represents all the labor done in the past. If he did not build the house yet, we are talking about the labor of the future. What the client and the bank did was exactly that, they exchanged present money for future money. This exchange we call loan or credit is the most important of all financial instruments and will be ubiquitous throughout the course.

For now, our problem is in the production of money. To lend the money to the client, the bank will have to borrow money, either from the depositing clients or other banks to whom other clients have deposited money and have not been able to lend it to other clients. Nevertheless, the house is a quantity of labor of the future, which means that, at the outset, there is still no money to represent it, so the money needed to lend to 'our' client may not exist yet. In this case, the bank will have to borrow from the central bank. The central bank is the entity responsible for issuing money in a modern economy. When it does, it means that the money has run out, and there is no more labor from the past available to be exchanged for the labor of the future, and this is how we produce money to match the value of labor.

Problem

2.5 Discuss the production of money by means of government bill.

2.4 The Value of Money

We have already focused on creating money and how it depreciates over time. In the previous section, we introduced credit or loan, but here we will define it strictly from a physicist perspective.

We call **loan or credit** the economic exchange of money in the present for money in the future. Let us remember that we define the economy as a system of exchange, and therefore, all economic acts must frame as such.

When we put ourselves in the position of the M observer who is going to lend money, that is, he will give money to a M' observer in the present to receive money in the future, we have a problem that we have already raised on in the section 2.2. The problem is that a specific amount of money in the future does not have the same value as the same amount of money has in the present. For the exchange to represent the same amount of labor for both observers, then the exchange must not be of numerically equal quantities. Let us then realize what quantities we are talking about to correspond to equal amounts of labor.

From Eq. (2.4) we know that the value of money in units of labor decreases from instant t_1 to instant t_2 by $y(t_2) = y_1 e^{-\alpha t_2}$. The expression is itself rigorous if we take time as continuous. But in order to cope with the overwhelming majority of human applications we have to transform time intervals in discrete intervals. Let's think backwards. Let's say that in every $t_2 - t_1$ period the money worth 1 at t_1 becomes $1-\alpha$ at t_2. Dividing $t_2 - t_1$ into n equal parts, in each of these parts, the money devalues $(1-\alpha/n)$. In the first, $(1-\alpha/n)$, in the first two $(1-\alpha/n)(1-\alpha/n)$, in the first three $(1-\alpha/n)(1-\alpha/n)(1-\alpha/n)$, ..., in the n periods $(1-\alpha/n)^n$. If we divide $t_2 - t_1$ into an increasing number of intervals, we have

$$\lim_{n\to\infty}\left(1-\frac{\alpha}{n}\right)^n = e^{-\alpha t} \quad (2.5)$$

that is the result of Eq. (2.4). For practical reasons, it is preferable to assume a α rate for a fixed period, for example, one year, and when we transform to continuous time, we use Eq. (2.5).

Going back to the way money is created, if the central bank knows that money will devalue α each year, it will have to ask other banks to return the same value by the end of that year, i.e., $1+\alpha$. In addition to this, the central bank adds its labor so that one economic exchange represents equal amounts of labor for both sides. Thus, rather than lending at the inflation rate, the central bank will lend to an *interest rate*. We will use r to represent an interest rate for one year period. We can describe economic exchange as an exchange between the central bank and another bank for the same labor quantity. The central bank delivering to the other bank money at present, say 1, while the other bank is delivering to the central bank the same value of money in the future, $1+r$. Thus, we have established from a physicist perspective how the time value depreciation of money works. Furthermore, this is the foundation of much of the entire financial system. To be crystal clear, we are just making coordinate transformations due to the reference frame's movement.

With this economic equivalence in mind, we can start dealing with a loan without having to repeat it all the time. We know that the value of 1 Euro today and the value of 1 Euro a year from now are different things and why they are different. Because of this difference, when we lend money, we 'charge interest' so that the exchange is fair, that is, so that the amount of labor exchanged is equal. Hence, from now on, we will refer to the value of money without reference to exchange.

From this point on, we will base the reasoning always using the concept of interest. Having 1 Euro today means that we can have $(1+r)$ Euro in a year, which is equivalent to say that having a Euro a year from now is having $1/(1+r)$ Euro today. A Euro two years from now is equivalent to having $1/(1+r)^2$ today and sequentially. Having 40 Euro a year from now is equivalent to having $40/(1+r)$ euros. The coefficient $(1+r)^{-n}$ is called *discount factor*.

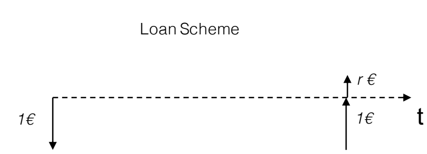


Fig. 2.4 Cash flow scheme of a loan.

The flow scheme such as that shown in Fig.2.4 is best suited for understanding how we deal with financial issues. The shapes of the scheme vary with the taste of those who make them, with the author's preference being the schemes in which downward arrows represent exits and up arrows represent entries and the size of the arrows represent amount. Thus, in the scheme of Fig.2.4, 1 Euro comes out at the first instant (down arrow). In the second instant, 1 Euro enters when the loan amount is returned (up arrow) plus one entry, the *r* Euro, corresponding to interest (up arrow).

This scheme works for the observer that is lending the money. For the receiving observer, the signals are opposite, and the loan is for this observer a *deposit* (see Fig. 2.5).

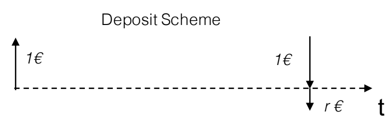


Fig. 2.5 Cash flow scheme of a deposit.

The value of the cash flow multiplied by the discount factor is called the *present value* of the flow.

From this point forward, we will designate market participants as *agents* so that we are more in line with the general literature on the subject.

As we have seen, there is a situation of 'reflection' between the two agents—the lending agent who delivers the borrowed money and the borrowing agent who receives the deposit money. The first will receive money in the future for the exchange, and the second will pay money in the future for the exchange. We call *assets* the financial instruments by which we will receive money. The loan is, in this case, an asset of the lending agent. On the other hand, *liabilities* are the financial instruments for which we will pay money. In this case, the loan is a liability of the debtor agent.

This classification is essential for us to map between our view of these basic concepts and communication between companies and the surrounding environment, a fundamental aspect of financial activity. Business communication with the exterior happens by using reports commonly referred to as *accounting*, that inventories assets - what the company will receive money for - and liabilities - what the company will have to pay. The difference between the two amounts is called the *capital* of the company, which bears a slight resemblance to the concept of capital that we introduced in the definition of economy. The company's capital is, roughly speaking, the money that belongs to the company and, consequently, its owners. We will set back to business owners later when we speak about shares.

Problem

2.6 Would you classify company capital as an asset or as a liability? Argue based on the definitions above.

As an important note, corporate assets and liabilities are not necessarily loans and deposits. Not even companies that live on loans and deposits - called 'banks' - are like that.

Problem

2.7 Is a building where a company has its headquarters an asset? Argue based on the definitions above.

We made this short foray into accounting concepts to understand what information the various market players put on the street for our knowledge. Of course, nothing subjected to report is entirely rigorous or unquestionable. For a straightforward reason: if we remind ourselves of what we define, nothing is about the past or the present. It is all about the future, whether the money is to be received or paid. Even in accounting terms, only current year results are likely to be specialized because the past does not matter. Only assets and liabilities are distinguishable. Thus, whatever we can say, any number we put in an accounting record always has an associated uncertainty factor. In other words, what we say is going to happen always has a *risk* associated with it, and this is the reason why anyone talks about finance. For now, we will stick with this idea.

Going back to the company's capital, we said that this is the company's money; its owners own it. Vital to us is that regulation requires that many companies, especially those above a given business turnover threshold, have their capital represented by a 'piece of paper.' Historically they were papers that attributed the right to its holder to receive a part of the company's profits; they were called *shares*. Some keep the shares in the original owners' hands and call themselves *private* (because they have no obligation to report other than their owners and other stakeholders). However, many decide to put the shares up for sale in organized markets where those interested in company ownership buy and sell these *securities*. We call these companies 'public' because they must report much more and be more transparent to these organized markets. The idea to retain is that these securities assign a fraction of the company's ownership to their holders. With it, the right to receive part of the profits in the percentage corresponding to the number of securities held. This profit payment is called *dividend*.

The organized markets where one trades these shares are called *stock exchanges*. They are the most popular financial markets, and on which we will pay particular attention, both for historical and practical reasons.

Symbol	Company Name	Last Price	Change	% Change	Volume
WBA	Walgreens Boots Alliance, Inc.	66.48	-0.01	-0.02%	4,609,632
CSCO	Cisco Systems, Inc.	43.75	-0.03	-0.07%	15,859,970
PG	The Procter & Gamble Company	81.52	0.09	+0.11%	5,068,223
UNH	UnitedHealth Group Incorporated	260.91	0.55	+0.21%	1,635,221
AXP	American Express Company	101.81	0.23	+0.23%	3,389,232
PFE	Pfizer Inc.	40.81	-0.12	-0.29%	17,634,149
BA	The Boeing Company	338.36	-1.05	-0.31%	1,877,236
MCD	McDonald's Corporation	158.14	-0.54	-0.34%	2,879,590
V	Visa Inc.	140.21	0.48	+0.34%	4,817,496
JNJ	Johnson & Johnson	130.22	-0.53	-0.41%	3,760,874

Fig. 2.6 NY Stock Exchange Listing. Source: Yahoo Finance

Examples of these markets are the New York Stock Exchange (NYSE), Euronext, or the London Stock Exchange (LSE), and we can identify Cisco Systems or Visa as public companies. As an example of private companies, whose capital is represented by shares but not traded in an organized market, we have Caixa Geral de Depositos and Auchan.

Problem

2.8 Use the definition of company's capital and discuss why stock price changes.

2.9 Discuss the possibility of using McDonald's Corp shares as money.

Of course, if a company owns shares of another company, those shares are assets of that company because it expects to be remunerated for holding those securities. Both lending money and buying a share underly the idea that there is an associated remuneration which, let us generalize, we will call *rate of return*. Thus, both assets are comparable if we look at them as alternatives in an investment choice. If we are the holders of a finite amount of money and want to use it, we will do what economists call the choice of alternatives, and we will take an option that will be as close as possible to the more rational one. In that sense, we will need to measure the uncertainty we have about that profitability. This match between expected return and risk is the basis of any investment decision that can be summed up in *for the* *same risk, I choose the highest return; for the same profitability, I choose the lowest risk*[1].

Having made this introduction to finance basics, we need to understand how uncertainty enters the set of deterministic statements we have made so far.

2.5 Risk

Let us go back to the loan we represented in the previous section. When we represented it, we took an unstated assumption that the debtor will repay the loan. Only in this way will the $(1+r)$ inbound flow in the future be equivalent to the 1 outflow in the present.

Now let us drop this assumption. Let us say that there will be a deviation between the loan's actual repayment (or, generally, the investment) and the initial expectation. This deviation is called loan risk but applies to all types of investments where a deviation from the initial expectation is possible.

Strictly speaking, there is no such thing as a risk-free investment. *A priori*, when an OECD state issues debt in its currency, this is considered to be as close as possible to a risk-free investment, although cases of losses in high-quality debt are known. More recently, the case of the haircut of the Hellenic Republic's debt was an example of a risk-free investment that ended up damaging creditors to half of what they lent. In this case, it was not debt in its currency (when the debt is in a different currency from the issued by the country is called *sovereign debt*). In financial jargon, when the debtor fails on debt repayment, it is said to *default*.

Problem

2.10 Discuss the importance of debt being in the own currency in the previous paragraph. Use the money making diagram.

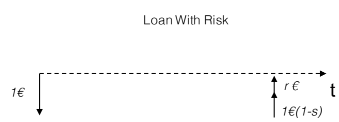
There is a risk in all human activity where there is an expectation regarding the future. It can be in our remaining life, what we hope to earn, the stability of our job, our home's value, in other words, what we do not know for sure. The idea to be retained is that risk is everywhere. We do what separates agents of the financial market from others, so our subject is exclusive 'how we deal with risk' from this point forward.

Looking at a loan seems a little tricky to deal with the borrower's risk of not paying back. Either the debtor pays, or he does not pay; this reduces the situation to the point where we only lend if we are sure about the payment, and that is absurd. The solution is to make several similar loans (a *portfolio*) as banks do, hence measuring

Fig. 2.7 News of the Hellenic Republic's default where the creditors did everything they could for not being declared default. Source: CNN

the probability of the borrower not repaying. Representing this probability by *s*, it comes (see Fig.2.4) that the future inflow ceases to be equivalent to the present outflow. To match the *r* interest rate, we need to add *s* to cover the risk issue.

Fig. 2.8 Loan with risk. To get back to the same scheme as before, we need to raise interest rates with *s*.



This additional is called the *risk spread* which is higher as higher is the risk of the debtor.

Problem

2.11 Look for news on the internet about the risk spread difference between the German Federal Republic and the Portuguese Republic. Justify this difference in qualitative terms.

With risk spread, we find the last parcel that justifies the existence of an interest rate on loan—first, inflation. Second, the labor of producing a loan. Third, the risk of the debtor not paying. All of this comes together in one equivalence. The outgoing flow in the present being equal to the present value of the incoming flow in the future so that the exchange is fair. We now need to know how we numerically find this probability, which will become our main subject.

Note that in all investments, there is a risk assessment. However, it is not always possible to have a quantitative assessment (what is the probability, for example, of default of the United States of America?) So we can only stick to qualitative and relative information and, from it, start to quantify the risk. Nevertheless, finding numbers from relative positions has never been a problem for a physicist.

Some situations are too obvious, such as lending money to a healthy business is less risky than lending to a bankrupt business. However, some people devote their time to making these qualitative assessments based on numbers, but often on subjective criteria for qualitative and grading classifications. These grading classifications are called *ratings* and these people work for companies providing this service called rating agencies.

Let us imagine that the size of the economy is a criterion to classify the Federal Republic of Germany's debt and the Portuguese Republic debt. The first is one of the world's largest economies, holding one of the most robust industries. The second is one of the poorest economies in Europe. From this information, we cannot tell if the probability of the first one defaulting is less than that of the second one? Yes, we can. We cannot say, however, how much smaller. Therefore, agencies say *the first is better to lend than the second* by issuing ratings that gradually rank debtors.

FitchRatings Search

Fitch Affirms Germany at 'AAA'; Outlook Stable

03 AUG 2018 04:04 PM ET

Fitch Ratings-London-03 August 2018: Fitch Ratings has affirmed Germany's Long-Term Foreign-Currency Issuer Default Rating (IDR) at 'AAA' with a Stable Outlook.

A full list of rating actions is at the end of this rating action commentary.

KEY RATING DRIVERS
The 'AAA' ratings reflect Germany's diversified, high value-added economy, strong institutions and history of sound public debt management. A large structural current account surplus, averaging 8.2% of GDP from 2014-2017, supports the country's net external creditor position. Germany's position as the primary benchmark issuer for the eurozone affords it significant financing flexibility. General government debt, at 64.1% of GDP in 2017, is higher than the current 'AAA' median of 41.2%, but is on a firmly downward path. Germany has run general government surpluses since 2014, and our 2018 forecast of a 1.4% of GDP surplus compares favourably with the 'AAA' median of 0.4%.

Fig. 2.9 Example of risk rating issued by Fitch Ratings, one of the most important agencies.

The fact that rating agencies rarely refer to a probability value, giving only qualitative and gradual ratings, does not prevent users of their information from building models based on these ratings. We will go into more detail on this subject when we talk about credit risk in greater depth. The reader should keep the notion that there exist people who worry about risk and give opinions summarized in a classification accepted by the financial markets.

An alternative/complementary form is the assignment of *scores* to each of the characteristics of a debtor—for example, married-1 / single-0. In the end, all characteristics and scores add in a final number to support the decision to lend to a debtor, typically a person. These models are called *scoring models* and are also a way to approach risk in a qualitative and gradual manner. Later we will also discuss this type of model.

In addition to the credit risk, we also have the so-called *market risk*, i.e., the risk of financial loss due to a price deviating from expectation; *operational risk*, also known as the 'fat finger risk', which is the risk of financial loss due to business operation anomalies (for example, instead of buying 100 shares, buying 100000 by mistake); *image or reputational risk* which is the risk of loss due to unexpected events affecting a company's image (e.g., selling spoiled food) and there are several other types of risk related to other types of unexpected events. As an idea to keep in mind is that risk is always present and can take on different types, the challenge is to eliminate them or, at least, eliminate the associated losses.

This section first introduced the subject of risk that will become ubiquitous on the remaining pages. Finance is nothing more than dealing with uncertainty. To conclude, when we mentioned earlier that the agents choose the one with the highest expected return for the same risk, and for the same expected return, the agents choose the one with the lowest risk; it is now easy to understand why.

Problem

2.12 Based on the example of the loan from Fig. 2.8 show that the risk-return match is what guarantees a fair exchange.

2.6 Instruments

We have already introduced two basic financial instruments, simple loans, and shares for practical reasons. The first difficulty for a physicist to enter financial markets relates to the financial instruments traded, the metalanguage associated with them, and mostly why they get traded in the first place. The first lesson to take is that if many people trade an instrument, stupid as it may seem to us at first glance, it is useful and necessary for them. Moreover, if necessary, speculators will appear to try to make money from the needs of others. Furthermore, this speculation is essential because it makes a price appear and produces *liquidity*, that is, to have someone that wants to buy when someone wants to sell.

Let us take the case of rice. Rice is the most important agricultural product by far as it is the staple food of about half of humanity. However, like all agricultural products, its production is surrounded by risk factors. The seed's price, the quantity of rain, the level of nitrates in the soil, the hours of light, and the final product's price are some examples of risk factors involved. Some factors derive from human activities; others do not. The key is to *turn all factors that are not financial into factors of financial nature*. Take, for example, rainfall. For the rice producer, insufficient rain will always be a problem because rice is a culture that needs much water, and, in principle, there is nothing he can do about the level of pluviosity. On the other side of the economy, too much rain is a problem for hotels that live off the tourist stream, and for them, there is nothing they can do to avoid it. In fact, by combining the two problems, the problem can be solved by targeting the financial consequences and mitigating - or even eliminate - the risk of excessive or deficient rainfall by moving the farmer's imponderable financial consequences to the hotel vice versa. Instead of trying to influence the rain, one influences the financial consequences.

From the most basic to the most sophisticated, all financial products' origin is based on such everyday things as rice production or hotel room sales. The farmer asks for a loan to buy seeds and, to guarantee the payment, needs to guarantee production by making an *option* over rainfall (literally 'buying rain'). He needs to guarantee the final product's price by making a *future*. He needs to ensure that the loan interest does not change by making a *swap*. In the end, all imponderables are turned into financial problems by applying products that interest other market agents, such as in the case of rainfall where hotels would want to 'sell' the rain the rice farmer would want to 'buy.'

To understand financial instruments from a practical point of view, let us start by defining the most basic instrument of all in terms of market jargon. However, it is the most abstract given that it summarizes what we have been saying so far. The instrument is the so-called *money market account*, that is, the current account through which I pass all the money. In practice, it is something that is only accessible to banks, the treasury account where they pass all flows from deposits and loans as described in the 2.3 section. By definition, a money market account is a financial instrument whose numerical value $B(t)$ is determined by

$$B(0) = 1 \tag{2.6}$$

$$dB(t) = \alpha(t)B(t)dt \tag{2.7}$$

which is no more than Eq. (2.2) with the rate α depending on time.

This instrument represents the cost of money at any given moment, and the $B(t)$ function is what we may call 'money' this point forward.

The rest of the chapter is about financial instruments. About what they are, the associated metalanguage and protocol, and mainly why they are useful. The fundamental to take to the following sections are the concepts associated with the value of money. Nevertheless, we will stress its importance again as we move forward.

2.7 Bonds and Interest Rate

A bond is a loan that can change hands like a banknote. Instead of being a loan in which one person lends directly to another, the person in need of the loan issues bonds whose value is a fraction of the total amount they need and then who wants to lend, subscribe or buy those securities.

Why is this better than the loan about which we spoke already? Because there is no obligation on the lender to be without the money until the end of the contract. Also, he does not lend everything the other part is asking. We can sell the bonds one buys a few days after or keep them until they are due (i.e., until the end). The end is called the *maturity date*.

Fig. 2.10 Bond with interest coupons attached to the right side. Source: Wikipedia

Bonds began to finance large projects, such as dams or canals, in situations where the investment period is much more extended than most people are available to stay without the money. Today their issuance is widespread, either by states or by companies and usually traded in organized markets as stock exchanges.

In the past, when electronic platforms for dealing with financial instruments did not exist, bonds were issued in paper with a series of detachable 'coupons' corresponding to each of the interest payments in exchange for which the issuer paid the interest (see 2.10). On each of these dates, the bondholder would detach one of the coupons and receive the interest money from the issuer. Therefore, even today, anything associated with the payment of interest on a bond still carries the designation of *coupon*, for example, coupon date or coupon rate.

In terms of financial instruments, bonds play a central role in history and the economy. They are also the way states finance themselves through financial instruments called *treasury bonds* and *treasury bills*.

A particular type of bond is called the *zero-coupon bond*. Furthermore, it is special because it allows us to define a broad set of concepts on the subject of loans in general and on the interest rate subject. A zero-coupon bond is a contract that guarantees the holder the payment of a currency unit over a T time horizon without any interim interest payment.

The value of the zero-coupon bond $P(t)$ at any time $0 < t < T$ is given by

$$P(T) = 1 \tag{2.8}$$

$$P(t) = e^{-\alpha(T-t)} \tag{2.9}$$

where α is the interest rate. At the time horizon T the jargon calls the bond *maturity*. The term is also used for most financial contracts and is, in practice, the time frame for the end of the contract. It is easy to understand that *maturity date* is the date on which the contract ends.

Problem

2.13 Show that setting a loan with an outflow amount at the beginning and an interest rate is equivalent to setting an inflow amount at the end with the same interest rate. How different are the amounts?

The value of the zero-coupon bond grows over time, like if it absorbs interest in it as time goes by. We say 'as the interest runs.' The concept of *accrued interest* is relatively intuitive, corresponding to the accumulation of each interest day, if our time unit is the day, in the bond. It is a slightly different view from that of Eq.(2.9), where it reads the bond as future flows and the accrued interest is an idea associated with the past, meaning that the interest has already absorbed up to t.

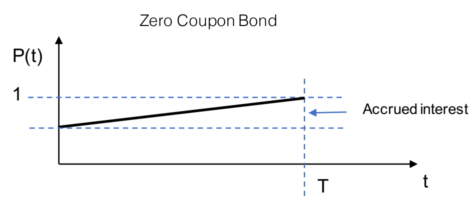


Fig. 2.11 Zero-coupon bond. The final value of the bond is 1 currency unit. The accrued interest is the interest, unpaid, which accumulates in the bond as time goes by.

Now let us start doing numerical calculations. The following lines are related to market protocols and not necessarily applied solely to zero-coupon bonds, but it is easier to understand based on these. Furthermore, since we need to do the math, let us start using them.

Let us consider a zero-coupon bond that pays 1 Euro after one month with an interest rate of 3% per year. How much is it worth today? To answer this question, let us introduce some more. Does the interest run every day? If so, are we talking about a month with 31 days or 30? What if it is in February? Do we have a leap year or not?

We have defined just now the simplest of instruments and have fallen into a series of practical problems. Luckily for us, the whole world had fallen into these problems a long time ago, so it has solved them using market protocols. In this case, it uses what is called by *the interest calculation basis*.

The interest calculation basis defines how one counts the days for interest calculation purposes, and it is associated with all financial contracts. There are dozens of interest calculation bases that have been created over the years as needed. However, some are more used than others. Protocol designates that interest calculation bases in the form *XXX / YYY*, where XXX is a designation for the days of the month to be considered and YYY is a designation for the days of the year to consider. The ACT designation indicates that the days to consider are those on the calendar. The ACT/365 base means that one considers the calendar days in the month, but the year is always 365 days, even if it is a leap year, i.e., one day is ignored. The base 30/360 means that we consider every month to have 30 days and the year to have 12 months, i.e., 360 days. There are ACT / 365, 30/365, ACT / ACT, and many other interest calculation bases that detail how one handles February, the 29^{th}. In practical terms, the overwhelming majority of the financial products we confront the bases we need are the ones we have mentioned here as significant.

So let us get back to our problem. Let us assume the calculation base is 30/360. So the one-month interest is $3\% \times 30/360 + x = 1$ where x is today's value and 1 is the zero-coupon bond at the end of the month. So $x = 0.9975$.

Solved Problem

2.14 Make the previous calculations using Eq.(2.9)

Using Eq.(2.9) the resulting value is 0.9997503. It seems to be a matter of rounding, but it is not. In fact, the expression of Eq. (2.9) is an expression that results from the so-called *interest compounding*. If the zero-coupon bond had a maturity of 1 year, today's value would be $1 = x(1+r)$, where r was the annual rate. If there were interest payments after 6 months, then the accounts would have to count the 6 month interest, which would be reinvested for the remaining 6 months. That is, $1 = x(1+r/2)^2$. If it were 3 in 3 months, $1 = x(1+r/4)^4$. And so on until we get to the point where we say you pay interest continuously, that is

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^{rt} \tag{2.10}$$

The r.h.s. of Eq.(2.10) is Eq. (2.7) solution for constant interest rate. The fact that value is not the same is because we always have to think that we have to use the money somewhere when we invest in something. Furthermore, that side is the abstract instrument we call the money market account. ■

We will now address the other bonds because, typically, zero-coupon bonds have relatively short maturities, often less than one year. When states finance themselves, they want to finance themselves for several years. So let us take a bond that pays interest every year and whose debt amount does not change over time, from issuance to maturity. The amount of debt, the amount on which one calculates interest, is called *principal amount*.

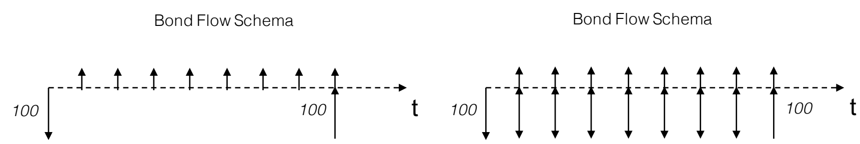


Fig. 2.12 'Regular' fixed rate bond (*coupon-bearing bond*). In each interest payment period, the holder receives the interest in each coupon, but the principal amount is only received at maturity.

Let us look at the schematics in Fig. 2.12. Both schemes refer to the same obligation. The difference is that on each coupon date, we have added two symmetrical flows with the principal amount that cancels out financially and therefore, they are not real. We did it because we may equate each period with a zero-coupon bond. In each period, we deliver the principal amount initially and receive the principal amount plus interest at the end of the period. That is, we can look at each bond as a zero-coupon bond portfolio.

Naturally, once we establish the base product, as described in Fig. 2.12, we are now free to set up the payment structure as we understand, with different rates, with different amounts, and so on. The variety of structures is so large that we must find a measure with which bonds are comparable. This measure is the *yield*.

Suppose we have a zero-coupon bond whose value, $P(t)$, at the instant t is given by Eq.(2.9). So, it comes to

$$\alpha = -\frac{\log P(t)}{T-t} \tag{2.11}$$

where α is the yield. However, was this not circular reasoning? Recall that we said that the advantage of bonds over loans is that we can trade them. That is, there is an additional set of factors, for example, risk, that may influence what other people interested in the bond - the market - think is the value of the bond, that may differ from that which is the 'pure' value obtained from Eq. (2.9). The value of $P(t)$ that appears in Eq. (2.11) is thus the empirical value which, in an ideal world without risks, would be equal to the bond interest rate.

Generalizing for another type of bond, so if we have a principal amount V then, the value of the obligation, $P(t)$, comes

$$P(t) = Ve^{-\alpha(T-t)} + \sum_{i}^{N} C_i e^{-\alpha(t_i - t)} \tag{2.12}$$

where C_i is the amount of each of the N coupons, i, which expires on date t_i and T is the bond's maturity. Since we can collect the value at which the bond is being bought and sold, we say that this value is $P(t)$ and we will calculate the α that equals Eq. (2.12). It is no longer a closed expression like Eq. (2.11), but it is possible to obtain the value by numerical methods. We call this α yield-to-maturity or, more commonly in investment theory, internal yield. As one can deduce, the yield is in its nature of a rate and is associated with a periodicity embedded in the time measurement units T, t, and t_i.

Problem

2.15 A bond with a residual maturity (up to the end of the contract) of 5 years is quoted at 95 cents, has a principal of 1 Euro and pays interest of 2% from today. Calculate yield-to-maturity using the numerical method you find appropriate.

We may slightly change Eq. (2.12), knowing that coupon amounts or any other amount have a present value regardless of how they occur, whether, by interest, principal payment, commission, or whatever cash flow occurs. As long as it is money, the way the discount factor applies is the same. Then the value of the obligation can be written as

$$P(t) = \sum_{i}^{N} C_i e^{-\alpha(t_i - t)} \tag{2.13}$$

where C_i becomes, in this case, a flow of any type, and we will agree that outflows are negative, inflows are positive. Continuing to generalize, since we can classify C_i amount as we wish as long as it is cash, then Eq. (2.13) applies to a bond like any instrument that has associated cash flows. Moreover, this is the measure that makes all financial positions comparable, called *present value* or, abbreviated and more common in jargon, *PV*. The concept is so important that we make a slight rearrange to the Eq. (2.13),

$$PV = \sum_{i}^{N} C_i e^{-\alpha(t_i)} \tag{2.14}$$

This expression applies to any set of C_i financial flows.

In conclusion, the bond model, starting with the zero-coupon bond, can be used for any investment, including several types of bonds. What will always be at stake will be the risk we assume on the predicting cash flows. As final notes, and related with this, bonds are often referred to as *fixed income securities*, even though they are far from being that. What is associated with the term ’fixed’ is that a yield or at least its form is defined *a priori*, unlike equity securities (shares) that depend on the firm’s performance associated with that capital. There is a direct relationship between yield and market price when measured separately, not worth mentioning.

As they have a face value, it is also widespread that bonds are quoted as a percentage of that face value. For example, a bond whose market price is 96% means one has to know the face value to know the cash price. However, contrary to what some literature states, although very common, it is not right for all bonds, so we recommend some caution.

Because agents trade bonds in an organized market, that gives us a valuable source of information about what other economic agents think about the system around us. Let us now imagine going to the market and looking at zero-coupon bond prices. As one might imagine, they were not all issued simultaneously, not all issued to the same maturity, not even by the same issuer. That allows us to build part of what is one of the essential tools in quantitative finance, the *yield curves* (see Fig. 2.13).

We should emphasize that there are people that the only thing they do in life is to build these curves. What we will explain here is an evident simplification of reality. Also, because zero-coupon bonds usually exist for short maturities, curves are constructed using more than one type of financial instrument. However, as we saw above, we can reduce all investments to one bond, so to get an idea of what we are talking about, let us use bonds.

Let us focus on Fig.2.13. The yield curve is a rate plot vs. time horizon from today. That curve gives us a benchmark of the rate over the time horizon that includes the market perspective (the aggregation of a set of multiple agents) of the future of interest rates (the economy in general). The superlative importance of yield curves across finance comes from there, from being a picture of what markets believe the future is. Conceptually, it would be enough to go to the bond market and calculate the various yields-to-maturity to build a similar chart. In practice, it is a bit more complicated than that. If we notice, associated with the chart is the issuer, the US federal treasury and, less explicitly, a currency: the US dollar. Meaning that this curve is only rigorous if the term can be used in this context and under these conditions. In addition to bonds, one has to resort to various instruments to extrapolate because the curve has to fit more generic issuers that pose a higher risk than the US Treasury. However, the general concept is that described here. There are information service vendors who provide yield curves already built in practice without the need for each person to build them.

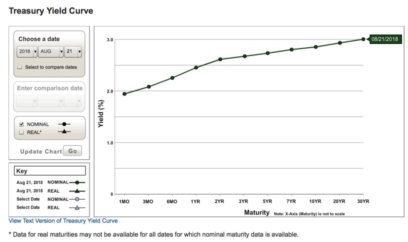


Fig. 2.13 Yield curve for USA federal tresury. Source: *US Department of the Treasury*

What do we need a yield curve? As we have already mentioned, we need it to have a time reference for making a loan. Let us do the following exercise, if we lend in 6 months, for 3 months, what rate should we apply? The problem is far from trivial. We are saying that we need to know the curve in 6 months to determine what rate we will apply to 3 months from then. The way to do this is by using a general principle that 'the present rate of the future is the future rate' by extracting it from the yield curve.

So to settle the question let us use the present rate for 6 months and the present rate for $6+3=9$ months. Dividing the 9 months into two periods, one 6 month and one 3 month, we have to

$$e^{\alpha_1 t_1} e^{\alpha_{12}(t_2-t_1)} = e^{\alpha_2 t_2} \tag{2.15}$$

where α_1 and α_2 are, respectively, the rates for 6 months, t_1, and for 9 months, t_2. α_{12} is exactly the rate we are looking for, which in 6 months will apply for 3 months. Solving, we have,

$$\alpha_{12} = \frac{\alpha_2 t_2 - \alpha_1 t_1}{t_2 - t_1} \tag{2.16}$$

At α_{12} we call it *forward rate* and α_1 and α_2 are taken from the yield curve.

Problem

2.16 Solve the forward rate expression for the discrete-time case.

At this stage, one should ask the extent to which the principle of 'the present future rate is the future rate' is valid. Well, it is not by all means. It is just a deterministic