Chapter 4

The Ginzburg-Landau Theory

Recall Chapter 3:

- Extensive discussion of the Ising model as one of the simplest "microscopic" models with a phase transition
- Introduction of several methods than can also be applied to other systems (e.g., transfer matrix method, series expansions, mean-field approximation, Monte Carlo simulations, others like renormalization will follow).
- Still, the Ising model is rather very special. It is not clear, to which extent our findings can be generalized to phase transitions in general.

In this chapter: General approach, based on symmetry considerations, which highlights the relation between phase transitions "of same type". The derivation is based on a mean-field point of view, but this can be relieved later on.

Ginzburg-Landau theories are popular starting point for developing field theories in statistical physics.

4.1 Landau expansion for scalar order parameter

4.1.1 Ising symmetry

Recall: Bragg-Williams approximation for Ising model

Close to $T = T_c$, $m = M/N$ is small $\Rightarrow$ expand in powers of m.

$$\Rightarrow \frac{F}{N} = -\frac{1}{\beta_c} m^2 + \frac{1}{\beta}\Big[\tfrac{1+m}{2}\ln(\tfrac{1+m}{2}) + \tfrac{1-m}{2}\ln(\tfrac{1-m}{2})\Big]$$
$$\approx -\frac{1}{\beta}\ln 2 + \frac{1}{2\beta_c}\big[\tfrac{\beta_c}{\beta} - 1\big]\, m^2 + \frac{1}{12\beta} m^4 + \cdots$$

The same form can already be inferred from general symmetry considerations

Requirement: $\frac{F}{N} = f(m)$ symmetric with respect to $m \leftrightarrow (-m)$.

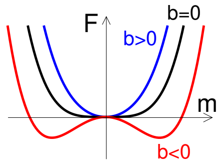
$$\Rightarrow \quad \boxed{\frac{F}{N} = a(T) + \frac{1}{2} b(T)\, m^2 + \frac{1}{4} c(T)\, m^4 + \frac{1}{6} d(T)\, m^6 + \cdots}$$

$\rightsquigarrow$ Landau expansion: Generally valid for systems with this symmetry!

Remark and Caveat: Strictly speaking, the expansion in powers of m is only allowed if F/N is analytic as a function of m. At phase transition points, this is not valid in the thermodynamic limit. Therefore, the Landau Ansatz represents an approximation and cannot be exact.

(Way out: Corresponding expansion for small subsystems
$\rightsquigarrow$ Ginzburg-Landau theory)

4.1.1.1 Case $c(T) > 0$



In that case, neglect $d(T)$

$\rightsquigarrow$ Graphical representation - see figure:

$\rightsquigarrow$ Continuous phase transition at $\boxed{b = 0}$

In the vicinity of T_c, one approximates $\boxed{b(T) = b_0\,(T - T_c)}$

Order parameter: $\frac{\partial F}{\partial m} = bm + cm^3 \stackrel{!}{=} 0$

$\Rightarrow\ m = \pm\sqrt{b_0/c}\,\sqrt{T_c - T} \quad (T < T_c)$

$\Rightarrow\ m \sim (T_c - T)^\beta$ with critical exponent $\boxed{\beta = 1/2}$ as in Section 3.5

Specific heat c_H:

$$\frac{S}{N} = -\frac{1}{N}\frac{\partial F}{\partial T} = -a'(T) - \tfrac{1}{2}b'(T)\,m^2 - \tfrac{1}{4}c'(T)\,m^4 - \tfrac{1}{2}b(T)(m^2)' - \tfrac{1}{4}c(T)(m^4)'$$

$$c_H = \frac{T}{N}\frac{\partial S}{\partial T} \qquad \text{For } T \to T_c:\ b = 0,\ b' = b_0,\ m^2 = \tfrac{b_0}{c}(T_c - T) \to 0 \text{ or } m^2 \equiv 0$$

$$(m^2)' = -\tfrac{b_0}{c} \text{ or } 0,\ (m^4)' = 0,\ (m^4)'' = 2(\tfrac{b_0}{c})^2 \text{ or } 0$$

$$\Rightarrow c_H = -Ta'' - Tb'\,(m^2)' - T\tfrac{c}{4}\,(m^4)'' = \begin{cases} -Ta'' + T\frac{b_0^2}{2c} & : T < T_c \\ -Ta'' & : T > T_c \end{cases}$$

$\rightsquigarrow$ Finite jump!

$\Rightarrow$ "$c_H \sim |T - T_c|^\alpha$ " with Critical exponent $\boxed{\alpha = 0}$ as in Section 3.5

Other exponents also the same as in Section 3.5

Reason: Results from the analytic expansion of F/N in powers of m. $\Rightarrow$ characteristic for mean-field exponents!

4.1.1.2 Case $c(T) < 0$

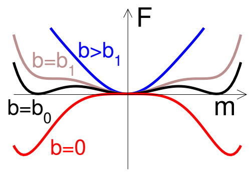
In that case, $d(T)$ cannot be neglected. Assume $d(T) > 0$

$\rightsquigarrow$ Graphical representation - see figure:

$b_1 = \frac{c^2}{4d}$: External minima form

$b_0 = \frac{3c^2}{16d}$: First order phase transition
(with $m_0^2 = 3|c|/4d$)

$b = 0$: Middle minimum at $m = 0$ disappears



Spinodals:

At $b \in [0 : b_0]$: metastable disordered states, "undercooling" is possible.

$b \in [b_0 : b_1]$: metastable ordered states, "overheating" is possible.

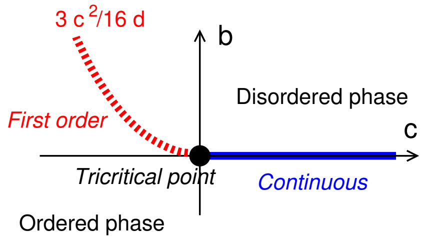
The spinodals $b = b_0$, $b = b_1$ mark the points where metastable states become unstable.

Example: M_nO, antiferromagnet
Before b changes sign, one already has a first order phase transition

4.1.1.3 Special case $b = c = 0$

$\rightsquigarrow$ Tricritical point

"Phase diagram" in the vicinity



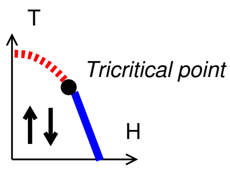
Practical relevance

If one has two intensive quantities that do not directly couple to the order parameter, a tricritical point may occur.

Example: Uniaxial antiferromagnet in a magnetic field

$b = b(T, H)$, $c = c(T, H)$

Possible phase diagram:



4.1.2 No Ising symmetry

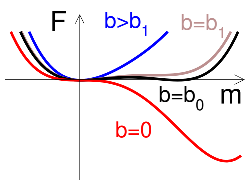
Example: Liquid-gas transition, liquid crystals,

Consider cases, where free energy F does not have to be symmetric with respect to an exchange $m \leftrightarrow -m$

Known: $\frac{F}{N} \to \infty$ for large $|m|$ $\Rightarrow$ At least one turning point $\bar{m}$ in between. Choose m axis such that $\bar{m} = 0$, hence $F'(\bar{m}) = 0$.

$$\rightsquigarrow \quad \boxed{\frac{F}{N} = a(T) + \frac{1}{2}b(T)\,m^2 - \frac{1}{3}c(T)\,m^3 + \frac{1}{4}d(T)\,m^4 + \cdots}$$

$b = b_1 = \frac{c^2}{4d}$: Second minimum forms
$b = b_0 = \frac{2c^2}{9d}$: First order phase transition (with $m_0 = 2c/3d$)
$b = 0$: First minimum disappears



$\rightsquigarrow$ Similar scenario as in Ising symmetric case 4.1.1.2
– First order phase transition,
– Spinodals at $b = 0$ and $b = b_1$,
– Metastable states in between

Conclusion: If Landau expansion contains a third order cubic term due to lack of symmetry, then the transition is first order!

NB: Consider as an example liquid-gas transition.
Line of first order transitions → consistent with argument!
Critical point: $b = c = 0$: In the presence of two control parameters (T and P), a point (P_c, T_c) with $c(T_c, P_c) = b(T_c, P_c) = 0$ may exist.
In that case, the phase transition is second order and Ising like!

4.2 Landau theory in systems with multicomponent order parameter

In this section, some examples are given how to construct Landau expansions from symmetry arguments for more complex systems with multicomponent order parameter.

4.2.1 Heisenberg model

System: Three dimensional spins on a lattice,
Interact with "Hamiltonian" $\mathscr{H} = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$

→ Invariant under rotation of $\vec{S}_i$

→ Invariants: $\vec{m}^2$, $(\vec{m}^2)^2$, $\cdots$
where $\vec{m} = \langle \vec{S} \rangle$: Order parameter per site

⇒ Landau expansion: $\frac{F}{N} = a + \frac{1}{2} b\, \vec{m}^2 + \frac{1}{4} c (\vec{m}^2)^2$

4.2.2 Heisenberg model with cubic anisotropy

Example: A real magnetic system on a cubic lattice.
Spins preferably orient along the main lattice directions.

Symmetry: $m_\alpha \leftrightarrow -m_\beta$ for all pairs (α, β)

→ Invariants: $\vec{m}^2$, $(\vec{m}^2)^2$, $(m_x^4 + m_x^4 + m_y^4)$

⇒ Landau expansion: $\frac{F}{N} = a + \frac{1}{2} b\, \vec{m}^2 + \frac{1}{4} c (\vec{m}^2)^2 + \frac{1}{4} d\, (m_x^4 + m_y^4 + m_z^4)$

4.2.3 Three component order parameter with uniaxial anisotropy

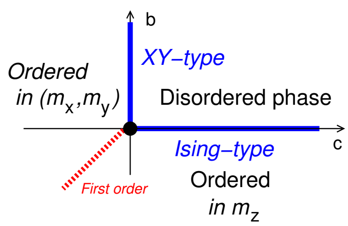
Symmetries: $m_z \leftrightarrow -m_z$ (m_x, m_y) invariant under (2D) rotation

→ Invariants: m_z^2, $m_x^2 + m_y^2$, m_z^4, $(m_x^2 + m_y^2)^2$, $m_z^2(m_x^2 + m_y^2)$

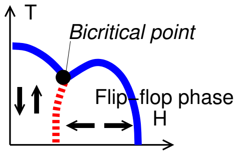
⇒ $\frac{F}{N} = a + \frac{1}{2} b\, m_z^2 + \frac{1}{2} c\, (m_x^2 + m_y^2) + \frac{1}{4} d\, m_z^4 + \frac{1}{4} e\, (m_x^2 + m_y^2)^2 + \frac{1}{4} f\, m_z^2 (m_x^2 + m_y^2)$

Discussion:

$b = 0,\ c > 0:$ Ising-type transition
$c = 0,\ b > 0:$ "XY"-symmetry
(m_x, m_y) order
$b = c = 0$: Ising- and XY-lines meet:
Bicritical point
$b < c < 0$: Different types of order compete
⇝ First order phase transition

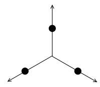


Example: Antiferromagnet with weak
uniaxial anisotropy in a homogeneous
external magnetic field H



4.2.4 Two component order parameter with trigonal symmetry

Symmetry: Invariance under a rotation of $2\pi/3$



Examples: Some adsorbate systems
Three-state Potts model:
$\mathscr{H} = -J\sum_{\langle ij\rangle}\delta_{q_i,q_j}$ with $q_i = 1, 2, 3$.

Possible way to determine invariants:

Rotation by $2\pi/3$ ≙ rotation matrix $\mathscr{D} = \begin{pmatrix} -1/2 & \sqrt{32} \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$

For any $f(\vec{m})$, the function $g(\vec{m}) = f(\vec{m}) + f(\mathscr{D}\vec{m}) + f(\mathscr{D}^2\vec{m})$ is invariant.

Apply this to polynomials $f(\vec{m})$ to get invariants of ...

2nd order:	$f(\vec{m}) = m_x^2,\ m_y^2$	$\to g(\vec{m}) \propto m_x^2 + m_y^2$
	$f(\vec{m}) = m_x\, m_y$	$\to g(\vec{m}) = 0$ (trivial)
3d order:	$f(\vec{m}) = m_x^3,\ m_x\, m_y^2$	$\to g(\vec{m}) \propto m_y(3m_x^2 - m_y^2)$
	$f(\vec{m}) = m_y^3,\ m_y\, m_x^2$	$\to g(\vec{m}) \propto m_x(3m_y^2 - m_x^2)$
4th order:	$f(\vec{m}) = m_x^4,\ m_y^4,\ m_x^2\, m_y^2$	$\to g(\vec{m}) \propto (m_x^2 + m_y^2)^2$
	$f(\vec{m}) = m_x\, m_y^2,\ m_y\, m_x^2$	$\to g(\vec{m}) = 0$

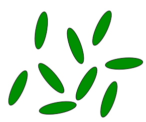
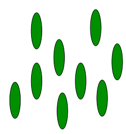
$$\Rightarrow \frac{F}{N} = a + \frac{1}{2}b(m_x^2+m_y^2) + \frac{1}{3}cm_x(m_x^2-3m_y^2) + \frac{1}{3}dm_y(m_y^2-3m_x^2) + \frac{1}{4}e(m_x^2+m_y^2)^2$$

Remarks:

- Cubic term ⇝ phase transition is first order!
- Six-fold symmetry: Trigonal symmetry and mirror symmetry
  ⇝ Cubic term disappears, phase transition may be continuous
- Exception: 3-State Potts model in two dimensions: Trigonal symmetry, but nevertheless continuous transition due to fluctuations !
  (So this may occasionally happen, but as a rule, phase transitions in systems with trigonal symmetry should be first order! For example, the phase transition in the 3-state Potts model in higher dimensions is first order)

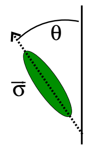
4.2.5 Liquid crystals

Example of a more
complex order parameter
Orientational order,
but no positional order

 ⟷ 

Isotropic phase   Nematic phase

Order parameter:



- Assume, there exists one preferred orientation
  $\rightsquigarrow$ Suitable order parameter is $S = \frac{1}{2}\langle 3\cos^2\theta - 1\rangle$:
  Disordered fluid: $\langle\cos^2\theta\rangle = 1/3 \quad \to \quad S = 0$
  Ordered fluid: $\langle\cos^2\theta\rangle = 1 \quad \to \quad S = 1$
- No preferred orientation $\rightsquigarrow$ Natural generalization
  Tensor: $Q_{\alpha\beta} = \frac{1}{2}\langle\sigma_\alpha\sigma_\beta - \delta_{\alpha\beta}\rangle$
  where $\vec{\sigma}_i$ points along the main axis of molecule i
  Note: Q is symmetric with $\mathrm{Tr}(Q) = 0$.

Landau expansion

- With preferred orientation: "Maier-Saupé model"
  $\frac{F}{N} = a + \frac{1}{2}bS^2 + \frac{1}{3}cS^3 + \frac{1}{4}dS^4 + \cdots$
  Due to the cubic term, the phase transition is first order.
- Without preferred orientation:
  Invariants under rotation: $\mathrm{Tr}(Q^2),\ \mathrm{Tr}(Q^3),\ \mathrm{Tr}(Q^4) = \frac{1}{2}(\mathrm{Tr}(Q^2))^2$.
  (Last identity holds because Q is symmetric and traceless)
  $\Rightarrow \quad \frac{F}{N} = a + \frac{1}{2}b\ \mathrm{Tr}(Q^2) + \frac{1}{3}c\ \mathrm{Tr}(Q^3) + \frac{1}{4}d\ \mathrm{Tr}(Q^4)$
  $\rightsquigarrow$ Again first order transition due to cubic term!

4.3 Ginzburg-Landau theory

Extension of Landau theory for inhomogeneous systems
Here: Discuss only systems with one-component order parameter

4.3.1 Ansatz

Homogeneous system $\rightsquigarrow$ Landau expansion
Different from previous section: Normalize with $1/V$ instead of $1/N$, i.e., $m = M/V,\ f := F/V$ etc. Expansion still has the same form.
$\Rightarrow\ F/V = a + \frac{1}{2}b\,m^2 + \frac{1}{4}c\,m^4 - h\,m$

Inhomogeneous system $\rightsquigarrow$ Search for generalization

Naïve Ansatz: $F = \int \mathrm{d}^d r\, f(m(\vec{r}))$ with $f(m) = a + \frac{1}{2}b\,m^2 + \frac{1}{4}c\,m^4 - h\,m$
Problematic, since the order parameter profile has no "stiffness", i.e., it adjusts instantaneously to $h(\vec{r})$
$\rightsquigarrow$ Spatial variations of $m(\vec{r})$ should be penalized

New Ansatz:
$$\boxed{\mathscr{F}[m(\vec{r}] = \int \mathrm{d}^d r\, \big(f(m) + \frac{1}{2}g\,(\nabla m)^2\big)}$$

Corresponds to lowest order expansion in m and ∇m, taking into account the symmetry $m \leftrightarrow (-m)$ and cubic symmetry in space!

4.3.2 Interpretation

Question: What is the meaning of $\mathscr{F}[m(\vec{r})]$? What does it describe?

(a) Not the free energy

- Not necessarily convex
- The free energy is a thermodynamic potential. It cannot depend on a microscopic order parameter field $m(\vec{r})$: Microscopic degrees of freedom must be integrated out!

Instead: A functional (a function from function space to $\mathbb{R}$), where the partition function has been partially evaluated (but not fully)!

(b) "Derivation" of the Ginzburg-Landau functional

(Not a rigorous derivation, rather a description of the object that $\mathscr{F}[m(\vec{r})]$ is supposed to represent)

Starting point, e.g., Ising model

Discrete spins S_i

"Coarse-graining": Averaging over blocks of size l_0, where l_0 has roughly the size of the correlation length far from T_c (but: chosen fixed, independent of T, not singular)

Slowly varying order parameter $m(\vec{r})$

No longer fluctuates on the scale of the lattice constant.
Fourier components with $k > 1/l_0$ have been integrated out.

Important: Block size l_0 must be chosen with care

Too large $\rightsquigarrow$ uncorrelated blocks, can be equilibrated independent of each other, nothing gained!

Too small $\rightsquigarrow$ correlations too strong and nonlocal, defining a "local" quantity $m(\vec{r})$ does not make sense!

Formal description: partial trace

Define $m(\vec{r})$: Average over block $v_{\vec{r}}$: $\quad m(\vec{r}) = \frac{1}{v_{\vec{r}}} \sum_{\vec{r}} S_i$

Now assume that $m(\vec{r})$ be given, then we have

$$\exp\Big(-\beta \mathscr{F}[m(\vec{r})]\Big) \stackrel{!}{=} \sum_{\{S_i\}} \mathrm{e}^{-\beta \mathscr{H}\{S_i\}} \prod_{\vec{r}} \delta\Big(\frac{1}{v_{\vec{r}}} \sum_{v_{\vec{r}}} S_i - m(\vec{r})\Big)$$

$\rightsquigarrow$ Calculate trace over all configurations which would yield the order parameter landscape $m(\vec{r})$ upon coarse-graining.

$\Rightarrow$ $\mathscr{F}[m(\vec{r})]$ has both energetic and entropic contributions!

Full Partition function:

$$\mathscr{Z} = \int \mathscr{D}[m(\vec{r})]\, \mathrm{e}^{-\beta \mathscr{F}[m(\vec{r})]} = \mathrm{e}^{-\beta F}$$

$\rightsquigarrow$ Functional integral over all smoothly varying functions!

(c) Comparison with density functional (for the experts)

Construction of density functional $\tilde{F}[\overline{m}(\vec{r})]$

- Definition of a microscopic order parameter field $m(\vec{r})$, e.g. as in (b): $m(\vec{r}) = \frac{1}{v_{\vec{r}}} \sum_{\vec{r}} S_i$
- Introduction of a conjugate field $h(\vec{r})$ that couples to $m(\vec{r})$
  $\Rightarrow$ Modified "Hamiltonian" $\tilde{\mathscr{H}}[h] = \mathscr{H} - \int \mathrm{d}^d r \, m(\vec{r}) \, h(\vec{r})$
  $\rightsquigarrow$ Thermodynamic potential: $\tilde{G}[h(\vec{r})] = -k_B T \ln \Big(\sum_{\{S_i\}} \mathrm{e}^{-\beta \tilde{\mathscr{H}}[h]} \Big)$
  $\overline{m}(\vec{r}) := \langle m(\vec{r}) \rangle = \frac{1}{\beta} \frac{\delta \tilde{\mathscr{G}}}{\delta h(\vec{r})}$ is almost always a unique function of $h(\vec{r})$
- Legendre transform: $\tilde{F}[\overline{m}(\vec{r}] = \tilde{G}[h(\vec{r})] - \int \mathrm{d}^d r \, \overline{m}(\vec{r}) \, h(\vec{r})$

Then we have (exactly): $F|_{h \equiv 0} = \min\limits_{\{\overline{m}(\vec{r})\}} \tilde{F}[\overline{m}(\vec{r})]$ (since $\frac{\partial \tilde{F}}{\partial \overline{m}} = h = 0$)

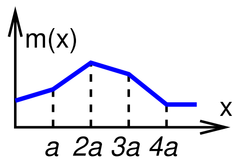
But: $\tilde{F}[\overline{m}(\vec{r})]$ and $\mathscr{F}[m(\vec{r})]$ are not the same functional!
In particular, $\tilde{F}[\overline{m}(\vec{r})]$ is generally nonlocal!
Moreover, $\overline{m}(\vec{r})$ (average local order parameter) does not refer to the same field as $m(\vec{r})$ (actual microscopic local order parameter)!

4.3.3 Brief digression: Dealing with functionals

I) Functional integral

$$\int \mathscr{D}[m(\vec{r})] \cdots = \lim_{\substack{a \to 0 \\ \text{lattice constant}}} \Big[\prod_{\vec{r}} \int_{-\infty}^{\infty} \mathrm{d}m_{\vec{r}} \Big] \cdots$$

e.g., in one dimension:
$\int \mathscr{D}[m(x)] \cdots = \int \mathrm{d}m_0 \, \mathrm{d}m_a \, \mathrm{d}m_{2a} \cdots$
$\rightarrow$ Path integral



II) Functional derivatives

Definition: $$\frac{\delta \mathscr{F}[m(\vec{r})]}{\delta m(\vec{r}')} = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \Big[\mathscr{F}[m(\vec{r}) + \epsilon \, \delta(\vec{r} - \vec{r}')] - \mathscr{F}[m(\vec{r})] \Big]$$

Examples:

- $\mathscr{F}[m(x)] = \int \mathrm{d}x \, f(m(x))$

$$\Rightarrow \frac{\delta \mathscr{F}}{\delta m(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big[\int \mathrm{d}x \, \big(f(m(x) + \epsilon \, \delta(x-y)) - f(m(x)) \big) \Big]$$
$$\overset{\text{Taylor}}{=} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big[\int \mathrm{d}x \, \big(f(m(x)) + \epsilon \, \delta(x-y) \, f'(m(x)) - f(m(x)) \big) \Big]$$
$$= \int \mathrm{d}x \, \delta(x-y) \, f'(m(x)) = f'(m(y))$$

- $\mathscr{F}[m(x)] = \int \mathrm{d}x \, \big(\frac{\mathrm{d}}{\mathrm{d}x} m(x) \big)^2$

$$\Rightarrow \frac{\delta \mathscr{F}}{\delta m(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big[\int \mathrm{d}x \Big(\Big(\tfrac{\mathrm{d}}{\mathrm{d}x} \big(m(x) + \epsilon \delta(x-y) \big) \Big)^2 - \big(\tfrac{\mathrm{d}}{\mathrm{d}x} m(x) \big)^2 \Big) \Big]$$
$$= 2 \int \mathrm{d}x \big(\tfrac{\mathrm{d}}{\mathrm{d}x} m(x) \big) \big(\tfrac{\mathrm{d}}{\mathrm{d}x} \delta(x-y) \big)$$
$$\overset{\substack{\text{partial} \\ \text{integration}}}{=} -2 \int \mathrm{d}x \, \delta(x-y) \, \tfrac{\mathrm{d}^2}{\mathrm{d}x^2} m(x) = -2 \tfrac{\mathrm{d}^2}{\mathrm{d}x^2} m(x) \Big|_{x=y}$$

Rules: $\dfrac{\delta}{\delta m(\vec{r}')}\int \mathrm{d}^d r\, m(\vec{r}) = 1$

$\dfrac{\delta}{\delta m(\vec{r}')}\, m(\vec{r}) = \delta(\vec{r}-\vec{r}')$

$\dfrac{\delta}{\delta m(\vec{r}')}\,\dfrac{1}{2}\int \mathrm{d}^d r\, (\nabla\, m(\vec{r}))^2 = -\Delta m$

Product rule, chain rule, etc.

III) Functional derivatives in the Ginzburg-Landau theory

Construct "Generating functional"

$$\mathscr{Z}[h(\vec{r})] = \int \mathscr{D}[m(\vec{r})]\, \mathrm{e}^{-\beta\left[\mathscr{F}_0[m(\vec{r})] - \int \mathrm{d}^d r\, h(\vec{r})\, m(\vec{r})\,\right]} =: \mathrm{e}^{-\beta F[h(\vec{r})]}$$

$\Rightarrow$ Order parameter: $\langle m(\vec{r})\rangle = \big(-\frac{\delta \mathscr{F}}{\delta h(\vec{r})}\big)_{h(\vec{r})\to 0^+}$

"Local" susceptibility: (meaning will become clear later)

$$\chi(\vec{r},\vec{r}') = \Big[\frac{\delta\langle m(\vec{r})\rangle}{\delta h(\vec{r}')}\Big]_{h\to 0^+} = -\frac{\delta^2 F}{\delta h(\vec{r})\delta h(\vec{r}')}\Big|_{h\to 0^+}$$
$$= \cdots = \beta\,\big(\langle m(\vec{r}) m(\vec{r}')\rangle - \langle m(\vec{r})\rangle\langle m(\vec{r}')\rangle\big)_{h\to 0^+}$$

Same in Fourier space with $m(\vec{k}) = \int \mathrm{d}^d r\, \mathrm{e}^{i\vec{k}\cdot\vec{r}} m(\vec{r})$

$$\mathscr{Z}[h(\vec{k})] = \int \mathscr{D}[m(\vec{k})]\, \mathrm{e}^{-\beta\left[\mathscr{F}_0[m(\vec{k})] - \frac{1}{(2\pi)^d}\int \mathrm{d}^d k\, h(-\vec{k})\, m(\vec{k})\,\right]} \equiv \mathrm{e}^{-\beta F[h(\vec{k})]}$$

$\Rightarrow$ Order parameter: $\langle m(\vec{k})\rangle = -\frac{1}{(2\pi)^d}\frac{\delta F}{\delta h(-\vec{k})}\Big|_{h\to 0^+}$

Susceptibility:

$$\chi(\vec{k},\vec{k}') = \Big[\frac{\delta\langle m(\vec{k})\rangle}{\delta h(\vec{k}')}\Big]_{h\to 0^+}$$
$$= \frac{\beta}{(2\pi)^d}\,\big(\langle m(\vec{k}) m(-\vec{k}')\rangle - \langle m(\vec{k})\rangle\langle m(-\vec{k}')\rangle\big)_{h\to 0^+}$$
$$= \frac{1}{(2\pi)^d}\int \mathrm{d}^d r\, \mathrm{d}^d r'\, \mathrm{e}^{i\vec{k}\cdot\vec{r} - \vec{k}'\cdot\vec{r}'}\chi(\vec{r},\vec{r}')$$

Specifically, if $\chi(\vec{r},\vec{r}') = \chi(\vec{r}-\vec{r}')$ (homogeneous system):

$\Rightarrow\ \chi(\vec{k},\vec{k}') = \tilde{\chi}(\vec{k})\,\delta(\vec{k}-\vec{k}')\frac{(2\pi)^d}{V}$ (NB: $\delta(0) = \frac{V}{(2\pi)^d}$)

with $\tilde{\chi}(\vec{k}) = \chi(\vec{k},\vec{k}) = \frac{V}{(2\pi)^d}\int \mathrm{d}^d r \mathrm{e}^{i\vec{k}\cdot\vec{r}}\chi(\vec{r})$

NB: Relation to global susceptibility $\chi = \frac{\partial \bar{m}}{\partial H}$:

Choose $h(\vec{r}) \equiv H = \text{const.}$, $\bar{m} = \frac{1}{V}\int \mathrm{d}^d r\, m(\vec{r})$

$$\Rightarrow\ \chi = \frac{1}{V}\int \mathrm{d}^d r \int \mathrm{d}^d r'\, \underbrace{\frac{\delta m(\vec{r})}{\delta h(\vec{r}')}}_{\chi(\vec{r},\vec{r}')}\ \underbrace{\frac{\partial h(\vec{r}')}{\partial H}}_{1} =: \int \mathrm{d}^d r\, \tilde{\chi}(\vec{r}) = \tilde{\chi}(\vec{k}=0)\,\frac{(2\pi)^d}{V}$$
$$= \cdots = \frac{\beta}{V}\big(\langle M^2\rangle - \langle M\rangle^2\big) \quad \text{with} \quad M = \int \mathrm{d}^d r\, m(\vec{r}).$$

4.3.4 Mean-field approximation and transition to the Landau theory

Preliminary remark: If the functional $\mathscr{F}[m(\vec{r})]$ were known, the expression for the partition function would be exact: $\mathscr{Z} = \int \mathscr{D}[m(\vec{r})] \mathrm{e}^{-\beta \mathscr{F}[m(\vec{r})]}$

(a) Mean-field approximation

Ansatz: Main contribution to the integral $\mathscr{Z} = \int \mathscr{D}[m(\vec{r})]\mathrm{e}^{-\beta\mathscr{F}[m]} =: \mathrm{e}^{-\beta F}$ comes from the minimum of $\mathscr{F}$ $\Rightarrow$ $F = \min_{\{m(\vec{r})\}} \mathscr{F}[m(\vec{r})]$

Specifically: Consider $\mathscr{F}[m(\vec{r})] = \int \mathrm{d}^d r \left[\frac{1}{2} g (\nabla m)^2 + f(m) - h(\vec{r})\, m(\vec{r})\right]$

Minimum $\frac{\delta F}{\delta m} \equiv 0 \;\Rightarrow\; -g\Delta m + f'(m) - h = 0$

Homogeneous system in the bulk ($h(\vec{r}) \equiv 0$, free boundaries)

$\rightsquigarrow$ $m(\vec{r}) \equiv \bar{m} = \text{const.},\; f'(\bar{m}) = 0,\; F = V f(\bar{m})$

$\rightsquigarrow$ Effectively a Landau theory

(Specifically: $f(m) = \frac{1}{2} b m^2 + \frac{c}{4} m^4 \;\Rightarrow\; \bar{m} = \begin{cases} 0 & : b > 0 \\ \pm\sqrt{|b|/c} =: \pm m_0 & : b < 0 \end{cases}$)

But: Ginzburg-Landau theory also allows to calculate mean-field profiles $m(\vec{r})$ in inhomogeneous systems! (see Sec. 4.3.6)

(b) Next step: Gaussian approximation

"Saddle point integration":

Main contribution to the integral $\mathscr{Z} = \int \mathscr{D}[m(\vec{r})]\mathrm{e}^{-\beta\mathscr{F}[m]} = \mathrm{e}^{-\beta F}$ stems from the minimum of $\mathscr{F}$ and small fluctuations around the minimum

Given $\mathscr{F}[m(\vec{r})] = \text{min. for } m(\vec{r}) = \bar{m}(\vec{r})$

$\rightsquigarrow$ Consider $m(\vec{r}) = \bar{m}(\vec{r}) + \eta(\vec{r})$, assume η is small, expand $\mathscr{F}[\bar{m} + \eta]$ up to second order in η

$$\Rightarrow \mathscr{F}[m(\vec{r})] = \mathscr{F}[\bar{m}(\vec{r})] + \int \mathrm{d}^d r \underbrace{\left.\frac{\delta \mathscr{F}}{\delta m(\vec{r})}\right|_{\bar{m}}}_{0:\bar{m}\text{ minimizes }\mathscr{F}} \eta(\vec{r}) + \frac{1}{2}\int \mathrm{d}^d r \mathrm{d}^d r' \left.\frac{\delta^2 \mathscr{F}}{\delta m(\vec{r})\delta m(\vec{r}')}\right|_{\bar{m}} \eta(\vec{r})\eta(\vec{r}')$$

$$\Rightarrow \mathscr{Z} = \mathrm{e}^{-\beta \mathscr{F}_{\min}} \underbrace{\int \mathscr{D}[\eta(\vec{r})]\, \mathrm{e}^{-\frac{\beta}{2}\int \mathrm{d}^d r\, \mathrm{d}^d r' \left.\frac{\delta^2 \mathscr{F}}{\delta m(\vec{r})\,\delta m(\vec{r}')}\right|_{\bar{m}} \eta(\vec{r})\,\eta(\vec{r}')}}_{\text{Gaussian integral}} =: \mathrm{e}^{-\beta F}$$

Gaussian integral can be solved analytically.

$\rightsquigarrow$ $(2\pi)^{\frac{V}{2}} \Big/ \sqrt{\det(\beta \frac{\delta^2 \mathscr{F}}{\delta m(\vec{r})\,\delta m(\vec{r}')}}$

Use $\frac{1}{\det A} = \prod_i \lambda_i^{-1} = \mathrm{e}^{-\sum_i \ln \lambda_i} = \mathrm{e}^{-\mathrm{Tr}(\ln A)}$ (with λ_i: Eigenvalues)

$$\Rightarrow F = \mathscr{F}_{\min} + \frac{1}{2\beta}\,\mathrm{Tr}\left(\ln \left.\frac{\delta^2 \mathscr{F}}{\delta m(\vec{r})\delta m(\vec{r}')}\right|_{\bar{m}} \right) + \text{const}$$

Furthermore: $\langle \eta(\vec{r}) \rangle \propto \int \mathscr{D}[\eta(\vec{r})]\, \eta(\vec{r})\, \mathrm{e}^{-\frac{\beta}{2}\int \mathrm{d}^d r'\, \mathrm{d}^d r'' \left.\frac{\delta^2 \mathscr{F}}{\delta m(\vec{r}')\,\delta m(\vec{r}'')}\right|_{\bar{m}} \eta(\vec{r})\,\eta(\vec{r}')} \equiv 0$

$\Rightarrow \langle m(\vec{r}) \rangle = \bar{m}(\vec{r}),\quad \frac{\partial \langle m(\vec{r})\rangle}{\partial h(\vec{r}')} = \frac{\partial \bar{m}(\vec{r})}{\partial h(\vec{r}')}$

$\rightsquigarrow$ Basically same results than in mean-field theory.
Non-mean field behavior only appears if fluctuations are large!

(c) Application: Correlation functions in mean-field approximation

Consider homogeneous system with $\bar{m}(\vec{r}) \equiv \bar{m} = \text{const.}$

Trick: Exploit $\tilde{\chi}(\vec{k}) = \frac{\delta\langle m(\vec{k})\rangle}{\delta h(\vec{k})}\Big|_{h=0} = \frac{\beta}{(2\pi)^d}\langle m(\vec{k})m(-\vec{k})\rangle\Big|_{h=0}$

$\rightsquigarrow$ Response of the system to a periodic perturbation with amplitude $h(\vec{k})$ gives correlation functions in Fourier space

$$C(\vec{r}) = \langle m(\vec{r})m(\vec{r}')\rangle - \bar{m}^2 \;\to\; C(\vec{k}) \sim \langle m(\vec{k})m(-\vec{k})\rangle$$

Specifically: Consider again expansion $m(\vec{r}) = \bar{m} + \eta(\vec{r})$

Euler-Lagrange equation: $bm - g\Delta m = h \;\Rightarrow\; b\eta + 3c\bar{m}^2\eta - g\Delta\eta + \mathscr{O}(\eta^2) = h$

$$\left.\begin{array}{lll} T > T_c \quad (\bar{m} = 0) & : \; b\eta - g\Delta\eta & \stackrel{!}{=} h(\vec{r}) \\ T < T_c \quad (\bar{m} = \sqrt{\frac{-b}{c}}) & : \; -2b\eta - g\Delta\eta & \stackrel{!}{=} h(\vec{r}) \end{array}\right\} + \mathscr{O}(\eta^2)$$

In Fourier space

$$\begin{array}{lll} T > T_c & : \; b\,\eta + g\,k^2\eta = h(\vec{k}) & \Rightarrow \eta(\vec{k}) = h(\vec{k})/(b + gk^2) \\ T < T_c & : \; -2b\,\eta + g\,k^2\eta = h(\vec{k}) & \Rightarrow \eta(\vec{k}) = h(\vec{k})/(2|b| + gk^2) \end{array}$$

$\Rightarrow$ Lorentz curve: $C(\vec{k}) \sim \chi(\vec{k}) \sim \frac{\delta\langle m(\vec{k})\rangle}{\delta h(\vec{k})\rangle} \sim \frac{\delta\langle \eta(\vec{k})\rangle}{\delta h(\vec{k})\rangle} \Rightarrow$ $\boxed{C(\vec{k}) \sim \dfrac{1}{k^2 + \xi^{-2}}}$

with $\boxed{\xi = \begin{cases} \sqrt{g/b} & : T > T_c \\ \sqrt{g/(2|b|)} & : T < T_c \end{cases}}$

Back transformation in real space (for calculation see below or 3.5.3.4)

$$\Rightarrow \quad \boxed{C(\vec{r}) \sim \int \mathrm{d}^d k \mathrm{e}^{i\vec{k}\cdot\vec{r}} C(\vec{k}) \sim \begin{cases} \mathrm{e}^{-r/\xi} & : r/\xi \gg 1 \\ r^{2-d} & : r/\xi \ll 1 \end{cases}}$$

Interpretation:

ξ is the correlation length, diverges at the critical point ($b = 0$).
At the critical point with $\xi \to \infty$, $C(\vec{r})$ decays algebraically!
Critical behavior: Exponents ν and η (Recall $b = b_0(T - T_c)$)

- Correlation length: $\boxed{\xi \sim |T - T_c|^{-\nu}}$, $\boxed{\nu = 1/2}$ $(\xi \sim 1/\sqrt{|b|})$
- "Anomalous dimension": $\boxed{C(r) \sim r^{2-d+\eta}}$, $\boxed{\eta = 0}$ at $T = T_c$

(exact: 2D Ising: $\nu = 1$, $\eta = 1/4$ 3D Ising: $\nu = 0.63$, $\eta = 0.04$)

(Addendum: Back transformation $C(\vec{k}) \to C(\vec{r})$ (similar to Sec. 3.5.3.4)

$C(\vec{k}) = \frac{1}{k^2+\xi^{-2}}$; $C(\vec{r}) \sim \int \mathrm{d}^d k\, \mathrm{e}^{-i\vec{k}\cdot\vec{r}} \frac{1}{k^2+\xi^{-2}}$, d dimensions

Use: ($\star$) $\int_{-\infty}^{\infty} \mathrm{d}p\, \mathrm{e}^{-ipx} \frac{1}{p^2+a^2} = \frac{\pi}{a}\mathrm{e}^{-|x|/a}$ (derived, e.g., via theorem of residues)

$d = 1$: $\int \mathrm{d}k \frac{1}{k^2+\xi^{-2}} \stackrel{\star}{=} \pi\xi \mathrm{e}^{-|x|/\xi}$

$d \geqslant 2$: $C(\vec{r}) \sim \mathrm{d}^d k\, \mathrm{e}^{-i\vec{k}\cdot\vec{r}} \frac{1}{k^2+\xi^{-2}} \stackrel{\vec{k}=\hat{\vec{k}}r}{=} r^{2-d} \int \mathrm{d}^d \hat{k} \mathrm{e}^{-i\hat{\vec{k}}\cdot\vec{e}_{\vec{r}}} \frac{1}{\hat{k}^2+(r/\xi)^2}$

| Choose x axis along $\vec{e}_{\vec{r}}$, Set $\hat{\vec{k}} =: (p, \vec{q})$

$$= r^{2-d} \int_{-\infty}^{\infty} \mathrm{d}p\, \mathrm{e}^{-ip} \int \mathrm{d}^{d-1}q \frac{1}{p^2+q^2+(\frac{r}{\xi})^2}$$

$$\stackrel{\star}{\sim} r^{2-d} \underbrace{\int_0^{\infty} \mathrm{d}q\, q^{d-2}\, \mathrm{e}^{-\sqrt{q^2+(\frac{r}{\xi})^2}} \frac{1}{\sqrt{q^2+(\frac{r}{\xi})^2}}}_{=:I(r/\xi)}$$

Consider asymptotic behavior of $I(x)$:

$x \to \infty: \sqrt{x^2+q^2} \approx x(1+\frac{1}{2}(\frac{q}{x})^2 = x + \frac{1}{2x}q^2$; $\frac{1}{\sqrt{q^2+x^2} \approx \frac{1}{x}(1-\frac{q^2}{2x^2}}$

$\Rightarrow I(x) \approx \mathrm{e}^{-x}\frac{1}{x}\underbrace{\int_0^\infty \mathrm{d}q\, q^{d-2}\mathrm{e}^{-\frac{1}{2x}q^2}}_{\Gamma(\frac{1-d}{2})x^{(d-1)/2}}(1+\mathscr{O}(\frac{1}{x})) \sim \mathrm{e}^{-x}x^{(d-3)/2}$

$x \to 0$, $d > 2$: $I(x) \approx \int_0^\infty \mathrm{d}q\, q^{d-3}\mathrm{e}^{-q} = \Gamma(d-2)$.

$d = 2$: Exact solution $\int_0^\infty \mathrm{d}q\mathrm{e}^{-\sqrt{q^2+x^2}}\frac{1}{\sqrt{q^2+x^2}} = K_0(x) \stackrel{x\to 0}{\to} -\ln(x)$

Apply this to $C(\vec{r}) \sim r^{2-d}I(r/\xi)$

$\Rightarrow r/\xi \gg 1: \; C(r) \sim r^{2-d+(d-3)/2}\mathrm{e}^{-r/\xi} = r^{(1-d)/2}\mathrm{e}^{-r/\xi}$

$r/\xi \ll 1: \; C(r) \sim \begin{cases} r^{2-d} & \text{for } d>2 \\ -\ln(r/a) & \text{for } d=2 \end{cases}$)

4.3.5 Validity region of the mean-field approximation

Mean-field approximation neglects fluctuations.

Question: When is this acceptable?

Estimate: Ginzburg criterion (see also Section 3.5.4.2)

Fluctuations of the order parameter in the range of the correlation length must be small compared to the order parameter!

Specifically: Compare $M = \int_{\xi^d} \mathrm{d}^d r\, m(\vec{r}$ at $t \propto (T - T_c)$

Request: $\langle M^2\rangle - \langle M\rangle^2 \ll \langle M\rangle^2$, where $\langle M^2\rangle - \langle M\rangle^2 = \chi\xi^d$ and $\langle M\rangle^2 = \langle m\rangle^2\xi^{2d}$

$\Rightarrow \chi\xi^{-d}\langle m\rangle^{-2} \ll 1 \quad \Rightarrow \quad \boxed{R|t|^{-\gamma+\nu d-2\beta} \ll 1}$

Specifically for $|t| \to 0: \quad (-\gamma + \nu d - 2\beta) > 0$

$\Rightarrow$ Mean-field approximation describes critical behavior correctly for

$\boxed{d > d_c = \frac{2\beta_{MF}+\gamma_{MF}}{\nu_{MF}}}$ d_c : "Upper critical dimension" ($\beta_{MF}, \gamma_{MF}, \nu_{MF}$: Mean-field exponents)

For $d < d_c$: Fluctuations dominate, mean-field approximation fails
For $d = d_c$: Logarithmic corrections
For $d > d_c$: Mean-field approximation captures critical behavior

Ising-type transitions: $\gamma_{MF} = 1$, $\beta_{MF} = \nu_{MF} = 1/2 \Rightarrow \boxed{d_c = 4}$

Significance of prefactor R

Mean-field approximation may oK even for $d < d_c$,

if $1 \gg t \gg R^{1/(\gamma+2\beta-d\nu)} = R^{1/\nu(d_c-d)}$: mean-field range

if $t \ll R^{1/(\gamma+2\beta-d\nu)} = R^{1/\nu(d_c-d)}$: critical range

(Example: Superconductivity - Critical range $\sim 10^{-14}K$
One practically always sees mean-field behavior.

Remark: Argument applies only if the direct interactions decay fast enough (faster than $1/r^d$)!

4.3.6 Conclusion: Relevance of Ginzburg-Landau theory

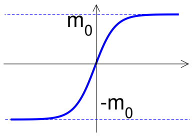
⋆ Ginzburg-Landau functional

- In principle "exact" starting point for perturbation expansions, field theoretic renormalization etc.
  Contruction from symmetry considerations → Universality
- Allows assessment of validity of mean-field approximation (previous section)

⋆ Treatment of inhomogeneous systems

e.g., surfaces, thin films, interfaces

Interface



Problem: Minimize $\mathscr{F}[m(\vec{r})]$ with boundary condition $\lim_{x\to\pm\infty} = \pm m_0$
(with m_0: Bulk order parameter)

$$\mathscr{F}[m(\vec{r})] = \int \mathrm{d}^d r \Big[\tfrac{1}{2} g(\nabla m)^2 + \tfrac{1}{2} b m^2 + \tfrac{1}{4} c m^4\Big]$$

⇒ Equation: $bm + cm^3 - g\Delta m = 0$, $m_0 = \sqrt{|b|/c}$, $\xi = \sqrt{\frac{g}{2|b|}}$

⇒ Solution: $m = m_0 \tanh(x/2\xi)$ (Check by insertion)

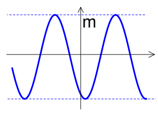
⋆ Allows description of modulated phases

("incommensurable phases", magnetic screw structures, lamellar phases in microemulsions or block copolymers)

Special case $\boxed{g < 0}$. In this case, $\mathscr{F}[m(\vec{r})]$ must include a stabilizing term of higher order

e.g., $\mathscr{F} = \int \mathrm{d}^d r \Big[f(m) + \frac{1}{2} g(\nabla m)^2 + \frac{1}{2} k(\Delta m)^4 \Big]$
(or $\frac{1}{4} k'(\nabla m)^4$)

If g is sufficiently small, $\mathscr{F}$ is minimized by a modulated order parameter.

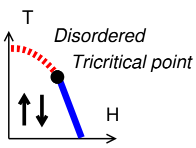


4.4 Multicritical phenomena

4.4.1 Examples

(a) Tricritical point

Example: Strongly anisotropic uniaxial antiferromagnet in a homogeneous external field (discussed earlier in Sec. 4.1.1.3)

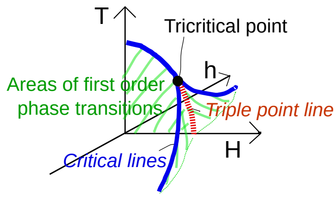


Blue solid: First order transition
Red dashed: Second order transition
Black point: Tricritical point

Why is this point called "tricritical"?

In an extended phase space, three critical lines meet there.

E.g., antiferromagnet: Choose as additional intensive variable the field h that couples to the order parameter (a staggered field)



Additional characteristics:

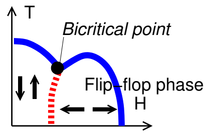
– In mean-field approximation different critical exponents than in the Ising model (see Sec. 4.4.2).
One obtains: $\alpha = 1/2,\ \beta = 1/4,\ \gamma = 1$, but still $\nu = 1/2,\ \eta = 0$.

$\Rightarrow$ Different upper critical dimension according to the Ginzburg criterion: $d_c = (\gamma + 2\beta)/\nu = 3!$

(b) Bicritical point

Two critical lines meet each other

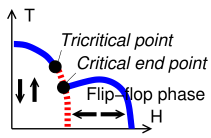
Example: Weakly anisotropic uniaxial antiferromagnet in a homogeneous external field (discussed earlier in Sec. 4.2.3)



(c) Critical end point

Critical line ends at a line of first order phase transitions

Example: Uniaxial antiferromagnet with intermediate anisotropy in a homogeneous external field



(d) Multicritical points of higher order

Example: Tetracritical point - four critical lines meet.

(e) Lifshitz point

Modulated phases compete with regular phases

(f) and many others ...

We will now illustrate the treatment of multicritical phenomena with the Ginzburg-Landau theory at two examples: The tricritical point and the Lifshitz point.

4.4.2 Tricritical point

(a) Landau expansion (already discussed in Sec. 4.1.1.3)

$\frac{F}{V} = a + \frac{1}{2}bm^2 + \frac{1}{4}cm^4 + \frac{1}{6}dm^6 - hm$

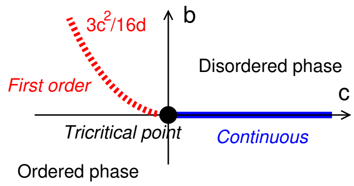
Tricritical point corresponds to $b = c = 0$

In order for this to happen, b and c should depend on two intensive parameters Δ, T.

$\rightsquigarrow$ $b = c = 0$ defines a point (Δ_t, T_t) in the (Δ, T)-plane

We already showed:

At $c < 0$, one has a first order phase transition at $b = 3c^2/16d$.
Now we discuss the critical behavior directly at the critical point.



(b) Critical behavior in the Landau theory

Preliminary remark: From $\frac{\partial F}{\partial m} = 0$, one concludes at $h = 0$:

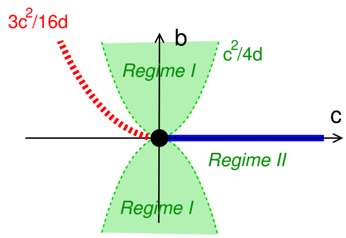
$bm + cm^3 + dm^5 = 0 \quad \overset{(c<0)}{\Rightarrow} \quad m^2 = \frac{|c|}{2d}\left(1 + \sqrt{1 - \frac{4bd}{c^2}}\right)$

$\rightsquigarrow$ Behavior different for the cases $|4bd/c^2| \ll 1$ and $|4bd/c^2| \gg 1$
"critical" and "tricritical" regime!

Graphical illustration:

I: "Tricritical regime"
II: "Critical regime"

Approaches to the tricritical point:
in I: $b \propto (T - T_t)$, $c \propto (T - T_t)$
in II: $b \ll (T - T_t)^2$, $c \propto (T - T_t)$



- Order parameter

  (I) Approach tricritical point with a finite angle to the phase transition line $\rightarrow$ in the tricritical region
  $\Rightarrow \frac{4bd}{c^2} \gg 1$ (since b, c approach zero linearly)
  $\Rightarrow m \approx \left(\frac{-b}{d}\right)^{1/4} \quad \Rightarrow \quad \boxed{\beta_t = 1/4}$

  (II) Approach tricritical point in the critical regime such that $1 \ll \frac{4bd}{c^2}$
  $\Rightarrow m \approx \left(\frac{|c|}{2d}\right)^{1/2} \quad \Rightarrow \quad \boxed{\beta_u = 1/2}$

- Specific heat c_H: $\frac{F}{V} = a + \frac{1}{2}bm^2 + \frac{1}{4}cm^4$ and $c_H = -T\frac{\partial^2 F}{\partial T^2}$

  (I) In the tricritical regime: $m \propto (T_t - T)^{1/4}$ and $b \propto (T - T_t)$
  $\Rightarrow c_H \sim \frac{\partial^2}{\partial T^2}(T_t - T)^{3/2} \sim (T_t - T)^{-1/2} \quad \Rightarrow \quad \boxed{\alpha_t = 1/2}$

  (II) In the critical regime: $m \propto (T_t - T)^{1/2}$, $b \ll |T - T_t|^2$, $c \propto (T - T_t)$
  $\Rightarrow c_H \sim \frac{\partial^2}{\partial T^2}(T_t - T)^3 \sim (T_t - T)^1 \quad \Rightarrow \quad \boxed{\alpha_u = -1}$

- Susceptibility

  From $bm + cm^3 + dm^5 - h = 0$, one gets $\frac{\partial m}{\partial h}\big|_{h=0} = \frac{1}{b+3cm^2}\big|_{h=0}$.

  (I) In the tricritical regime:

  $\frac{\partial m}{\partial h} \sim \frac{1}{b} \sim (T_t - T)^{-1} \qquad \Rightarrow \qquad \boxed{\gamma_t = 1}$

  (II) In the critical regime: $b \ll |T - T_t|^2,\ c \propto (T - T_t)$

  $\frac{\partial m}{\partial h} \sim \frac{1}{b+3cm^2} \overset{cm^2 \sim (T-T_t)^2}{\sim} (T - T_t)^{-2} \qquad \Rightarrow \qquad \boxed{\gamma_u = 2}$

- Correlation functions

  The exponents ν, η do not change at the tricritical point, since the mean-field correlations do not depend on c (e.g., $\xi \sim \sqrt{|g/b|}$)

  $\Rightarrow \quad \boxed{\nu_t = 1/2,\ \eta_t = 0}$

Summary: Mean-field exponents in the tricritical regime:

$$\beta_t = 1/4,\ \gamma_t = 1,\ \alpha_t = 1/2,\ \nu_t = 1/2,\ \eta_t = 0$$

(c) Application: Ginzburg criterion

Recall Sec. 4.3.5: The Landau theory is good, if $d\nu - 2\beta - \gamma > 0$ for the mean-field exponents ν, β, γ. Inserting the values for the critical exponents at the tricritical point, one obtains $d > d_t$ with $\boxed{d_t > 3}$

Thus the upper critical dimension at the tricritical point is only 3!
In three dimensions, critical fluctuations only lead to logarithmic corrections to the behavior predicted by the Landau theory.

4.4.3 Lifshitz point

(a) Ginzburg-Landau theory for modulated phases

Practical relevance: Often used to describe materials that spontaneously form modulated nanostructures, e.g.,

– Modulated magnetic superstructures in crystals
  (Hornreich et al 1975 – lattice spin model: ANNNI model)
– Amphiphilic systems and microemulsions
– Block copolymer nanostructures
– Domains in lipid membranes

Also postulated to exist in the QCD phase diagram by some models

Ginzburg Landau theory

Modulated phases are possible, if the coefficient g of the square gradient term in the Ginzburg-Landau functional becomes negative. In this case, a stabilizing term of higher order must be included, e.g., $\frac{1}{2}v(\Delta m)^2$

$$\rightsquigarrow \quad \mathscr{F} = \int \mathrm{d}^d r \left[\tfrac{1}{2}bm^2 + \tfrac{1}{4}cm^4 - hm + \tfrac{1}{2}g(\nabla m)^2 + \tfrac{1}{2}v(\Delta m)^2\right]$$

Phase behavior: To find the transition to a modulated phase, we calculate the structure factor $S(\vec{k}) \propto \chi(\vec{k})$ ($\vec{k}$-dependent susceptibility)

Minimize $\mathscr{F}$ $\rightarrow$ Euler-Lagrange equations

$$\Rightarrow \quad b\,m + c\,m^3 - g\Delta m + v\,\Delta^2 m = h$$

Fourier transform $\vec{r} \to \vec{k}$ and linearization in m

$\Rightarrow \quad b\, m(\vec{k}) + g\, \vec{k}^2 m(\vec{k}) + v\, k^4\, m(\vec{k}) = h(\vec{k})$

$\Rightarrow \quad \chi(\vec{k}) \propto \frac{\partial m(\vec{k})}{\partial h(\vec{k})} = \frac{1}{b+gk^2+vk^4}$

Analysis

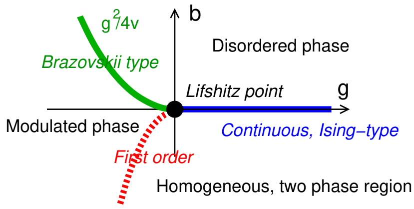
If $g < 0$, then $\chi(\vec{k})$ has a maximum at $k^* = \sqrt{-g/2v}$,

$\Rightarrow \ \chi(k^*) = \frac{1}{b-g^2/4v}$

In that case, $\chi(k^*)$ diverges at $b = g^2/4v$

$\rightsquigarrow$ Homogeneous phase is unstable, transition to a modulated structure with characteristic wave vector k^*

Phase diagram



Discussion

– In mean-field approximation: Two types of continuous transitions meet at the multicritical Lifshitz point: A regular Ising-type transition at $g > 0$, $b = 0$ and and a "Brazovskii"-type transition at $g < 0$, $b = \sqrt{g^2/4v}$ between a disordered phase and a modulated structure.

– At the Lifshitz point, the wave vector k^* of the modulated structure becomes zero - i.e., the wave length diverges.

(c) Critical behavior at Lifshitz points

- Exponents $\alpha, \beta, \gamma, \delta$ are the same as in the Ising model, as they do not depend on g.

  $\boxed{\alpha_L = 0,\ \beta_L = 1/2,\ \gamma_L = q,\ \delta_L = 3}$

- At the Lifshitz point ($g = 0$), we have $\chi(\vec{k}) \sim \frac{1}{b+vk^4} \sim \frac{1}{b(1+k^4\xi^4)}$

  $\rightsquigarrow$ Not a Lorentz curve, but $\xi = (v/b)^{1/4}$ is clearly the characteristic length scale in the system! Diverges as $b \propto (T - T_L) \to 0$.

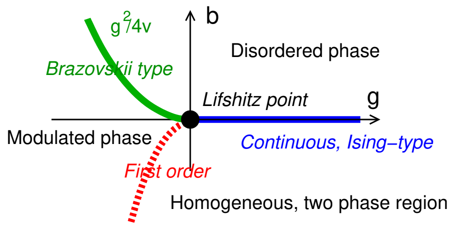
  $\xi \propto b^{-1/4} \quad \Rightarrow \quad \boxed{\nu_L = 1/4}$

  At $b = 0$, we have $\chi(\vec{k}) \propto k^{-4} =: k^{-(2-\eta_L)} \quad \Rightarrow \quad \boxed{\eta_L = -2}$

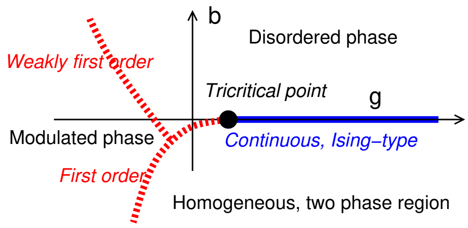
- Upper critical dimension: $d_c \nu_L - 2\beta_L - \gamma_L > 0 \quad \Rightarrow \quad \boxed{d_c = 8}$

(d) Fluctuation effects

- Upper critical dimension is very large ⇝ Fluctuation effects are strong!
- In fact, the lower critical dimension (the minimum dimension where a Lifshitz exists), is believed to be $d_l = 4$.
- In three dimensions or less, the Lifshitz point becomes unstable and probably turns into a regular tricritical point
  (Numerical evidence for the case of a block copolymer melt: Vorselaars, Spencer, Matsen, PRL 2020).
- Also, the Brazovskii transition becomes first order due to fluctuations by a mechanism called "Brazovskii mechanism" (ordered modulated domains break up).



Mean-field phase behavior



Real phase behavior in 3D