Chapter 7

Landau theory

7.1 Landau theory and phase transitions

At a first-order phase transition, an order parameter like the magnetization is discontinuous. At a critical point, the magnetization is continuous – as the parameters are tuned closer to the critical point, it gets smaller, becoming zero at the critical point. However, experiments on the liquid-gas phase transition and on three-dimensional magnets (and exact computations like Onsager and Yang's for the two-dimensional Ising model) both point that even though the magnetization is continuous, its derivative is not. In mathematical language the magnetization is a continuous function, but not analytic. For example, at $h \to 0+$ in the Ising magnet in 3d, the magnetization vanishes as $T \to T_c$ from below as

$$M \propto (T_c - T)^{.315} \qquad T < T_c, \quad \text{3d Ising} . \tag{7.1}$$

where all evidence suggests that the exponent is not even a rational number. In the two-dimensional Ising model, the exact computations give

$$M \propto (T_c - T)^{1/8} \qquad T < T_c, \quad \text{2d Ising.} \tag{7.2}$$

Even though the exponent is rational, the function is decidedly not analytic.

This was (and remains) very strange compared to most of physics. The partition function of any finite system is a continuous function of all the parameters. Thus if any non-analyticity occurs, it must be a property of taking an infinite number of degrees of freedom. We usually take this limit out of necessity – it's not possible to follow 10^{23} (or for that matter even 100) particles individually, even with a computer. Even Monte Carlo simulations can at best do thousands of particles. Now we're saying that at a critical point, the limit we so desperately need to take is suspect. Since dimensional-analysis arguments rely on analyticity, these are also suspect. Of course, at the end of the day all formulas are dimensionally consistent. What happens though is that at and near critical points, a hidden parameter is necessary for describing the physics. This

parameter arises from the short-distance physics – even if we are interested in describing long-distance physics, critical physics necessarily involves all length scales!

To understanding how that happens requires considerable effort – this is why Wilson won a Nobel Prize, and why many others provided essential ingredients. The first major step toward theoretical understanding came from Landau, and his approach is still called today *Landau theory*, or *Landau-Ginzburg theory*. Sometimes it is also called Ginzburg-Landau theory, because the two wrote a paper applying these ideas to superconductivity. However, the original insight came from a solo paper of Landau's in 1937; Ginzburg later understood how to see what goes wrong with Landau theory, explained below in section 7.7. It took several decades for Wilson and others to figure out how to fix it. (To confuse the history more, allegedly Lifshitz did the writing, if not the physics, of many of Landau's papers!)

Landau theory is an *effective theory* for what happens at and near the critical point. The experimental fact that very different systems can have quantitatively identical critical behavior suggests that one does not need to worry about every single detail of the system to understand this behavior. We gave an explicit example of how if we ignored many details of the liquid-gas system, we could obtain a lattice gas that was identical to the Ising model. This provides a suggestion as to why the universality occurs; Landau theory is the first serious attempt to derive a theory that will describe the critical behavior quantitatively.

Landau theory *only* describes the universal behavior of a system; by construction, it cannot for example give non-universal numbers like the value of T_c for a given system. But one of the miracles of critical behavior is that it can give precise results for the universal behavior. It is important to emphasize that Landau's original (genius) idea for an effective theory was and remains completely correct. It's just that the naive computations do not give the right answers. To be precise, in the next section, I will describe how the effective theory can arise from taking a specific approximation called mean-field theory. This approximation breaks down in low dimensions, for reasons explained by Ginzburg. But one of the beautiful aspects of Landau theory is that it makes deriving the consequences of mean-field theory really easy. The whole point is that the effective theory is independent of the details, so one can just guess what it is based on the symmetries and degrees of freedom of the system.

Landau theory is an *effective theory of the order parameter*. To be precise about it, one first decides what the appropriate order parameter is to describe the phase transition. In one phase, the order parameter is non-vanishing, in another it vanishes. In a ferromagnetic spin system, this very naturally is the magnetization $\vec{M}$. In an antiferromagnetic systems, as discussed in earlier chapters, there are a variety of possibilities, such as the staggered magnetization, which describes a transition away from Néel order. For example, in the Ising model the order parameter is very naturally the magnetization $M(x)$, the continuum limit of the expectation value of the spin $\langle M_k \rangle = \langle \sigma_k \rangle$ at the point k.

It is worth noting that in work in the last decade describing a "breakdown of Landau's

paradigm", what sometimes is meant is that multiple order parameters are needed to describe a given transition; subtle physics causes different order parameters to vanish at the same point. Another possibility is that no local order parameters change values at a phase transitions. One example of such is known as "topological order", where only non-local order parameters characterize the transition. Examples of this will be given later in this book.

One of Landau's insights was an easy way to see how the non-analyticity arises. The basic assumption of Landau theory is that *at a fixed value of the order parameter, the free energy as a function of the order parameter is analytic*, both in the parameters such as J and T, and in the order parameter itself. The non-analyticity at a phase transition then comes because in the partition function one must sum over all possible values of the order parameter.

The Ising model provides the canonical way of illustrating the simplest version of the idea: *Landau theory without fluctuations*. For reasons that will later be clear, such a theory is called *mean-field theory*. The partition function is of course defined as the sum over all configurations

$$Z = \sum_{\{\sigma_j = \pm 1\}} e^{-\beta E} \ . \tag{7.3}$$

The sum can be be done by first fixing an overall value of magnetization per site (here $M = \sum_j \sigma_j / N$ for N sites), and then summing over all values of M:

$$Z = \sum_M \sum_{\{\sigma_j = \pm 1 | \sum_j \sigma_j = MN\}} e^{-\beta E} \tag{7.4}$$

The point of Landau theory is then to define

$$e^{-\beta V f(M)} \equiv \sum_{\{\sigma_j = \pm 1 | \sum_j \sigma_j = MN\}} e^{-\beta E} \tag{7.5}$$

where V is the volume of the system. For large enough N, M essentially becomes a continuous variable, so the sum over M can be replaced with an integral. This yields

$$Z = \int_{-1}^{1} dM e^{-\beta V f(M)} \tag{7.6}$$

Similarly, for an n-component spin ($n = 2$ is usually called the XY model, $n = 3$ the Heisenberg), one would define $f(\vec{M})$.

One piece of notation needs some explanation. Previously I defined the expectation value of the spin to be the magnetization M, with no brackets. When rewriting the partition function as (7.6), one needs to considers all possible magnetizations in order to sum over all configurations, not just the expectation value. While this is a little confusing, it is not really that outrageous. Under standard assumptions, the integration

in the partition function is dominated by configurations where $F(M(x))$ is a minimum. This means that the expectation value $\langle M(x)\rangle$ is precisely this minimum. It is then conventional to omit the brackets and just call this minimum M.

This procedure is quite analogous to the analysis of chapter 1, where I showed how the thermodynamic notion of free energy as a function of fixed energy:

$$\mathcal{F}(E) \equiv E - TS(E) \tag{7.7}$$

is equivalent to the one from statistical mechanics

$$F = -k_B T \ln(Z) \ . \tag{7.8}$$

The connection comes from noting that under very standard assumptions

$$Z = \int dE\, e^{-\beta(E-TS(E))} = \int dE e^{-\beta \mathcal{F}(E)} \ .$$

Under the further assumption that the sum over states in the partition function is dominated by states that have energy all around the average energy $\langle E\rangle$, the two definitions are related by

$$F \approx \mathcal{F}(\langle E\rangle) \ .$$

Thus in Landau theory, the order parameter plays a similar role to the energy in thermodynamics. The function $f(M)$ then plays the role of the free energy per site.

Once the order parameter has been identified (or perhaps conjectured), the next step in Landau theory is to find an expression for f valid near the critical point. The definition in (7.6) is exact, but a priori, it will be some complicated functional of the order parameter. One can develop the mean-field approximation using the exact definition, and indeed I will do this for the Ising model in the section 7.3. But Landau's next genius move was observing that with the analyticity assumption, no serious calculation is necessary. Since different systems have identical critical behavior, this provides a strong hint that universal properties will not depend on complicated details. Moreover, an order parameter is chosen so that its expectation value vanishes at the critical point. Thus with Landau's assumption at at fixed value of the order parameter is analytic, the free energy can be expanded in a Taylor series around the critical point. The symmetries of the theory determine the form of this expansion.

First consider the Ising model neglecting all local fluctuations in the fields, so that $M(x)$ is constant in space. If there is no magnetic field, the partition function is invariant under flipping the sign of the spins $\sigma_k \to -\sigma_k$. Thus the free energy in this limit must be an even function of M. The Taylor series for the free energy density for fixed M is then

$$f(M) = a - hM + bM^2 + cM^4 + \ldots \tag{7.9}$$

for some numbers a, b and c. The linear term arises because the magnetic term in the Ising energy is $h\sum_j \sigma_j$, so that with the assumption of uniform magnetization, so it

results in a contribution of $-hM$ to the energy density. The other coefficients a, b, c will depend on which microscopic model is being studied. For the Ising model, these can be related to the parameter J by using the mean-field approximation; this is discussed in section 7.3. The sign of c must be positive, in order that the free energy be bounded from below. If perchance the underlying physical model of interest results in a negative c, then one must extend the expansion so that the highest-order term has positive coefficient.

For classical spin models with more components, the symmetry is larger, but the argument is similar. The interaction term $\vec{S}_i \cdot \vec{S}_j$ is invariant under rotation in the n-component space of spin degrees of freedom. Thus in the absence of an external magnetic field, the free energy function must be invariant under rotations of $\vec{M}$ and so a function of $|M|^2 \equiv \vec{M} \cdot \vec{M}$. In the absence of spatial fluctuations, then

$$f(\vec{M}) = a - \vec{H} \cdot \vec{M} + b|M|^2 + c|M|^4 + \dots \tag{7.10}$$

This indeed reduces to (7.9) when setting $n = 1$. In field-theory language, this is often called the (linear) $O(n)$ model, because the spin-rotation symmetry is the Lie group $O(n)$.

These examples are merely the simplest examples of Landau theory; literally hundreds of generalizations have been considered over the decades. One can write down Landau theories for models with gauge fields, fermions, phonons, spatial anisotropy, impurities more exotic symmetries, and pretty much everything a physicist can imagine for degrees of freedom. Although of course the analysis depends on these details, the philosophy is very much the same as to that developed by Landau and those that followed.

7.2 Critical exponents (and more) by neglecting fluctuations

In the mean-field approximation of Landau theory, the effect of fluctuations is completely ignored – the magnetization is treated as being completely uniform in space. With this approximation, Landau theory gives a very simple way of not only understanding why there is a critical point, but also in computing critical exponents. The approximation is typically not valid in lower dimensions, and the renormalization group is needed get the correct numbers. Typically, and in the examples discussed here, this dimensionality above which the approximation is valid is four; there presumably is cosmic significance that this is precisely the dimensionality of space-time in our observable universe.

Neglecting fluctuations not only provides the first step in computing the critical exponents in any dimension, but provides an huge amount of intuition. For example, I will show here how the critical exponents are independent of many of the details of the theory. This provides *a posteriori* support for the whole Landau approach. In fact, simple arguments then give critical exponents *exactly* in this approximation.

The partition function is given by (7.6) using the approximation (7.9) for the free energy density at fixed magnetization. Since we are interested in the behavior in the

Figure 7.1: $f(M)$ for various values of the couplings in the Landau theory for the Ising model

large-volume limit, the integral over M is dominated by magnetizations where $f(M)$ is a minimum. These obey

$$\left.\frac{\partial f}{\partial M}\right|_{M=M_0} = 0 \ . \tag{7.11}$$

In the Landau theory for the Ising model (7.9), this yields

$$2bM_0 + 4cM_0^3 = h \tag{7.12}$$

This is the equation of state, relating the magnetization to the couplings h, b and c in the Landau theory. It is valid when the couplings are such that M is small, so that the free energy is reasonably approximated by the expansion around $M = 0$.

The free energies for various regimes of the couplings are illustrated in the figure. If $h \neq 0$, then there is the function $f(M)$ may have two minima, but the lowest one is where the magnetization has the same sign as h. The interesting behavior occurs at $h = 0$. Since c must be positive, the results depend crucially on the sign of b. If b is positive, then the minimum of $f(M)$ must be at $M = 0$ for $h = 0$. If b is negative, the minima are elsewhere:

$$M_0 = \pm\sqrt{-\frac{b}{2c}} \ . \tag{7.13}$$

The two degenerate minima are a consequence of the spin-flip symmetry occurring at $h = 0$. For $h \to 0^+$ the minimum with the + sign wins, and likewise for the minus sign. Since the partition function is dominated by configurations with magnetization near M_0, this means that the expectation value of the magnetization is

$$\langle M \rangle = M_0 \ .$$

The critical temperature thus corresponds to having $b = 0$, with $T < T_c$ and $T > T_c$ corresponding to b negative and positive respectively. It is useful to define the *reduced*

temperature

$$t \equiv \frac{T - T_c}{T_c} \tag{7.14}$$

so that $t = 0$ at the critical temperature. The reduced temperature is a dimensionless quantity giving a measure of the distance from criticality. Since Landau theory assumes that the free energy at fixed magnetization is analytic, then b is an analytic function of t. Since $b = 0$ at $t = 0$, in the region of the critical point one can take

$$b = Bt + \mathcal{O}(t^2) \ , \tag{7.15}$$

The other parameter c in the free energy at fixed magnetization does *not* vanish in general at the critical temperature. (If it does vanish for some reason, then one must keep the M^6 term, and the universal behavior will be different.) Thus

$$c = C + \mathcal{O}(t) \ . \tag{7.16}$$

where C, like B is independent of t.

Putting this together means that when fluctuations are neglected,

$$\langle M \rangle = M_0 \propto \sqrt{T_c - T} \propto t^{1/2} \tag{7.17}$$

for $h = 0$ and $T < T_c$. The coefficient depends on the details of the theory (the couplings B, T_c and c), but the exponent does not! The exponent is also independent of dimensionality. As is obvious from the plots, the magnetization has the same form as determined by experiment and numerical computation

$$\langle M \rangle = \begin{cases} A(-t)^\beta & T < T_c \\ 0 & T > T_c \end{cases}$$

for some non-universal parameter $A = \sqrt{-B/(2CT_c)}$. The value $\beta_{\rm MFT} = 1/2$ predicted by mean-field theory is different from the values 1/8 and .315 known for two and three dimensions respectively. However, in four and higher dimensions, the numerically determined value of β is indeed 1/2, as predicted by Landau theory.

Landau theory also gives the qualitatively correct phase diagram in any $d > 1$. Since for $T > T_c$, $M_0 = 0$ for $h = 0$, bringing h through zero here still leaves M_0 continuous. However, for $T < T_c$, the value of M_0 jumps from $+\sqrt{-b/2c}$ to $-\sqrt{-b/2c}$ as h is decreased through zero. This is indeed a first-order transition. The only discontinuities or non-analyticities in M_0 as a function of the parameters occur at $h = 0$, so the dashed line in the figure is not a phase transition. It is the point at which the second minimum of the potential vanishes, and so the model cannot be thought of as a ferromagnet. This gives a qualitative way of distinguishing between a ferromagnet, where interactions favor spins lining up locally, and paramagnet, where the behavior is essentially determined by the external magnetic field.

Figure 7.2: Phase diagram for Landau theory for the Ising model

The non-analyticity of the magnetization as the temperature approaches T_c defines one critical exponent. Another non-analyticity occurs in the magnetization as a function of h precisely at the critical temperature:

$$\langle M \rangle \propto h^{1/\delta} \qquad \text{for } T = T_c \tag{7.18}$$

for positive h. This definition of the exponent δ is by historical convention. Calcluating it by neglecting fluctuations in Landau theory is easy using the equation of state (7.12). Since $b = 0$ when $T = T_c$, this gives $h \propto M^3$, so that

$$\delta_{\text{MFT}} = 3 \ .$$

Another exponent describes the behavior of the zero-field susceptibility near the critical point:

$$\chi_0 = \frac{\partial \langle M \rangle}{\partial H}\Big|_{h=0} \propto |t|^{-\gamma} \tag{7.19}$$

There are several interesting thing about this exponent. Notice that in the typical case of γ positive, this means that the zero-field susceptibility diverges – a very small change in magnetic field near the critical point causes a large change in magnetization. This is a consequence of the discontinuity in the magnetization at the first-order transition. Another interesting thing is that even though in principle the value of γ could depend on whether the temperature is above or below the critical temperature, in practice it does not. Neglecting fluctuations, differentiating (7.12) gives

$$2(Bt + 6c\langle M \rangle^2)\chi_0 = 1 \ .$$

Since $\langle M \rangle^2$ is either vanishes or is proportional to t, the constant of proportionality in (7.19) will depend on whether t is positive or negative. In both cases, however the exponent is

$$\gamma_{\text{MFT}} = 1 \tag{7.20}$$

There are other critical exponents one can define, but I defer their definitions.

Universality goes deeper than critical exponents. The equation of state in the Ising model can be written in the form

$$h = f(M, t) \ .$$

If there are even more couplings in the problem, then there will be even more variables. *Widom's scaling hypothesis* says that near a critical point, the functional dependence takes on a much simpler form:

$$h = M^{\delta}\Phi(t/M^{1/\beta}) \ . \tag{7.21}$$

Thus the particular combination $hM^{-\delta}$ is a *function of just one variable*, not two. Thus even though the identification of t with some microscopic parameter is not universal, once this is determined different systems in the same universality class will exhibit the same functional dependence on $t/M^{1/\beta}$. The critical exponent will change once fluctuations are included, but the functional form in (7.21) will still apply. Neglecting fluctuations, the equation of state for Ising then gives Φ to be

$$hM^{-3} = \Phi_{\rm MFT}\left(t/M^2\right) = B\frac{t}{M^2} + C \ . \tag{7.22}$$

Recall that in the critical region B and C are independent of t. Thus t appears only in the combination t/M^2 here.

7.3 Landau theory from the mean-field approximation

The complicated part in doing statistical mechanics is of course dealing with interactions. By definition, an "interaction" means a term in the energy involving more than one fluctuating variable. Even when the interactions are nearest-neighbor, only in very special cases can the partition function be computed exactly. *mean-field theory* is an approximation that simplifies the effect of interactions greatly, effectively reducing the computation to that of independent degrees of freedom. In this approximation, one replaces the effect of other spins on a given spin with the effect of the *average* on the given spin. This average is the "mean field". One can then determine the mean field self-consistently.

While Landau's approach explained in the previous section does not make such a calculation strictly necessary, it is often quite useful to do so. It gives a way to relate the microscopic parameters (e.g. the coupling J) to those in the mean-field theory (e.g. a, b and c in (7.9), and so is very helpful qualitatively. It also allows a better understanding of the approximations the Landau approach subtly makes, for example the role of dimensionality.

Mean-field theory is best understood by a specific example, which in this book almost inevitably means the Ising model. Here one writes

$$\sigma_i\sigma_j = (\sigma_i - M + M)(\sigma_j - M + M) = M^2 + M(\sigma_i - M) + M(\sigma_j - M) + (\sigma_i - M)(\sigma_j - M) \ .$$

(I have reverted here to using M to denote the expectation value.) If the fluctuations are small, the degree of freedom σ_j is typically very close to the average value M. The

last term is quadratic in this small quantity, and the mean-field approximation amounts to neglecting it. Using this approximation into the interaction part of the energy gives

$$\begin{aligned} E_{MFT} &= -h\sum_j \sigma_j - J\sum_{<ij>}\sigma_i\sigma_j & (7.23)\\ &\approx JN_lM^2 - h\sum_j\sigma_j - JM\sum_{<ij>}(\sigma_i+\sigma_j)\ , & (7.24)\\ &= \frac{Nz}{2}JM^2 - (h+JzM)\sum_j\sigma_j & (7.25)\end{aligned}$$

where z is the number of nearest neighbors each site has, and N_l is the total number of links, which is related to the number of sites N by $N_l = zN/2$. The mean-field approach therefore reduces the interactions to behave as an effective magnetic field, albeit one that depends on the magnetization. This effective magnetic field is sometimes called the *molecular field*, because it arises from the surrounding spins.

The magnetization still of course follows from taking the expectation value

$$M = \langle\sigma_k\rangle = \frac{\sum_j \sigma_k e^{-\beta E_{\rm MFT}}}{\sum_j e^{-\beta E_{\rm MFT}}} = \frac{1}{N\beta}\frac{\partial \ln(Z_{\rm MFT})}{\partial h}\ .$$

Since $E_{\rm MFT}$ depends on M as well, this amounts to a self-consistent equation for M, the equation of state. This equation is easy to derive for Ising, because $E_{\rm MFT}$ contains no interactions, so

$$Z_{\rm MFT} = e^{-\beta JM^2Nz/2}\left(2\cosh(\beta(h+JzM))\right)^N\ . \quad (7.26)$$

Thus the equation of state for the mean-field approximation to the Ising model is

$$M = \tanh(\beta(h+JzM))\ . \quad (7.27)$$

Another way of obtaining the same relation is to find the value of M that minimizes the free energy per site

$$f_{\rm MFT} = -\ln Z_{\rm MFT}/(\beta N) = \frac{1}{2}zJM^2 - \frac{1}{\beta}\ln(2\cosh(\beta(h+JzM)))\ . \quad (7.28)$$

Setting $\partial f_{\rm MFT}/\partial M = 0$ indeed yields (7.27), just as in the direct mean-field calculation.

This equation of state looks pretty different from that derived from Landau theory. But recall that Landau theory applies only to the region around the critical point, where the magnetization and magnetic field are small. Setting $h = 0$ initially and expanding tanh in a Taylor series gives

$$M = \beta JzM - \frac{1}{6}(\beta JzM)^3 + \ldots$$

Obviously $M = 0$ is a solution, but not necessarily the only one. If $\beta Jz - 1 > 0$, then there is another. This occurs for β large enough, i.e.

$$T < T_c^{\rm MFT} \equiv Jz \ . \tag{7.29}$$

Then there are minima of the free energy at

$$M = \pm\sqrt{\beta Jz - 1}(\beta Jz)^3 = \sqrt{\frac{T_c^{\rm MFT} - T}{\beta^2 T_c^3}} = t^{1/2}\frac{T}{T_c^{\rm MFT}} \ . \tag{7.30}$$

The mean-field approximation applied to the Ising model therefore gives a way of estimating the critical temperature. This typically is an *over*estimate, because the fluctuations being neglected tend to favor disordering the system, and so lower the temperature at which order does occur. For example, we saw in chapter 4 that the exact critical temperature for the 2d Ising model on the square lattice is $2J/\ln(1+\sqrt{2}) \approx 2.27J$, substantially smaller than the mean-field value $4J$. Similarly, in three and four dimensions the numerically determined values are $4.51J$ for the cubic lattice and $6.68J$ for the hypercubic, as compared to the mean-field values of $6J$ and $8J$ respectively.

Mean-field theory also gives approximate relations between the microscopic parameter J and the Landau parameters a, b, c. Expanding (7.28) around $M = 0$ gives

$$f_{\rm MFT} = -T\ln 2 + \frac{1}{2}zJ(1-\beta Jz)M^2 + \frac{1}{12\beta}(\beta Jz)^4 M^4 + \dots$$

Matching this to (7.9) relates the coefficients of the Taylor series in Landau's expansion to the Ising parameters J and T. Since this expansion is (at best) valid near the critical point, they should be related to the parameters B and C defined by $b = Bt + \mathcal{O}(t^2)$ and $C = c + \mathcal{O}(t)$, giving

$$B = \frac{Jz}{2}, \qquad C = \frac{Jz}{12} \ . \tag{7.31}$$

The actual numbers are not particularly important, except for the fact that both are positive. The fact that b is positive means that the ordered phase is indeed $t < 0$, while having $C > 0$ is essential for the expansion to make sense. In some examples like the Blume-Capel model in the homework, it is possible for C to be negative. Then one must include the M^6 term in the expansion, changing the mean-field critical exponents.

One of the nice things about the mean-field theory approximation is that it renders any model solvable. For example, the mean-field solution of the Ising model with longer-range interactions only changes non-universal parameters such as T_c. Namely, generalize the nearest-neighbor interaction to

$$\sum_{j,j'} J_{jj'}\sigma_j\sigma_{j'}$$

where $J_{jj'}$ is an arbitrary interaction. The entire mean-field computation goes through as before, except now zJ is replaced with $\sum_{j'} J_{jj'}$.

This example illustrates why the effect of dimensionality is very minimal in mean-field theory – it essentially only changes the strength of the mean field, and hence T_c, but not the universality class. The physics only depends on $\sum_{j'} J_{jj'}$, not whether the interactions are frustrated, or are short or medium or long range – the mean-field approximation is not sensitive to this information. It seems obvious it will miss much interesting and subtle physics. Nevertheless, given the crudity of this approximation, it remains remarkable how useful it often is. Moreover, the arguments here suggest that the approximation should become better in higher dimensions. Since the number of nearby degrees of freedom increases exponentially with dimension,[1] replacing individual interactions with their average becomes a better and better approximation. Even better, I will show in section 7.7 that in a suitably high dimension (≥ 4 for many theories) mean-field theory is not merely a good approximation: it gives *exact* results.

7.4 The correlation length

mean-field theory requires making a very dramatic approximation, neglecting fluctuations. Since the fundamental issue in statistical mechanics and thermodynamics is the study of fluctuations, it is hardly acceptable to stop here. However, it is a tribute to the fundamental simplicity of the field and to the insights of Landau that so much can be extracted from so little. It gives a qualitatively correct picture in any dimension higher than one, and exact quantitive results in high enough dimensions. It also gives a framework for understanding how the non-analyticities arise in general.

Here I will use mean-field theory to compute the correlation length and verify that it indeed has the advertised property: at the critical point it diverges. A self-consistent check on Landau theory worked out in the next section will then be to use this result to see if the fluctuations are indeed small.

Thus now it is necessary to start developing a framework for understanding when and how mean-field theory goes wrong. A very useful quantity in this study is the *correlation length* ξ. Roughly speaking, the value of ξ gives a notion of "how far the interactions reach", e.g. if one flips a single spin from say down to up, that flip affects the probabilities for finding other spins down or up out to a distance ξ. It tells one how far away thermal fluctuations have an effect. At very high temperature, the correlation length is small because the fluctuations are so strong that they wipe out any correlations. At very low temperature, the fluctuations are (in high enough dimension) very small, so again they result in a very small correlation length. In the middle (i.e. when the temperature is at the same scale as the energy scales coming from the couplings), then it is possible for the correlation length to be much larger. To give away the ending, what we will show is near critical points, the correlation length increases, diverging at the critical point. Thus

[1] In fact, when studying physics on irregular graphs instead of a lattice, one *defines* the effective dimensionality of the system as how the number of sites $n(R)$ within a region of radius R depends on R, i.e. $n(R) \sim R^{d_{\text{eff}}}$. Thus the Bethe lattice/Cayley tree has $d_{eff} = \infty$.

fluctuations are *extremely important* near critical points.

As discussed in chapter 1, the behavior of correlators is one of the best ways of understanding and characterizing different phases of matter. In this context it is natural to look at the correlator of the operator used to define the order parameter, i.e. in a ferromagnet we want to look at correlators, e.g. $\langle \sigma_j \sigma_k \rangle$ in Ising. In an ordered phase this will go to a non-zero constant at $|j-k| \to \infty$, while in a disordered phase it goes to zero in this limit. Of interest here is how correlators *approach* this limit, so it useful to define $r = |j-k|$ and

$$G(r) = \langle \sigma_j \sigma_k \rangle - \langle \sigma_j \rangle \langle \sigma_k \rangle \tag{7.32}$$

so that

$$\lim_{r \to \infty} G(r) = 0 \ .$$

Then the correlation length is defined so that at very large r,

$$G(r) \sim e^{-|a-b|/\xi} \tag{7.33}$$

This exponential depends is the dominant behavior at large r; in the omitted prefactor there may very well contain powers of r. These do not affect the value of the correlation length ξ. This thus defines a precise notion of how far away degrees of freedom can be and still have an effect on each other.

The subtracted correlator $G(r)$ almost always has the form (7.33). As shown in earlier chapters, this can be computed easily and exactly for the one-dimensional Ising model, and in free-fermion models. In the former, the correlation length is simply

$$\xi = -\frac{1}{\ln(\tanh(\beta J))} \ . \tag{7.34}$$

In one dimension there is no ordered phase, as follows from the Peierls argument given in chapter 1, and from the Mermin-Wagner theorem described in a subsequent chapter. Nonetheless, the correlation length gets larger and larger as the temperature is lowered. It is easy to understand why. At exactly zero temperature, there are no thermal fluctuations, so the partition function is dominated by the two ordered configurations. At any non-vanishing temperature, fluctuations disorder the one-dimensional system. However, the entropy contribution to the free energy grows only logarithmically with system size, so for very low temperatures $\beta J \gg 1$, the energy contribution still wins up to long length scales. Thus the correlation length grows exponentially with βJ for large βJ, as is easy to check from the exact result (7.34).

A diverging correlation length means that the mean-field approximation is highly suspect in this regime; the Ginzburg criterion discussed in section 7.7 makes the issue precise. In fact, at some values of the coupling, the correlation length becomes infinite, so that (7.33) no longer implies: $G(r)$ decays algebraically with distance. To give a preview of the following, here is an extremely important piece of physics:

The correlation length diverges at a critical point. This implies physics at *all length scales* needs to be considered in understanding critical points.

This is why it took so long and was so difficult to quantitatively characterize critical behavior. Simple assumptions about fluctuations do not work!

7.5 Including spatial dependence in Landau theory

In the mean-field approximation, a complicated interacting many-body system is turned into a non-interacting by effectively averaging over all the interactions. By construction, there is no way of computing a correlation length, since as long as the system is translation invariant, the mean field will be translation invariant. The next step is therefore to include spatial dependence into Landau theory. This is fairly straightforward to do, and is the task of this subsection. In 7.6, I compute the correlation length in a mean-field fashion. This calculation allows a precise way to understand how and why mean-field theory does not work in low dimensions, via the Ginzburg criterion discussed in section 7.7.

The approach is similar to before. However, here the system is divided up into a large number of smaller subsystems, each with N_s spins. In the Ising case each subsystem has a magnetization

$$M_s = \frac{1}{N_s} \sum_{j \in s} \sigma_j \ .$$

Then the partition sum can be divided up in a more elaborate version of the earlier argument:

$$\sum_{\{\sigma_j = \pm\}} e^{-\beta E} = \sum_s \sum_{M_s} \sum_{\sigma_j \in s | \sum_{j \in s} = M N_s} e^{-\beta E} \ .$$

The beautiful thing about having large numbers of degrees of freedom is that these many subsystems each still can contain many spins. Thus each M_s can be taken to be range continuously from -1 to 1. Moreover, since there are many subsystems, their locations of the systems can be labeled by a continuous variable $\vec{x}$, and so M_s can be taken as a function $M(\vec{x})$.

In Landau theory for the Ising model, one therefore considers a function $f(M(\vec{x}))$, the free energy density at fixed magnetization. Then the partition function is given by

$$Z = \int [DM(\vec{x})] e^{-\beta \int d^d x f(M(\vec{x}))} \ . \tag{7.35}$$

The integral is called a *functional integral*. One must integrate over all different field configurations, i.e. do an integral from -1 to 1 at *every* point $\vec{x}$ in space. Many times the measure $[DM(\vec{x})]$ is not well-defined, and so can send mathematicians into fits of derision. Of course, in the context here each x is really a finite region of N_s spins, so

one can use this to give a precise definition of the measure. That does not automatically render the measure well-defined, as one must then show that the continuum limits is a sensible one. In the cases of interest here, the continuum limits are sensible, and thus without further ado I proceed.

A *field* is a degree of freedom for every point in space (or space-time). This is what $M(\vec{x})$ is! It is akin to an electromagnetic field, but an even simpler object, since it has no space or space-time indices. Landau theory provides a beautiful way of understanding how and why field theory is a useful description of critical phenomena and beyond. The effect of the magnetization varying in space is to include terms in the free energy depending on spatial derivatives. When spatial variations are small, it is easy to include their effect in the free energy density. Although of course the precise symmetry depends on the underlying lattice, it is natural to assume that if the interactions are invariant under the appropriate subgroup of rotations, then the theory in the continuum long-distance limit should be fully invariant under rotations. Exceptions to this are possible but rarely (never?) end up affecting universal properties, except as very small corrections. The free-energy density for Ising in the limit of small fluctuations is then

$$f(M(x)) = a + \kappa \nabla M \cdot \nabla M - hM + bM^2 + cM^4 + \dots \tag{7.36}$$

where the dot product here is in real space; the generalization to n-component spins is obvious. The coupling κ in many contexts is the *stiffness*, because when positive it raises the free energy for spatial variations and so favors a uniform M. If κ for some physical system is negative, one must include terms with four derivatives. Otherwise, higher-derivative terms typcially can be neglected when the fluctuations are small. This assumption can and will be checked.

It is usually convenient to define the field to be a suitably averaged *difference* between a fluctuating degree of freedom and its expectation value. This definition is designed to characterize *fluctuations*: its expectation value by definition is zero. The Landau field theory for the Ising model involves a single field $\phi(\vec{r})$. The simplest definition would then be

$$\phi(\vec{r}) \sim \sigma_{\vec{r}} - \langle M \rangle \ . \tag{7.37}$$

Since σ is a discrete variable, this obviously is not going to turn into a continuous function. To obtain something nicer, one needs to average over enough spins in a region around $\vec{r}$ to give something near to the expectation value. One can define precise procedures for averaging and then prove various interesting things. The beauty of Landau theory is that none of the precise choice are supposed to matter: one simply assumes that only very gross behavior, such as the types of degrees of freedom and their symmetries, matters. The genius of this ideas is that it works most of the time!

The first step is to rewrite the free energy density in terms of ϕ. For $T > T_c$ and vanishing magnetic field, the expectation value of the magnetization vanishes, so one simply replace M with ϕ in f:

$$\beta f(\phi(\vec{r})) = \kappa |\nabla \phi|^2 + b\phi^2 + c\phi^4 + \dots \qquad \text{for } T > T_c \ . \tag{7.38}$$

One then does the partition sum (i.e. the functional integral) over ϕ just as for M; I will explain below how to define this sum precisely. Note that for convenience the couplings have been rescaled to absorb the factor β.

For $T < T_c$ and $h = 0$, the expectation value of M is non-vanishing, obeying $M^2 = -b/(2c)$. To define ϕ one must therefore specify which minimum is of interest. As long as the flucutations are small, this is consistent to do. Here the two minima are related by the discrete symmetry $\sigma_j \to -\sigma_j$, which translates in the field theory to $\phi(\vec{r}) \to -\phi(\vec{r})$). (The low-temperature phase therefore provides an example of what is called "spontaneous symmetry breaking".) This gives here

$$\beta f(\phi) = \kappa|\nabla\phi|^2 - 2b\phi^2 + \mathcal{O}(\phi^3) \qquad \text{for } T < T_c \tag{7.39}$$

for either minimum. The term odd in ϕ appears because the potential is not symmetric around the minimum, as is obvious from the plotting $bM^2 + cM^4$ for b negative. Note that since b is negative for $T < T_c$, the quadratic term indeed contributes something positive to the free energy for both $T > T_c$ and $T = T_c$, which is of course a consequence of expanding around the minimum.

7.6 Diverging correlation length from mean-field theory

Including spatial variation in Landau theory is so straightforward, it is hard to see what could go wrong, but the correlation length provides a clue. In order to define a continuous field, it was necessary to average over a box containing a large number of spins, but of a size still much smaller than that of the full system. Moreover, neglecting higher derivatives is reasonable only if the so-defined field does not vary quickly in space. If the correlation length is smaller than that of the linear dimension of the box, then these seem like a reasonable approximations. However, if the correlation length is larger than the box dimension, one might worry that averaging over all the spins in a given box will not give an accurate approximation of the full interactions between different boxes.

Thus in this section I find the correlation length by computing the correlator

$$G(r) = \langle\phi(\vec{r})\phi(0)\rangle \ . \tag{7.40}$$

in the mean-field approximation. Namely, I neglect field-theory interactions by keeping only terms quadratic in the field. For reasons that will soon be obvious, the resulting simplified model is often called the *Gaussian model*. It turns out to be easier (and useful later) to compute the more general correlator

$$G_{\alpha,\alpha'}(r) = \langle e^{\alpha\phi(\vec{r})+\alpha'\phi(0)}\rangle \tag{7.41}$$

so that

$$G(r) = \frac{\partial^2}{\partial\alpha\partial\alpha'} G_{\alpha,\alpha'}(r)\Big|_{\alpha=\alpha'=0} \ . \tag{7.42}$$

The way of making progess here (and in basically all of theoretical physics) is to work in Fourier space. The transformed field is defined as

$$\widetilde{\phi}(\vec{k}) = \int_V d^d x\, \phi(\vec{x}) e^{i\vec{k}\cdot\vec{x}} \tag{7.43}$$

In a finite box of volume V, the values of $\vec{k}$ are quantized. The functional integral can then simply be written as a *product* of integrals over each $\phi(\vec{k})$:

$$\int [D(M(x))] \to \int [D\phi] \equiv \prod_k \int d\widetilde{\phi}(\vec{k}) \ . \tag{7.44}$$

where the product is over all possible values consistent with the boundary conditions. Because the form of f, the integrand $e^{-\beta f}$ is very small for configurations with large $\phi(x)$. Thus is sensible to extend the integrals to all values of $\phi(x)$ instead of restricting it to lie between -1 and 1, and so take each integral over $\widetilde{\phi}(\vec{k})$ to run from $-\infty$ to ∞ with equal weight for all k.

$$Z = \prod_k \int_{-\infty}^{\infty} d\widetilde{\phi}(\vec{k}) e^{-\beta \int d^d x f(\phi(\vec{k}))} \ . \tag{7.45}$$

This is simply a collection of integrals, one for each momentum. For the correlator, this is generalized to

$$G_{\alpha,\alpha'}(r) = \frac{1}{Z} \prod_k \int_{-\infty}^{\infty} d\widetilde{\phi}(\vec{k}) e^{-S_{\alpha,\alpha'}} \ , \qquad S_{\alpha,\alpha'} = -\alpha\phi(\vec{r}) - \alpha'\phi(0) + \beta \int d^d x f(\phi(\vec{k})) \ . \tag{7.46}$$

Since the integrals are over values of $\phi(\vec{k})$, the next step is to rewrite the free-energy density in Fourier space. Another very nice consequence that the spatial derivatives simply yield powers of $\vec{k}$. The downside is that for terms higher than quadratic in ϕ, the expressions look nasty, with integrals over all the different momentum. Thus I make here the drastic approximation to neglect all terms higher than quadratic in ϕ. A field theory containing only terms up to quadratic in the fields is typically called *free*, because going to Fourier space effectively decouples all degrees of freedom at different values of $|\vec{k}|$. For example, (7.38) reduces to $\beta f_{\text{free}}(x) = \kappa|\nabla\phi|^2 + b\phi^2$. Thus

$$\beta \int d^d x f_{\text{free}}(x) = \int dk((\kappa k^2 + b)\widetilde{\phi}(\vec{k})\widetilde{\phi}(-\vec{k}) \ .$$

The partition function (7.45) and the correlator (7.46) become simply the product of Gaussian integrals!

All correlators can be computed in a free field theory, by doing Gaussian integrals or their fermionic analogs. In this free Gaussian field theory, the correlator is computed simply by completing the square in the exponent. Namely, for $T > T_c$,

$$S_{\alpha,\alpha'}\Big|_{\text{free}} = V \sum_{\vec{k}} \left((\kappa k^2 + b)\widetilde{\phi}(\vec{k})\widetilde{\phi}(-\vec{k}) - \alpha e^{i\vec{k}\cdot\vec{r}}\widetilde{\phi}(\vec{k}) - \alpha'\widetilde{\phi}(\vec{k}) \right)$$

can be rewritten as

$$S_{\alpha,\alpha'}\Big|_{\text{free}} = V \sum_{\vec{k}} \left((\kappa k^2 + b) \left(\widetilde{\phi}(\vec{k}) - \frac{\alpha e^{-i\vec{k}\cdot\vec{r}} + \alpha'}{2(\kappa k^2 + b)} \right) \left(\widetilde{\phi}(-\vec{k}) - \frac{\alpha e^{i\vec{k}\cdot\vec{r}} + \alpha'}{2(\kappa k^2 + b)} \right) - \frac{|\alpha e^{-i\vec{k}\cdot\vec{r}} + \alpha'|^2}{4(\kappa k^2 + b)} \right) . \tag{7.47}$$

For $T < T_c$, b is replaced with $-2b$. Each $\phi(\vec{k})$ in (7.46) is integrated independently from $-\infty$ to ∞, and so with the free $S_{\alpha,\alpha'}$, the added to $\phi(\vec{k})$ in the integrand can be shifted away. The Gaussian integrals can now easily be done, but to compute $G_{\alpha,\alpha'}$ there is no need to bother – the same integral also appears in the denominator! These two cancel, and so the only remaining piece is the term added to complete the square. Thus

$$G_{\alpha,\alpha'}(r) = \exp\left(V \sum_{\vec{k}} \frac{|\alpha e^{-i\vec{k}\cdot\vec{r}} + \alpha'|^2}{4(\kappa k^2 + b)} \right)$$

with $b \to -2b$ when $T < T_c$. Now there is no need for putting the system in a finite box and quantizing k, and so the sum can be converted to an integral. Taking the derivatives then gives for either case

$$G(r) = \frac{1}{2\kappa} \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}} \frac{1}{k^2 + \xi^{-2}} \tag{7.48}$$

where

$$\xi = \begin{cases} \sqrt{\frac{\kappa}{b}} & T > T_c \\ \sqrt{\frac{\kappa}{-2b}} & T < T_c \end{cases} \tag{7.49}$$

Note that $G(r)$ is a Green's function for the wave equation, because it satisfies the equation

$$(-\nabla^2 + \xi^{-2})G(r) = \frac{1}{2\kappa}\delta(\vec{r}) . \tag{7.50}$$

This integral can now be done to show that ξ is indeed the correlation length. In one dimension, it can be done by residue:

$$G(r)\Big|_{d=1} = \frac{1}{4\kappa\pi} \int_{-\infty}^{\infty} dk \ \frac{e^{ikr}}{k^2 + \xi^{-2}} = \frac{\xi}{4\kappa} e^{-r/\xi} , \tag{7.51}$$

yielding indeed exponential decay for any finite correlation length. At $\xi \to \infty$, the Green's function here blows up, indicating that there is no phase transition in 1d in this model.

Ironically, the integral in (7.48) in three dimensions turns out to be easier than two. It can be done simply by putting the integral in spherical coordinates:

$$\begin{aligned} G(r)\Big|_{d=3} &= \frac{1}{2\kappa(2\pi)^2}\int_0^\infty dk\, k^2 \int_0^\pi d\theta \sin\theta\, e^{ikr\cos(\theta)} \frac{1}{k^2+1/\xi^2} \\ &= \frac{1}{(2\pi)^2\kappa r}\int_0^\infty dk\, k \sin(kr)\frac{1}{k^2+1/\xi^2} \\ &= \frac{1}{8\pi\kappa r} e^{-r/\xi} \end{aligned} \tag{7.52}$$

where the k integral is done by residue. Thus indeed for $b \neq 0$, the correlator falls off exponentially as advertised, with correlation length given by (7.49). As the coupling is tuned to the critical point, the correlation length diverges. Defining a critical exponent ν to characterize this divergence via

$$\xi \propto t^{-\nu} \tag{7.53}$$

gives in mean-field theory

$$\nu_{\rm MFT} = 1/2 \tag{7.54}$$

both above and below the critical temperature.

In two dimensions, the trick to evaluate the correlator is to define the x direction to be the direction of $\vec{r}$, and then to evaluate the integral in Cartesian coordinates. One then does the k_y integral first by residue, and then looks up the integral (9.6.25) in the marvelous (and free on the web) book by Abramowitz and Stegun (AS) on special functions [2]

$$\begin{aligned} G(r)\Big|_{d=2} &= \frac{1}{2\kappa(2\pi)^2}\int_{-\infty}^\infty dk_x e^{ik_x r}\int_{-\infty}^\infty dk_y \frac{1}{k_x^2+k_y^2+\xi^{-2}} \\ &= \frac{1}{8\pi\kappa}\int_{-\infty}^\infty dk_x \frac{e^{ik_x r}}{\sqrt{k_x^2+\xi^{-2}}} \\ &= \frac{1}{4\pi\kappa} K_0(r/\xi) \end{aligned} \tag{7.55}$$

where K_0 is the modified Bessel function. Its behavior for small and large r is given again in the good book (AS 9.6.12 and 9.7.2):

$$K_0(r/\xi) \sim \begin{cases} -\ln(r/\xi) & r \to 0 \\ \sqrt{\frac{\pi\xi}{2r}}\, e^{-r/\xi} & r \to \infty \end{cases} \tag{7.56}$$

[2] There's a pretty accurate way to tell a theoretical physicist from a pure mathematician in the 21st century: typically the former like special functions while the latter try (usually successfully) to avoid them.

so that again the decay is exponential at large distances. At short distances (or at the phase transition where the correlation length diverges), the correlator diverges logarithmically. This fact turns out to have many remarkable consequences I will describe later in the book.

A diverging correlation length at a critical point is not a fluke of mean-field theory. In fact, it is the defining physical characteristic of a critical point. Critical correlations are not characterized by any length scale like a correlation length. The only way $G(r)$ can depend on the distance between the two fields is via a power law, so that for $\xi = 0$ and r much larger than the lattice separation,

$$G(r) = \frac{A}{\kappa} r^{-2x_\phi} \tag{7.57}$$

for some dimensionless constant A; in $d = 3$, (7.52) gives $A = 1/8\pi$. Since k is of dimension energy and G is dimensionless, κ is of dimension (energy)$^{d-2}$. It follows immediately that

$$x_{\phi,\text{Gaussian}} = \frac{d-2}{2} \ . \tag{7.58}$$

in any $d > 2$.

It is fairly obvious from (7.48) that in any dimension $G(r)$ will fall off exponentially for large r/ξ, since the imaginary exponential oscillates wildly as a function of k in this limit, and the integral can be done by saddle point. In fact, the integrals in (7.48) for the correlator can be computed in any dimension, showing that the correlation length obeys (7.49) in general. The standard physicist way to do this (see e.g. Goldenfeld) is to solve the Green function equation (7.50) away from the origin, and then fix the solution by showing that it has the correct behavior at $r = \infty$. While this technique gives the correct answer with enough care, I always find it a bit bothersome, since it's being rather cavalier with assumptions about analyticity and so forth. Since I like special functions, and wasted a day of my life finding the trick, here I'll show how compute it directly. One interesting fact of later importance is that the expressions below make sense for any real value of $d > 1$, and so allows a "continuation" of the correlator to non-integer dimensions.

In terms of d-dimensional spherical coordinates, the integrand in (7.48) depends only on the magnitude k and a single angle θ. Doing the integral over the remaining $d-2$ coordinates gives the area of a $(d-2)$-dimensional sphere of radius 1, which is $S_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$. Thus for $d \geq 2$,

$$G(r) = \frac{S_{d-2}}{2\kappa(2\pi)^d} \int_0^\infty dk \, \frac{k^{d-1}}{k^2 + \xi^{-2}} \int_0^\pi d\theta \sin^{d-2}\theta \, e^{ikr\cos\theta} \tag{7.59}$$

The trick is to evaluate these integrals explicitly is to rewrite them in two-dimensional Cartesian coordinates $X = k\cos\theta$, $Y = k\sin\theta$, giving

$$G(r) = \frac{S_{d-2}}{2\kappa(2\pi)^d} \int_0^\infty dY \, Y^{d-2} \int_{-\infty}^\infty dX \frac{e^{iXr}}{X^2 + Y^2 + \xi^{-2}} \ .$$

Doing the X integral by residue and then changing variables $Y = \sinh(t)/\xi$ gives

$$\begin{aligned} G(r) &= \frac{\pi S_{d-2}}{2\kappa(2\pi)^d} \int_0^\infty dY \, \frac{Y^{d-2}}{\sqrt{Y^2+\xi^{-2}}} e^{-r\sqrt{Y^2+\xi^{-2}}} \\ &= \frac{\pi S_{d-2}\xi^{2-d}}{2\kappa(2\pi)^d} \int_0^\infty dt \, (\sinh t)^{d-2} e^{-r\cosh(t)/\xi}. \end{aligned}$$

The good book (AS 9.6.23) gives the latter integral in terms of a Bessel function, yielding

$$G(r) = \frac{1}{4\pi\kappa(2\pi\xi r)^{(d-2)/2}} \, K_{(d-2)/2}(r/\xi) \ . \tag{7.60}$$

As a check, note that (AS 10.2.16,10.2.17) $K_{1/2}(z) = e^{-z}\sqrt{\pi/2z} = K_{-1/2}(z)$, so that (7.60) in $d = 1$ and $d = 3$ indeed reduces to (7.51,7.52). In general, the asymptotic value of Bessel function remains $\sqrt{\frac{\pi\xi}{2r}}e^{-r/\xi}$, so ξ defined by (7.49) is the correlation length in any dimension. Similiarly, for any $d > 2$, $K_{(d-2)/2}(r/\xi) \propto r^{(2-d)/2}$, and the appropriate behavior (7.57) is recovered.

7.7 How mean-field theory goes wrong: the Ginzburg criterion

In using the mean-field approximation to derive the location of the critical point, as well as the critical exponents $\nu_{\rm MFT}$, $\gamma_{\rm MFT}$ and $\delta_{\rm MFT}$, fluctuations were neglected. In other, terms in the energy beyond "one-body" are neglected. For example, in the Ising model, the terms $J(\sigma_i - M)(\sigma_j - M)$ in the energy are ignored, while the terms kept contribute JM^2 or $JM\sigma_i$. In order for this approximation to be consistent in the ordered phase $T < T_c$, the the average of the terms kept must be larger than average of the "fluctuations" $\langle J(\sigma_i - M)(\sigma_j - M)\rangle$ neglected. The analogous statement in the original continuum approach in section 7.1 is that everything is independent of space. Once the derivative term is included in the effective free energy, the way to suppress fluctuations is to take $\kappa \to \infty$.

Since these correlations are substantial out to a distance $|i - j| \sim \xi$, it is necessary to average as well over a spatial region $\mathcal{V}$ of volume ξ^d. Thus

$$\sum_{\mathcal{V}} \langle(\sigma_i - M)(\sigma_j - M)\rangle \ll \sum_{\mathcal{V}} \langle M\rangle^2 \sim \xi^d \langle M\rangle^2 \ , \tag{7.61}$$

where $\sum_{\mathcal{V}}$ is the sum over all sites in the region $\mathcal{V}$. Note that this criterion can equivalently be stated by saying that the average of the fluctuations in the order parameter must be much smaller than the order parameter itself.

Translating this requirement into the language of mean-field theory using the field ϕ with free energy (7.39) gives

$$\int_{\mathcal{V}} d^d r\, G(r) \ll \xi^d \langle M \rangle^2 \tag{7.62}$$

Since $G(r) \to 0$ exponentially fast for $r > \xi$, the integral on the left-hand side can be extended to be over all of space. This integral was already done in section (7.6), since it is just the Fourier transform at $k = 0$:

$$\int d^d r\, G(r) = \widetilde{G}(k = 0) = -\frac{1}{2b} = \frac{\xi^2}{\kappa} \ .$$

The right-hand side simplifies to

$$\xi^d \langle M \rangle^2 = -\xi^d \frac{b}{2c} = \xi^{d-2} \frac{\kappa}{4c} \ .$$

Putting these into (7.62) gives the *Ginzburg criterion*

$$\xi^{4-d} \ll \frac{\kappa^2}{4c} \ . \tag{7.63}$$

This provides a powerful consistency condition.

No matter what the couplings are, the correlation length diverges as the critical temperature is approached. Even if one takes κ very large to suppress the fluctuations, close enough to the critical point, the Ginsparg criterion is violated when $d < 4$. mean-field theory does not describe critical behavior quantitatively for $d < 4$! This dimension $d = 4$ is called the *upper critical dimension* for ϕ^4.

I emphasize that the Ginszburg criterion does not show that Landau's effective field theory approach is wrong. It shows only that the mean-field approximation does not work in dimensions below the upper critical dimension. In other words, one cannot neglect fluctuations in any effective approach in $d < 4$.

A remarkable converse is that this argument provides a strong piece of evidence that in $d \geq 4$, mean-field theory works. The resulting critical exponents are not merely a good approximation, but exact. This indeed has been shown rigorously to be true in many cases. People sometimes say that such theories are "trivial". What they mean is that even if one adds interactions like the ϕ^4 term, the critical behavior is that of free-field theory. Intuitively the fact that mean-field theory works better in higher dimensions is not a complete shock; as the dimensionality increases the number of nearby degrees of freedom increases exponentially, and so replacing interactions with the mean field might be expected to work better and better. However, this intuition does not at all explain why the mean-field exponents are exact in any dimension $d \geq 4$, nor the remarkable fact that such a simple approach as Landau's yields them.

Going back to $d < 4$, the question is then what use mean-field theory is. In $d = 1$, it gives an qualitatively wrong answer, since there is no critical point; the correlation length does diverge at $T \to 0$, but this really isn't a critical point in any other sense, since there are no fluctuations of any sort. In $d = 2$, it gives qualitatively correct answers for Ising, but numerically well off, i.e. it predicts $\eta = 1/2$, while the exact answer is $1/8$. Moreover, for the XY/Heisenberg models (or any with spontaneously broken continuous symmetry), the Mermin-Wagner theorem means that there is no phase transition of the usual order-disorder sort, so again mean-field theory isn't even qualitatively correct. In $d = 3$, it's much better. So the question can be refined to: in $d = 3$ and perhaps $d = 2$, can mean-field theory be modified in some fashion to fix it? The answer, more or less, is no. A radically new approach is needed to the problem. This is the renormalization group, discussed in the next chapter.