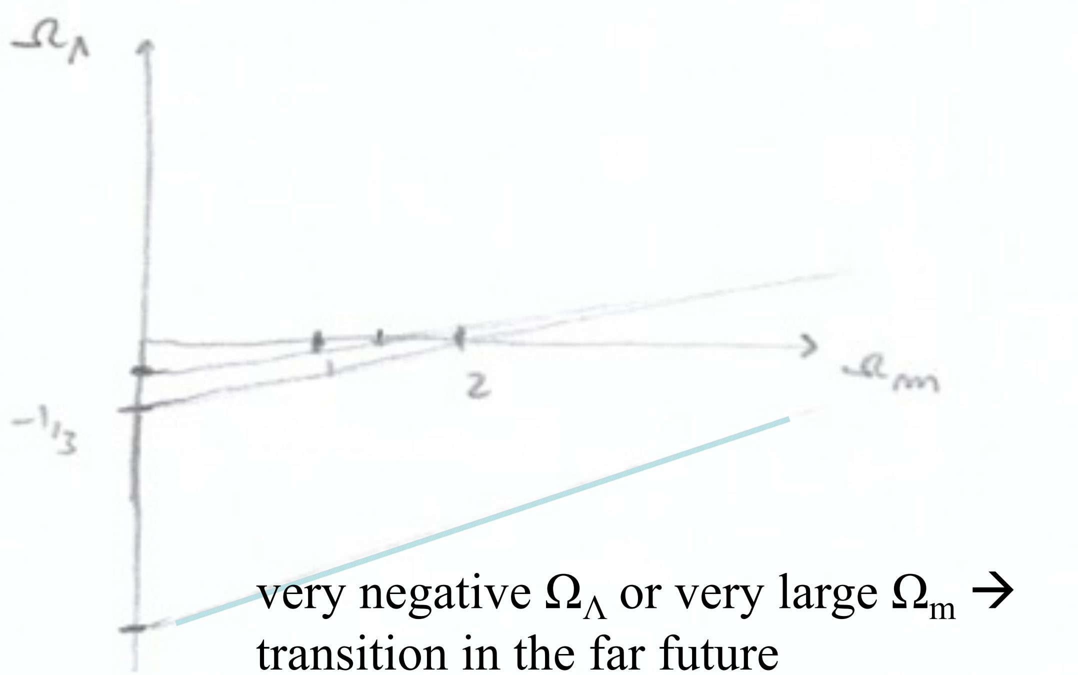
$$\Omega_m(-1) + 2 = -6\,\Omega_\Lambda$$

$$\Leftrightarrow \quad \Omega_\Lambda = \frac{1}{6}\Omega_m - \frac{1}{3}$$

$a = 4$

$$-3\,\Omega_m + 4 = -60\,\Omega_\Lambda$$

$$\Omega_\Lambda = \frac{1}{20}\Omega_m - \frac{1}{15}$$



Cosmologies with this property (e+c with transition in the future) lie on these straight lines (one for each value of transition).

The collapse in the past region (a < 1)

Let us consider examples of collapse in the past.

Note: since the universe is expanding today, these cases imply c+e (i.e., bouncing models with no big bang), instead of e+c → **GR allows models without Big Bang**

$$\Omega_m \underbrace{(1-a)}_{>0} + a = -\Omega_\Lambda \underbrace{(a^3 - a)}_{<0}$$

⇒ The collapse area will have zeropoint at $\Omega_\Lambda > 0$ and slope > 0

$a = 0.1$

$$\Omega_m \, 0.9 + 0.1 = \Omega_\Lambda \, 0.1$$

$$\Rightarrow \Omega_\Lambda = 9 \Omega_m + 1$$

$a \ll 1$

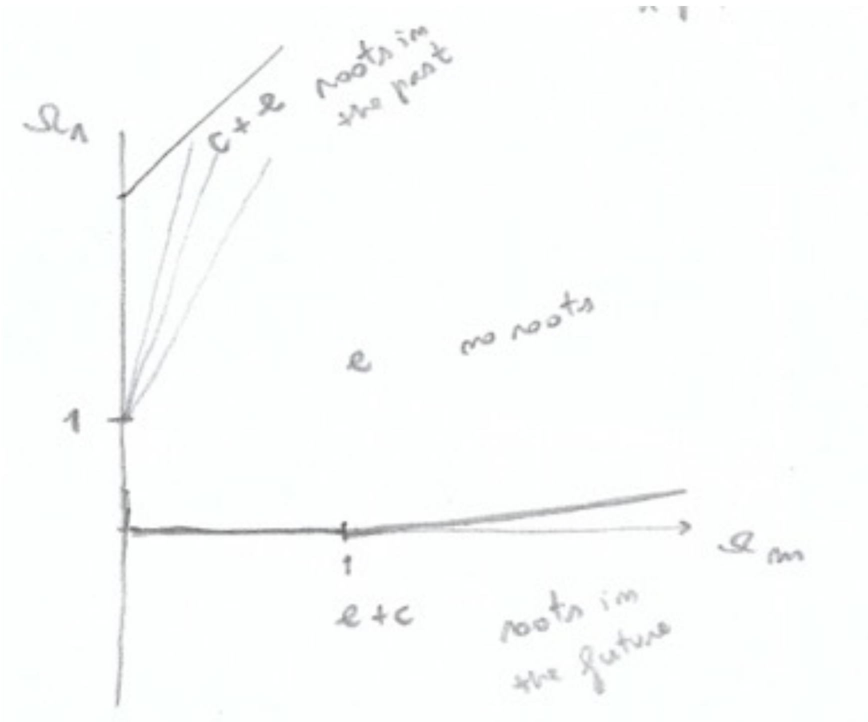
$$\Omega_m + a = \Omega_\Lambda \, a$$

If $a \ll 1$, line starts at $\Omega_\Lambda = 1$

$a = 0.9$

$$\Omega_m \, 0.1 + 0.9 = -\Omega_\Lambda \, (-0.17)$$

$$\Omega_\Lambda = 0.6 \, \Omega_m + 5$$



Cosmologies with this property (c+e with transition in the past) lie on these straight lines (one for each values of transition).

Note: a measurement of the transition redshift would constrain the cosmology → finding the line where the “real” cosmology is → **values along the same line are degenerate with respect to this observable** (the **transition redshift**)

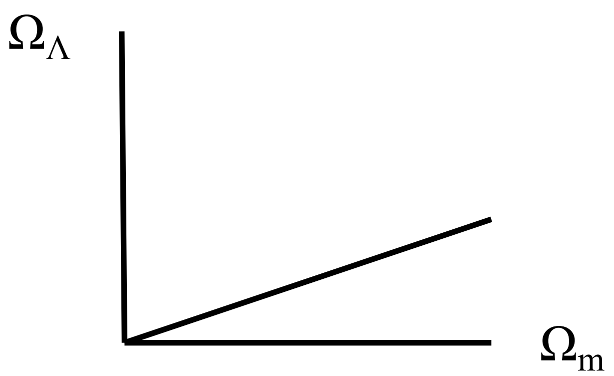
The no-acceleration line ($\Omega_m - 2\Omega_\Lambda = 0$)

Introducing the three species in the second Friedmann equation, we can find a constraint for the cosmologies that do not have acceleration today:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$\Omega_m a^{-3} + \Omega_K [1+3(-1/3)] a^{-2} + \Omega_\Lambda (1-3) = 0 \quad (\text{for } a=1)$$

This is then $\Omega_m - 2\Omega_\Lambda = 0$



Note: a measurement of the acceleration of the universe would constrain the cosmology → finding the line where the "real" cosmology is → **values along the same line are degenerate with respect to this observable** (the **acceleration**)

Note: the acceleration line intersects the curvature line. Two independent measurements (of the acceleration and the curvature) would allow us to find the intersection point of the two lines → **breaking the degeneracy of the cosmological parameters.**

The loitering line

Universes with a c+e transition but with no acceleration at the transition redshift, cannot leave the transition point → they remain trapped at that point with zero H(a) and zero acceleration.

They are called loitering cosmologies and lie on a line separating the no-big bang universes from the big bang universes.

Let us find out what are the scale factors at which the acceleration of a universe can go to zero. Again, from the second Friedmann equation, these are the scale factors that verify:

$$\Omega_m a^{-3} - 2\Omega_\Lambda = 0 \rightarrow a^3 = \frac{\Omega_m}{2\Omega_\Lambda}$$

(the scale factor is different for each universe)

Now, we are looking for cases where this happens at a transition, i.e., which verify f(a)=0

Inserting in f(a):

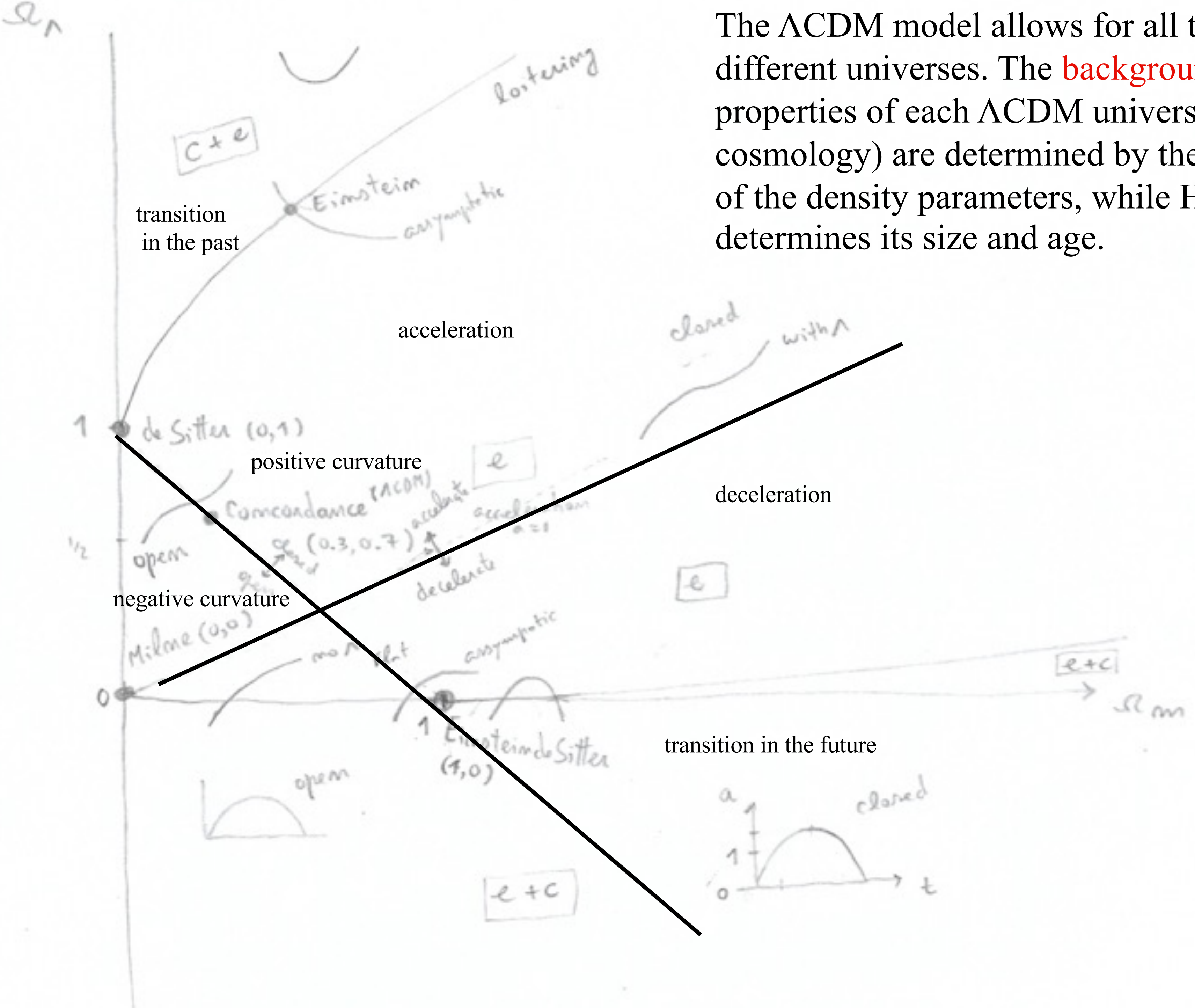
$$\Rightarrow \quad \Omega_m (1-a) + a = -\Omega_\Lambda (a^3 - a)$$

$$\Rightarrow \quad \Omega_m + \left(\frac{\Omega_m}{2\Omega_\Lambda}\right)^{1/3} (1 - \Omega_m - \Omega_\Lambda) - \frac{\Omega_m}{2}$$

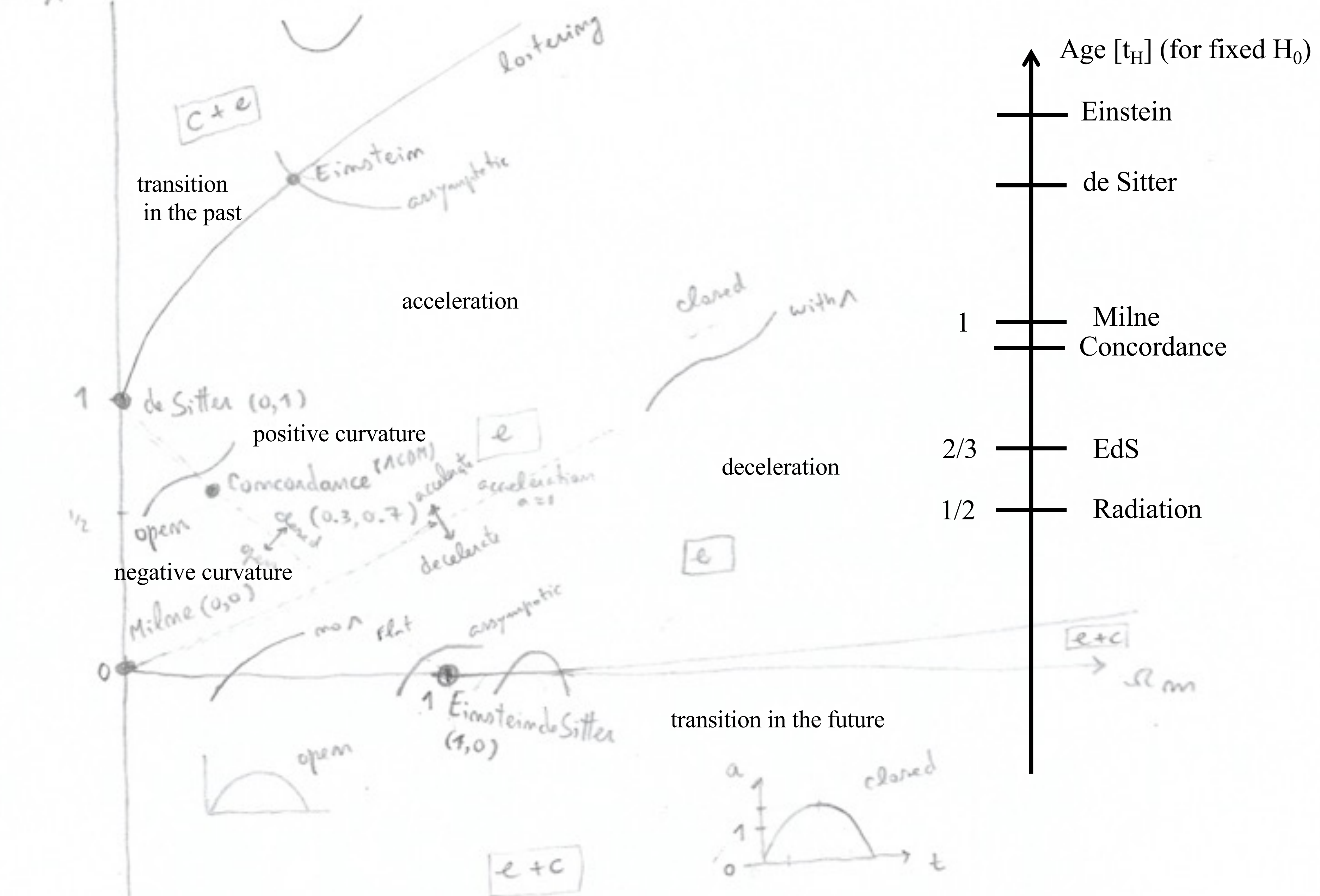
$$\Rightarrow \quad \Omega_m + 2(1 - \Omega_m - \Omega_\Lambda)\left(\frac{\Omega_m}{2\Omega_\Lambda}\right)^{1/3} = 0 \quad \longrightarrow \quad \text{It is the loitering equation}$$

This is a curve in the (Ω_m , Ω_Λ) plane. The well-known static **Einstein universe** is on this curve.

The ΛCDM model allows for all these very different universes. The background properties of each ΛCDM universe (or cosmology) are determined by the values of the density parameters, while H_0 determines its size and age.

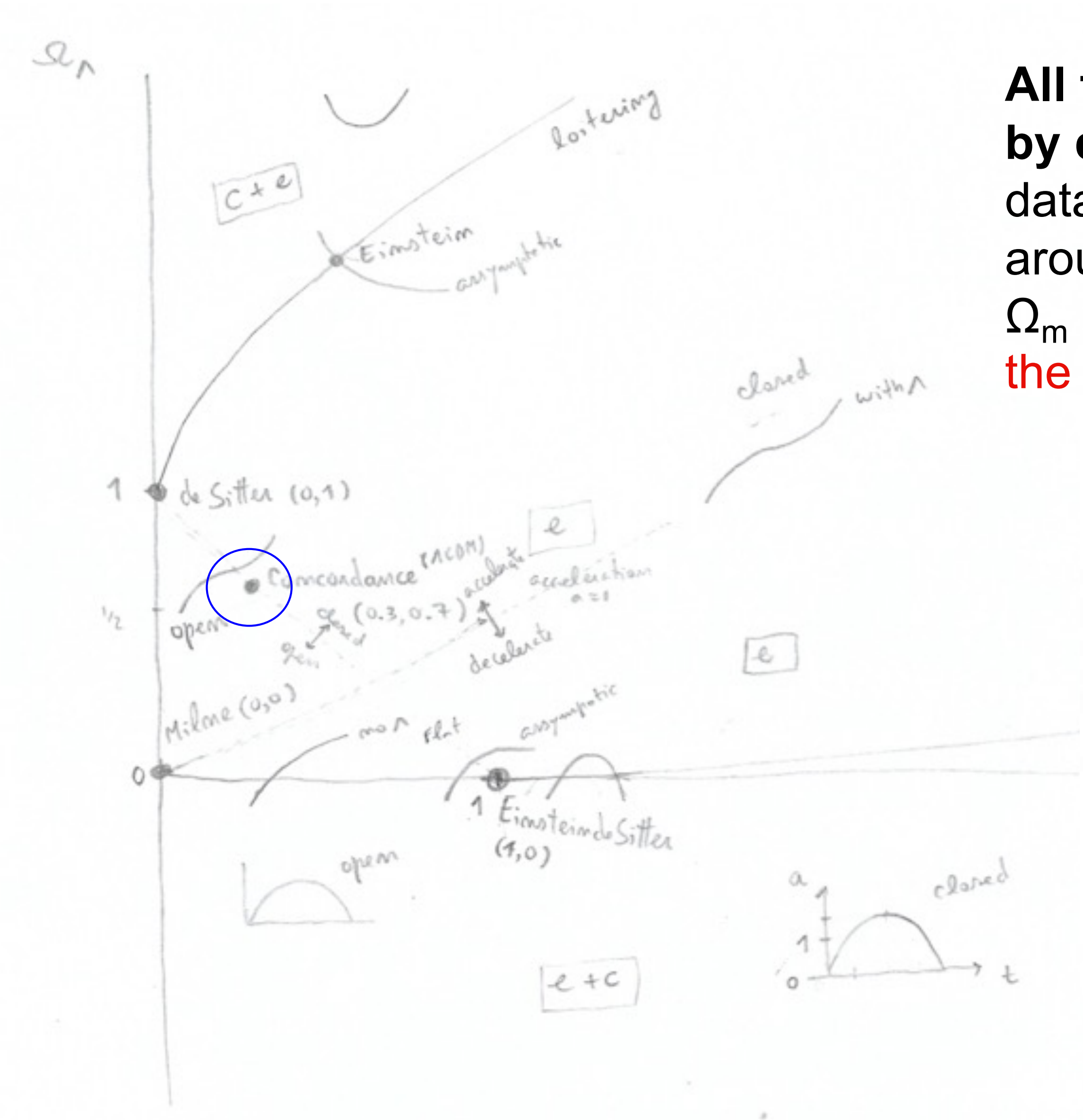


Some well-known cosmologies are:

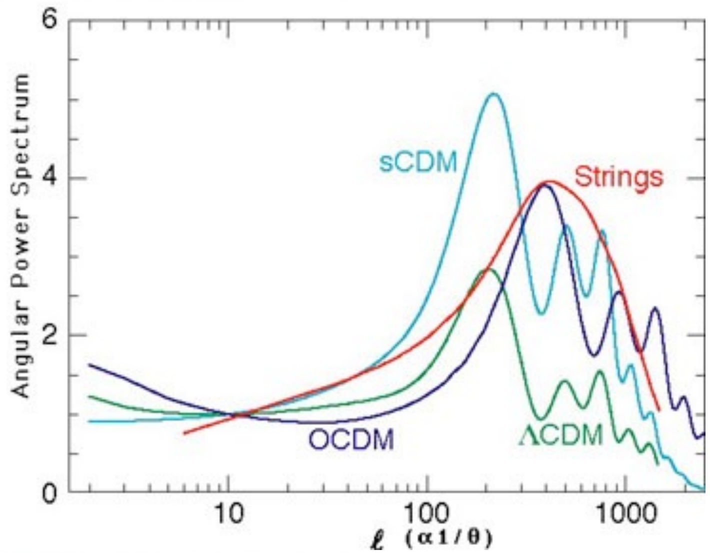


Note: you can compute background properties (age and distances) of these cosmologies, using the on-line cosmology calculator: *http://www.astro.ucla.edu/wright/CosmoCalc.html*

The concordance cosmology

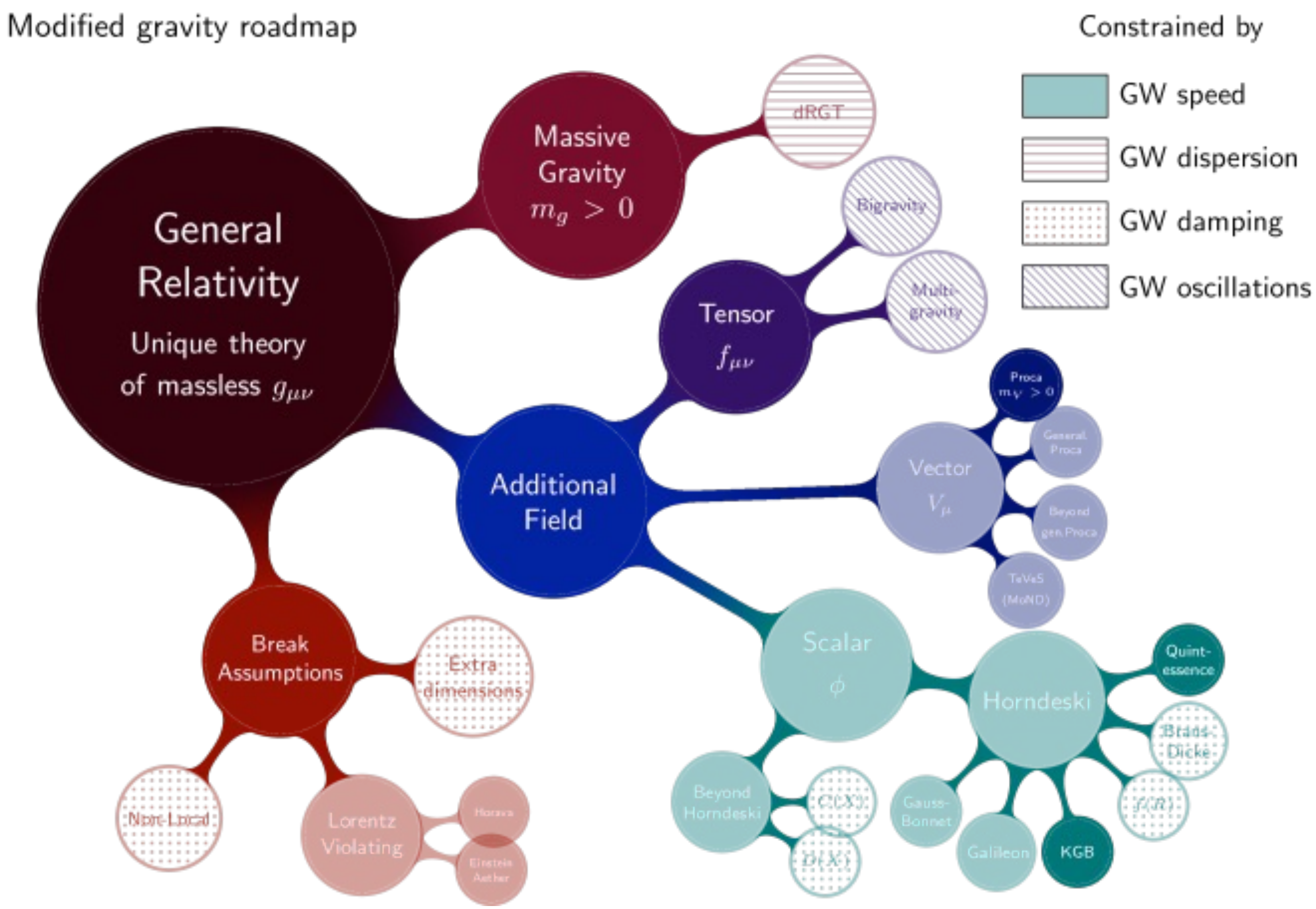


All these cosmologies are ruled out by data. The high-precision of current data only leaves a small uncertainty around the region defined by $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$ (and so $\Omega_K = 0$): the concordance model



However, there is still room for new models,
because modern cosmological models are not spread out through-out this plane, since they need to be close to the concordance model.

They consist mainly of different evolutions for ρ_{DE} (a) and w_{DE}(a) (instead of being constant), but that lead to the same Ω_{DE} ~ 0.7 and w(a=1) ~ -1



Ezquiaga & Zumalacarregui 2018, https://arxiv.org/abs/1807.09241v1

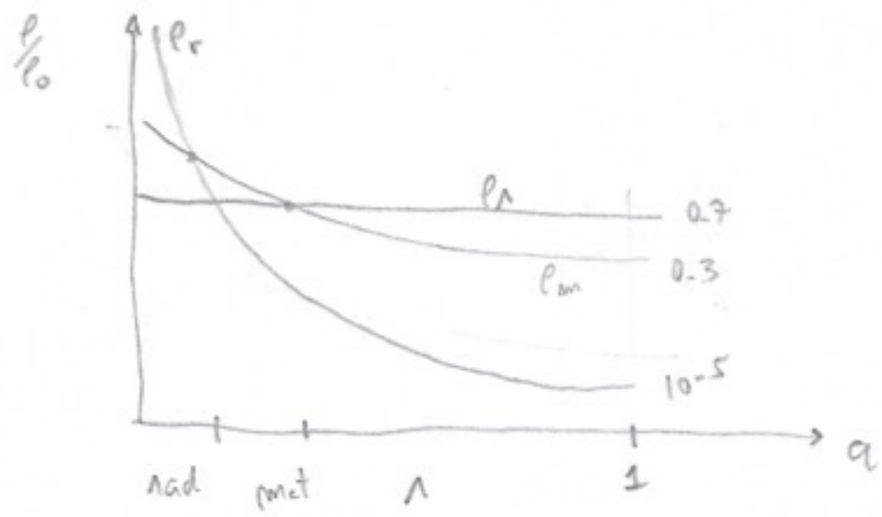
The values of the density parameters determine the behavior of the homogeneous Universe (also called the **background**).

Even though there are several open problems, the cosmology favoured by the observations is the so-called **concordance cosmology** (given in round numbers):

ΛCDM with $\mathbf{\Omega_m = 0.3, \Omega_\Lambda = 0.7, \Omega_K = 0, \Omega_r = 8 \times 10^{-5}, h = 0.7}$

Epochs of domination

Given these values and the functional forms of the densities, there is a sequence of **epochs of domination** in the evolution of the Universe: **the total density of the Universe is dominated by radiation, matter, and finally Λ .**



We can easily find the scale factor (or redshift) when the two transitions occur:

radiation / matter a_{eq}

$$\rho_m(a_{eq}) = \rho_{rad}(a_{eq})$$

$$\rho_{0m}\, a_{eq}^{-3} = \rho_{0r}\, a_{eq}^{-4}$$

$$\Omega_m\, a_{eq}^{-3} = \Omega_r\, a_{eq}^{-4}$$

$$a_{eq} = \Omega_r / \Omega_m = 2.67 \times 10^{-4}$$

$$z_{eq} = 3\,749$$

matter / dark energy a_Λ

$$\rho_m(t_\Lambda) = \rho_\Lambda$$

$$\Omega_m\, a_\Lambda^{-3} = \Omega_\Lambda$$

$$a_\Lambda = (\Omega_m / \Omega_\Lambda)^{1/3} = 0.75$$

$$z_\Lambda = 0.33$$

Age of the concordance universe

Knowing the values of the cosmological parameters, we can compute **the age of the concordance universe**
(hence age, if measurable, is another quantity - like distances, curvature, transition redshifts, horizon sizes, etc - that can constrain the parameters)

For this, we just need to compute the integral found from the Friedmann eq:

$$\left(\frac{\dot a}{a}\right)^2 = H^2(a) \Leftrightarrow \frac{\dot a}{a} = H(a) \Leftrightarrow da = a\,H(a)\,dt \Rightarrow t(a) = \int_{a_i}^{a_t} \frac{da}{a\,H(a)}$$

$$\Leftrightarrow t(a) = \underbrace{\frac{1}{H_0}}_{t_{\rm Hubble}} \int_{a_i}^{a_f} \frac{da}{\underbrace{a\,E(a)}_{\text{funcț } E(a)}} \longrightarrow E(a) = \left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_k a^{-2} + \Omega_\Lambda\right]^{1/2}$$

or, in terms of redshift $1+z = 1/a \rightarrow da = -\frac{dz}{(1+z)^2}$

To have a rough estimate of the age, let us compute the duration of each of the three epochs, considering the simplification that only one species is relevant during each of the epochs:

Radiation epoch

• $0 < a < a_{eq}$

Ω_r only

$$t_{eq} = \frac{1}{H_0} \frac{1}{\sqrt{\Omega_r}} \int_{z_{eq}}^{\infty} \frac{dz}{(1+z)^3} = \frac{1}{H_0\sqrt{\Omega_r}} \frac{1}{2} \frac{1}{(1+z)^2} \Big|_{\infty}^{z_{eq}}$$

$$= \frac{1}{2} \frac{1}{H_0 \sqrt{\Omega_r} (1+z_{eq})^2} .$$

$z_{eq} = 3749 \rightarrow t_{eq} = 4.0 \times 10^{-6}\, t_H = 55\,000$ yr

($h = 0.7 \rightarrow t_H = 13.97$ Gyr)

Matter epoch

• $a_{eq} < a < a_\Lambda$

Ω_m only

$$t_\Lambda - t_{eq} = t_H \frac{1}{\sqrt{\Omega_m}} \int_{z_\Lambda}^{z_{eq}} \frac{dz}{(1+z)^{5/2}} = \frac{t_H}{\sqrt{\Omega_m}} \frac{2}{3} \frac{1}{(1+z)^{3/2}} \Big|_{z_{eq}}^{z_\Lambda}$$

$z_\Lambda = 0.33 \rightarrow t_\Lambda = 0.61\ t_H = 8.52$ Gyr

Dark energy epoch

• $a > a_\Lambda$

$$t_0 - t_\Lambda = t_H \frac{1}{\sqrt{\Omega_\Lambda}} \int_0^{z_\Lambda} \frac{dz}{1+z} = \frac{t_H}{\sqrt{\Omega_\Lambda}} \ln(1+z) \Big|_0^{z_\Lambda} = \frac{t_H}{\sqrt{\Omega_\Lambda}} \ln(1+z_\Lambda)$$

$t_\Lambda = 0.34\ t_H = 4.76$ Gyr

The radiation epoch is very short,

the matter epoch is the longest one,

the dark energy epoch did not start so recently as we might think

age of the Universe = 0.95 t_H = 13.28 Gyr

Characteristic sizes

We can also compute various characteristic sizes and distances in the concordance Universe:

(remember a comoving distance is dx = dt/a = da/a²H)

- the **particle horizon H_p** at a given time is the distance travelled by light since the big bang up to that time.

It is thus given by (comoving):

$$H_p(a) = \int_0^a \frac{c}{a'^2 H(a')} da'$$

- the **event horizon H_e** today is the maximum comoving distance that light can travel from today until the end of the Universe (t = ∞). This implies that light emitted today by an object farther than that distance will never reach us.

It is given by (comoving):

$$H_e(a=1) = \int_1^\infty \frac{c}{a'^2 H(a')} da'$$

- the **size of the observable Universe** at a given time is the distance between the observer at that time and the decoupling redshift (the last scattering surface that released the CMB radiation), beyond which the Universe is opaque.

It is thus given by (comoving):

$$D_c(a) = \int_{0.00091}^{a} \frac{c}{a'^2\, H(a')} da'$$

- the **Hubble radius**, given by (proper):

$$r_H(a) = \frac{c}{H(a)}$$

All these quantities are computed from the **Hubble function**, which in the concordance cosmology is given by:

$$H(a) = H_0 \left(\frac{0.3}{a^3} + \frac{8 \times 10^{-5}}{a^4} + 0.7 \right)^{1/2}$$

Using the concordance values for the density parameters, h=0.7, and

$1+z_{eq} = 3750 \rightarrow a_{eq} = 2.67 \times 10^{-4}$
$1+z_{dec} = 1101 \rightarrow a_{dec} = 9.1 \times 10^{-4}$

we can compute all these quantities.

Feature	a_{eq}	a_{dec}	a_0
Horizon_particle			
comoving [Mpc/h] ([Mpc])	73 (104)	197 (281)	9738 (13911)
proper [Mpc/h]	0.019	0.18	9738
Hubble radius			
comoving [Mpc/h] ([Mpc])	64 (91)	143 (204)	3000 (4286)
proper [Mpc/h]	0.017	0.13	3000
Observable Universe			
comoving = proper [Mpc/h] ([Mpc])	-	-	9541 (13630)
Horizon_event			
comoving = proper [Mpc/h] ([Mpc])	-	-	3422 (4889)

Notice that an event horizon exists because the Universe is accelerating. In EdS the event horizon is infinite, all emission will eventually reach the observer.