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Soluble Renormalization Groups and Scaling Fields for Low-Dimensional Ising Systems

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A variety of one-dimensional Ising spin systems, including staggered and parallel magnetic fields, alternating and second neighbor interactions, four-spin coupling, etc., are discussed in terms of renormalization group theory. A continuous range of distinct renormalization groups is constructed in exact closed form, analyzed in detail, and compared with exactly calculated thermodynamic properties. Fixed point linearization yields relevant, irrelevant, and marginal operators. All groups yield identical "critical" behavior (at $T = 0$) with $\eta = 1$, $\delta = \infty$, $\gamma = \nu = 2 - \alpha$, and with identical linear scaling fields. A generalization of Wegner's analysis to discrete groups yields explicit power series for the *non*linear scaling fields; these are seen to depend on the particular renormalization group and, hence, are physically *nonunique*. A planar, multiconnected "truncated tetrahedron" model of effective dimensionality $\log_2 3$ is analyzed via a decoration and star-triangle group revealing highly singular behavior as $T \to T_c = 0$.

1. INTRODUCTION

The general formulation of Wilson's renormalization group approach to the study of critical point behavior has been presented by Wilson [1] and others [2] and developed by Wegner [3] to discuss the corrections to asymptotic scaling [3]. The principal model so far discussed with an *exactly* realizable renormalization group, in contrast to a perturbation theoretic realization by ϵ-expansions, etc. [4–10] is Baker's model [11] which has the same structure as Dyson's hierarchical model [12, 13]. For this class of models, however, no exact solutions of the renormalization group equations have been found, nor have the models themselves been solved exactly by any other methods.

In this paper [14] we discuss a variety of models for which renormalization groups can be constructed in exact closed form as recursion relations in finite-dimensional parameter spaces. The simplest model is the nearest neighbor linear

Ising chain [15, 16]; we also consider more elaborate one-dimensional Ising models. Although the thermodynamics of these systems are readily calculable by traditional methods (such as the matrix method), and their "critical points" all occur at zero temperature [16] the explicit analysis of the associated renormalization groups illuminates a number of important features of the general theory. In particular we discuss (a) the *nonuniqueness* of the renormalization group; (b) its semigroup property; (c) the behavior of the constant or "background" term in the Hamiltonian; (d) the existence and nonexistance of fixed points for particular types of critical behavior; (e) the exact critical operators, including examples of relevant, marginal, irrelevant, and pseudo-relevant variables; (f) asymptotic scaling; (g) the "irrelevancy" and "nonlinear" corrections to asymptotic scaling; (h) the nonlinear scaling fields, which are shown to be nonunique, and techniques for calculating them.

In addition we discuss one model, a planar, multiconnected "truncated tetrahedron" Ising lattice, (which is not of simple one-dimensional form) where the exact renormalization group equations yield precise information on the critical ($T \to 0$) behavior which seems otherwise inaccessible. Although this last model represents the only new result concerning critical behavior as such, we believe the study throws light on various important aspects of the renormalization group approach and improves ones general understanding of its operation.

For the reader's convenience we summarize at this point, the main aspects of the paper. In Section 2 we review the renormalization group approach with emphasis on certain general features and questions. Readers familiar with the theory may wish to peruse this section lightly or merely refer back to it for notation, etc. as the occasion arises.

In Section 3 the simple linear Ising chain with nearest neighbor interactions is discussed in the presence of both a uniform magnetic field, H, and a staggered magnetic field, $H^{\dagger}$. A renormalization group transformation is defined by using the spin dedecoration (or "iteration") transformation [17, 18]. This yields explicit algebraic recursion relations. Various fixed points exist, including one describing the ferromagnetic critical point (at $T = 0$ with pair spin coupling $J > 0$) and the antiferromagnetic critical point (at $T = 0$ with $J < 0$). The appropriate critical exponents are correctly found by linearization of the renormalization group. Specifically, one obtains $\eta = 1$, $\delta = \infty$, and $\gamma = \nu = 2 - d$. (Compare with Ref. [16].) Both relevant and irrelevant critical variables or operators appear. In addition, when alternating coupling strengths J_1 and J_2 along the length of the chain are introduced (Section 4), a marginal variable appears. More elaborate, but still exactly soluble models discussed in Section 4 include the linear chain with both first and second nearest neighbor couplings and a "ladder" of spins including a four-spin interaction.

By considering different spatial rescaling factors b, produced by dedecorating firstly alternate spins and secondly, a pair out of every three spins, etc., the semi-group property of the renormalization transformation is explicitly checked. The importance of the constant or spin-independent term in the Hamiltonian appears in the calculations; it is seen to go to an appropriate limit at the fixed points. The scaling predictions of the renormalization group can be checked against the exact solutions and the appropriate scaling functions may be derived.

In Section 5 we describe two "truncated tetrahedron" Ising models. These are respectively three- and four-coordinated planar structures containing polygons of sizes 3, 6, 12, 24,... . However, despite the number of polygonal closures, these lattices (or pseudo lattices) have a connection number [19] of only three; as a result the critical point still occurs at $T = 0$. Nevertheless, the singularity there is extremely strong; its character (in zero field) can be deduced from the exact renormalization group which is constructed. We have not, otherwise, been able to solve these models.

Since the exact renormalization group equations for the linear models are at hand, Wegner's analysis [3] of the corrections to scaling can be applied. As demonstrated in Section 6, this enables us to generate explicitly, appropriate nonlinear scaling fields (combinations of temperature T, magnetic field H, etc.). However, these are seen to be nonunique: the free energy can also be extended outside the linear critical region in scaling form using other nonlinear fields. Wegner's discussion [3] assumes that the renormalization group is given as a continuous group on the spatial rescaling factor. In our analysis the rescaling factors are always integral and the renormalization group is discrete. Accordingly we present a method for calculating the scaling fields for a discrete group.

We also show, in Section 3, that it is possible to "miss" a fixed point describing a particular critical point if care is not taken. Specifically, the antiferromagnetic fixed point may be transformed away and an irrelevant operator is apparently turned into a relevant one. This illustrates that a useful renormalization group should be chosen so as to "focus" on the critical point of interest.

In a related context, we generalize in Section 7, an Ising spin renormalization group devised by Wilson [20] thereby obtaining an infinity of renormalization groups for the Ising chain parametrized by one continuous parameter. These groups (which in one limit include the original dedecoration group) generate the same low temperature fixed point with the same critical exponents, but have different nonlinear scaling fields and different high temperature fixed points. The distinct character of these renormalization groups is specifically related to the presence of a variable spin-rescaling factor. In general such a feature is needed in order to realize a fixed point with a particular value of the exponent η. However, for dimensionality $d = 1$ and a critical point with $\eta = 1$, a "fixed spin" renormalization group, such as the simple dedecoration group, suffices as the earlier analysis shows.

2. Renormalization Group Formalism

In this section we review briefly the general renormalization group formalism [1–3]. This will serve to exhibit the features to be explored in the explicit analysis and to introduce the notation.

2.1. *Renormalization Groups*

A system with given interactions at a specified temperature, T, and subject to particular external fields H, $H^{\dagger}$,... is described by its reduced Hamiltonian

$$\bar{\mathscr{H}} = \bar{\mathscr{H}}(\{s\}_N) = -\mathscr{H}(\{s\}_N, H, H^{\dagger}, ...)/k_B T, \tag{2.1}$$

where $\{s\}_N = \{s_1, s_2, ..., s_i, ..., s_N\}$ denotes a set of N local degrees of freedom associated with, say, lattice sites $\mathbf{R}_i$. In our discussion these degrees of freedom, or *field variables*, will be simple Ising spins, $s_j = \pm 1$. The reduced Hamiltonian, or what is equivalent, the Boltzmann factor $e^{\bar{\mathscr{H}}}$, is to be a translationally invariant function of the field variables (at least asymptotically in the thermodynamic limit $N \to \infty$); it will be parametrized by a set of "initial" fields or interaction parameters $\{k\} = \{k_0, k_1, ...\}$ describing the various (translationally invariant) terms appearing in $\bar{\mathscr{H}}$. It is normally convenient to take k_j as the coefficient multiplying some particular additive term in $\bar{\mathscr{H}}$ but for critical points occurring at $T = 0$ other choices will often be more convenient. The zeroth field, k_0, may be identified with the constant or "spin-independent" term in $\bar{\mathscr{H}}$, say $E_0/k_B T$. In general the set $\{k\}$ must be of infinite dimensionality. The solubility of the renormalization groups we describe resides in the fact that they may be realized exactly in a *finite* field space $\{k_0, k_1, ..., k_m\} \equiv \mathbb{K}_m$.

A renormalization group transformation $\mathbb{R}$ carries a Hamiltonian $\bar{\mathscr{H}}$ (we will omit the adjective "reduced" hereafter) into a transformed or *renormalized* Hamiltonian

$$\bar{\mathscr{H}}' = \mathbb{R}[\bar{\mathscr{H}}], \tag{2.2}$$

which is again translationally invariant and may, likewise, be parametrized by the set $\{k\}$. On a finite system the renormalization operator $\mathbb{R}$ acts to reduce the number of degrees of freedom from N to

$$N' = N/b^d, \tag{2.3}$$

where d is the spatial dimensionality while the *spatial rescaling factor* b exceeds unity. In the examples we will discuss, b will be an integer but this is not generally necessary.

The renormalization group must also (under all currently explored formulations [21, 22]) keep the partition function

$$Z_N[\mathscr{H}] = \mathrm{Tr}_N\{e^{\mathscr{H}}\} \tag{2.4}$$

invariant so that

$$Z_{N'}[\mathscr{H}'] = Z_N[\mathscr{H}]. \tag{2.5}$$

The operation Tr_N denotes the trace operation, integration, summation, etc., appropriate to the field variables $\{s\}_N$. The free energy per degree of freedom

$$f[\mathscr{H}] = \lim_{N\to\infty} N^{-1} \ln Z_N[\mathscr{H}], \tag{2.6}$$

thus satisfies the basic covariance relation

$$f[\mathscr{H}'] = b^d f[\mathscr{H}]. \tag{2.7}$$

This relation ultimately leads to critical point homogeneity and scaling. The presence of the thermodynamic limit in (2.6) and (2.7) is essential to any discussion of critical points (which do not exist for finite N) but, in general, leads to severe technical problems in constructing a mathematically rigorous definition of $\mathbb{R}$.

In a *linear* renormalization group the basic correlation function

$$G[\mathbf{R}_i - \mathbf{R}_j\,; \mathscr{H}] = \langle s_i s_j \rangle_{\mathscr{H}}\,, \tag{2.8}$$

(where $\langle\cdot\rangle$ denotes the standard statistical average taken with $\mathscr{H}$ in the thermodynamic limit) transforms according to

$$G[\mathbf{R}; \mathscr{H}] = c^2 G[\mathbf{R}/b; \mathscr{H}'], \tag{2.9}$$

where $c = c[\mathscr{H}]$ is a *spin rescaling factor* determined by $\mathbb{R}$. However, *non*linear groups have been successfully employed numerically [23], and have recently been studied theoretically [24]. We will also exhibit an exact nonlinear renormalization group (in Section 7).

Under iteration

$$\mathbb{R}^2[\mathscr{H}] = \mathbb{R}[\mathbb{R}[\mathscr{H}]],\ldots, \mathbb{R}^l[\mathscr{H}] = \mathbb{R}[\mathbb{R}^{l-1}[\mathscr{H}]], \tag{2.10}$$

one obtains renormalization operators with spatial rescaling factors $b_2 = b^2,\ldots, b_l = b^l$ which satisfy the semigroup property

$$\mathbb{R}^{l+l'} = \mathbb{R}^l\mathbb{R}^{l'}, \qquad b_{l+l'} = b_l b_{l'} = b^{l+l'}. \tag{2.11}$$

2.2. *Fixed Points and the Critical Spectrum*

To utilize a renormalization group one looks for a fixed point $\mathscr{H}^*$ defined by

$$\mathbb{R}[\mathscr{H}^*] = \mathscr{H}^* \tag{2.12}$$

and presumes that one may expand as

$$\mathbb{R}[\mathscr{H}^* + wQ] = \mathscr{H}^* + w\mathbb{L}Q + O(w^2), \tag{2.13}$$

where w is a scalar parameter and $\mathbb{L} = (\delta\mathbb{R}/\delta\bar{\mathscr{H}})^*$ is a *linear* operator on Hamiltonians. One then studies the eigenvalue problem

$$\mathbb{L}Q_j = \Lambda_j Q_j, \tag{2.14}$$

to obtain the spectrum of critical operators or variables Q_j, and corresponding eigenvalues Λ_j, which in view of the semigroup property (2.11) are expected to have the form

$$\Lambda_j(b) = b^{\lambda_j}, \tag{2.15}$$

where the λ_j are *independent* of b.

The form of dependence of $\Lambda_j(b)$ on b also follows if $\mathbb{R}$ can be defined for $b \to 1$ so that the infinitesimal generator

$$\mathbb{G} = \lim_{b\to 1+} (\mathbb{R}_b - \mathbb{I})/(b - 1) \tag{2.16}$$

can be constructed. If we redefine $l = \ln b$ the renormalization group Eq. (2.2) then becomes

$$d\bar{\mathscr{H}}/dl = \mathbb{G}[\bar{\mathscr{H}}]. \tag{2.17}$$

This is the form presumed by Wegner [3] in his analysis of corrections to scaling, etc., but we will be working with discrete groups only.

If the set of eigenoperators or critical variables Q_j for the fixed point $\mathscr{H}^*$ is complete one may write

$$\bar{\mathscr{H}} = \mathscr{H}^* + \sum_j h_j Q_j, \tag{2.18}$$

thereby parametrizing $\bar{\mathscr{H}}$ in terms of eigenfields or (linear) *critical fields* $\{h\} = \{h_0, h_1, \ldots\}$.

In the cases we will study the completeness of the scaling field expansion (2.18) [or its equivalent] will not be in doubt; but in general one should probably expect no more than some sort of asymptotic or weak completeness, e.g., for suitable expectation values taken with $\bar{\mathscr{H}}$ in the vicinity of $\mathscr{H}^*$. The mapping from the initial fields $\{k\}$ to the scaling fields $\{h\}$ is in general nontrivial but can often be simplified (at least to leading order) by symmetry considerations.

One of the critical operators, say Q_0, may be identified as the constant or spin-independent term with field k_0 or h_0. Since such a term in $\bar{\mathscr{H}}$ can always be removed from under the trace in (2.4) it follows that it transforms in a trivial way under the renormalization group so that its eigenvalue is always $\Lambda_0 = b^d$ or $\lambda_0 = d$.

Under action of the renormalization group the expansion (2.18) yields

$$\mathbb{R}[\bar{\mathscr{H}}] = \bar{\mathscr{H}}' = \mathscr{H}^* + \sum_j h_j \Lambda_j Q_j + O(h^2), \tag{2.19}$$

which may be recast as the *recursion relations*

$$h_j' = \Lambda_j h_j[1 + O(h_0, h_1, ...)]. \tag{2.20}$$

Iteration l times yields

$$h_j^{(l)} \approx \Lambda_j^l h_j, \tag{2.21}$$

provided one stays within the linear region where the $O(h^2)$ terms in (2.19) can be neglected. It now clearly makes sense to classify the critical variables as (a) *relevant* with $\Lambda_j > 1$ ($\lambda_j > 0$), which grow in importance, (b) *irrelevant* with $\Lambda_k < 1$ ($\lambda_k < 0$) whose contribution diminishes under iteration, and (c) *marginal* with $\Lambda_i = 1$ ($\lambda_i = 0$) which remain (in linear order) of constant magnitude. Normally the most relevant operator (with largest Λ_i) would identify the order parameter and its ordering field $h = H/k_B T$. Another relevant operator would be the "energy" and its field would be the reduced temperature $t = (T - T_c)/T_c$, etc. However, in the cases we will study, where $T_c = 0$, the corresponding temperature variable must evidently be expressed in different form.

Furthermore, in general, there are a variety of distinct fixed points with distinct scaling fields and exponents. In general different identifications will be necessary at different fixed points. In particular we will observe infinite temperature and infinite field fixed points where, clearly, appropriate new variables must be adopted.

2.3. *Scaling and Nonlinear Scaling Fields*

On using (2.21) in combination with (2.15) and the basic covariance relation (2.7) for the free energy, one obtains, for the linear region,

$$f(h_0, h_1, ..., h_j, ...) \approx b^{-ld} f(b^{l\lambda_0} h_0, b^{l\lambda_1} h_1, ..., b^{l\lambda_j} h_j, ...), \tag{2.22}$$

where $f[\mathscr{H}]$ has been expressed as a function of the scaling fields. Since b^l may be indefinitely large one can choose l so that as a relevant field, say $h_1 = t$, becomes small one has $b^{l\lambda_1}t = 1$ whereupon this relation can be written in the standard scaling form

$$f(h_0, t, \dots, h_j, \dots) \approx t^{2-\alpha} Y(h_0/t^{\phi_0}, \dots, h_j/t^{\phi_j}, \dots), \tag{2.23}$$

where

$$2 - \alpha = d/\lambda_1, \tag{2.24}$$

the crossover exponents are given by

$$\phi_j = \lambda_j/\lambda_1, \tag{2.25}$$

and the scaling function is

$$Y(y_0, y_2, \dots, y_j, \dots) = f(y_0, 1, y_2, \dots, y_j, \dots). \tag{2.26}$$

For irrelevant scaling fields h_k the crossover exponent ϕ_k is negative so that $h_k/t^{\phi_j} = h_k t^{|\phi_j|} \to 0$ as $t \to 0$ and one expects [21] to be able to set the corresponding argument y_k in (2.26) to zero and ignore the h_k dependence of f in the critical region. Expansion with respect to y_k should yield systematic "irrelevancy" corrections to asymptotic scaling laws [3]. As we will show explicitly in Section 3.5, however, "nonlinear" corrections arising from the $O(h^2)$ terms in (2.19) may well be more important.

For a linear renormalization group, where (2.9) applies, one obtains a functional equation for $G(\mathbf{R})$ at $\mathscr{H} = \mathscr{H}^*$ with solution

$$G^*(\mathbf{R}) \approx D/R^{2\omega} \qquad \text{as} \quad R \to \infty, \tag{2.27}$$

where

$$c^* = c[\mathscr{H}^*] = b^{-\omega}. \tag{2.28}$$

This typically critical-point decay law identifies the critical exponent η via

$$d - 2 + \eta = 2\omega = -\ln c^*/\ln b. \tag{2.29}$$

This identification shows that in using a linear renormalization group one must usually treat c^* as a sort of eigenvalue which is to be adjusted to obtain a nontrivial fixed point describing the critical behavior of interest. In combination with (2.19) and (2.21) one obtains the correlation scaling relation

$$G(\mathbf{R}; h_0, h_1, \dots) \approx R^{-(d-2+\eta)} D(Rt^{\nu}, \dots, h_j/t^{\phi_j} \dots), \tag{2.30}$$

with $\nu = 1/\lambda_1$, which through (2.24) implies the *hyperscaling relation* [22]

$$d\nu = 2 - \alpha. \tag{2.31}$$

Finally, the *nonlinear scaling fields*

$$g_j(h_0, h_1, ...) = h_j[1 + O(h_0, h_1, ...)] \tag{2.32}$$

are introduced [3] as the exact (formal) solutions of the *full*, nonlinear recursion relations (2.20) which behave as

$$g_j^{(l)} = \Lambda_j^{(l)} g_j, \tag{2.33}$$

where there are now *no* correction terms. If $\mathscr{H}$ is parametrized in terms of these nonlinear scaling fields (assuming this to be possible) we obtain the *exact* nonlinear homogeneity relation

$$f(g_0, g_1, ..., g_j, ...) = b^{-ld} f(b^{l\lambda_0} g_0, b^{l\lambda_1} g_1, ..., b^{l\lambda_j} g_i, ...), \tag{2.34}$$

in contrast to the asymptotic homogeneity (2.22).

Wegner [3] has shown how to generate the nonlinear scaling fields as power series, given differential recursion relations corresponding to (2.17). In certain cases logarithms also appear. We will show how the same can be accomplished for a discrete group. As mentioned, we will also show explicitly that whereas distinct renormalization groups describe a particular critical point in terms of the *same* linear scaling fields, the corresponding nonlinear fields are in general different, and hence, nonunique.

3. Dedecoration Renormalization Groups

3.1. *Decoration Transformations*

Decoration or iteration transformations of Ising models have been treated generally by Fisher [18]. Using such a transformation, one can replace a central spin (or any other physical system) coupled to two neighboring spins, s_1 and s_2, by a single bond joining the two "external" spins. The transformation is effected by first taking a trace over the internal degrees of freedom of the physical system coupled to the two "external" spins. One is then left with a conditional partition function $\psi(s_1, s_2; T)$ depending on the two external spin variables. The assembly is then replaced (see Fig. 1) by a single Ising bond of strength $J = k_B TK$ coupling

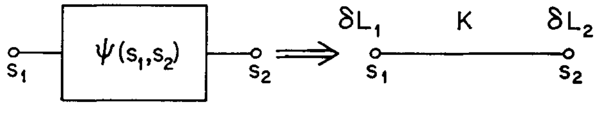

FIG. 1. The generalized decoration transformation.

s_1 and s_2, by augmented (reduced) magnetic fields $\delta H_1 = k_B T\,\delta L_1$ and $\delta H_2 = k_B T\,\delta L_2$ acting on s_1 and s_2, and by a spin-independent function $g(T)$ which contributes an additive term $-kT \ln g$ to the total free energy. The transformation equations, as derived by Fisher [18] are

$$e^{4K} = \psi_{++}\psi_{--}/\psi_{+-}\psi_{-+}\,, \tag{3.1}$$

$$e^{4\delta L_1} = \psi_{++}\psi_{+-}/\psi_{--}\psi_{-+}\,, \qquad e^{4\delta L_2} = \psi_{++}\psi_{-+}/\psi_{--}\psi_{+-}\,, \tag{3.2}$$

and

$$g^4 = \psi_{++}\psi_{--}\psi_{+-}\psi_{-+}\,, \tag{3.3}$$

where $\psi_{++} = \psi(+1, +1; T)$, and so on.

We apply this procedure to the nearest neighbor Ising chain of N spins in a magnetic field [15] which is described by the familiar Hamiltonian,

$$\mathscr{H} = -J \sum_{i=1}^{N} s_i s_{i+1} - H \sum_{i=1}^{N} s_i - NE_0(J, H). \tag{3.4}$$

The function $E_0(J, H)$ represents a "zero-spin" or "background" contribution to the energy which plays a role in the description of the linear Ising chain in terms of a renormalization group; initially, we may take E_0 to be a constant. It is most convenient to work with the reduced Hamiltonian $\bar{\mathscr{H}}$ defined by

$$\bar{\mathscr{H}} = -\mathscr{H}/k_B T = K \sum_{i=1} s_i s_{i+1} + L \sum_{i=1} s_i + NC(K, L), \tag{3.5}$$

where

$$K = J/k_B T, \qquad L = H/k_B T, \qquad C = E_0/k_B T. \tag{3.6}$$

By dedecorating *every other spin* along the chain [15] we now generate an elementary example of a renormalization group transformation with $b = 2$. The effect of this dedecoration transformation on the partition function $Z_N(C, K, L)$ is given by

$$Z_N(C, K, L) = Z_{N/2}(C', K', L'), \tag{3.7}$$

where K' represents the new (effective) coupling between the *remaining* spins while L' represents the corresponding transformed ordering field. If, temporarily, we ignore the spin-independent terms C and C', the dedecoration group simply maps the point (K, L) specifying the initial Hamiltonian onto a new point (K', L') describing the renormalized Hamiltonian.

In the process just sketched the physical subsystem removed by decoration was a single spin. Other renormalization groups can be constructed by applying the dedecoration transformation to more complicated objects. In the following analysis we will study the effects of removing two adjacent spins out of every three spins in a linear chain, and of removing alternate "rungs" in a "ladder" (or double chain) of spins.

3.2. *Recursion Relations for Dedecoration with $b = 2$*

Dedecoration of every other spin in a linear Ising chain generates a renormalization group with spatial rescaling factor $b = 2$. The conditional partition function for decoration is

$$\psi(s_1, s_2) = 2\cosh[K(s_1 + s_2) + L]. \tag{3.8}$$

It is convenient to work with the new set of variables,

$$w = e^{-4C}, \qquad x = e^{-4K}, \qquad \text{and} \qquad y = e^{-2L}. \tag{3.9}$$

In terms of these variables the dedecoration relations (3.1) to (3.3) yield the basic recursion relations

$$w' = w^2xy^2/(1 + y)^2\,(x + y)(1 + xy), \tag{3.10}$$

$$x' = x(1 + y)^2/(x + y)(1 + xy), \tag{3.11}$$

$$y' = y(x + y)/(1 + xy), \tag{3.12}$$

and (3.7) likewise applies in terms of w, x, and y.

Evidently the parameters x and y move in the (x, y) plane *independently* of w; the spin-independent term w is thus "driven" by the interaction terms. This is a general feature of the renormalization group. Clearly, fixed points may be obtained by study of the last two recursion relations alone. Restricting attention to the unit square ($0 \leqslant T, H \leqslant \infty$), for $J > 0$ we find a "paramagnetic" *line of fixed points* at $x^* = 1$, independent of y, corresponding to $T = \infty$ (or equivalently, $J = 0$). There is also an isolated "fully aligned" or "frozen," infinite-field fixed point at $(x^*, y^*) = (0, 0)$. Lastly there is "ferromagnetic" fixed point at $(0, 1)$ as shown in Fig. 2. Since we are interested mainly in the "critical" behavior of the system near $T = 0$, $H = 0$, where the correlations become long-ranged, we will study the "ferromagnetic" fixed point, $(0, 1)$. Under iteration of the transformation an initial Hamiltonian specified by (x_0, y_0) describes a discontinuous trajectory in the (x, y) plane; typical trajectories are sketched (as if continuous) in Fig. 2.

On linearizing about the ferromagnetic fixed point ($x^* = 0$, $y^* = 1$) with $\Delta y = y - y^* = y - 1$, the recursion relations become

$$x' \approx 4x, \qquad \Delta y' \approx 2\Delta y. \tag{3.13}$$

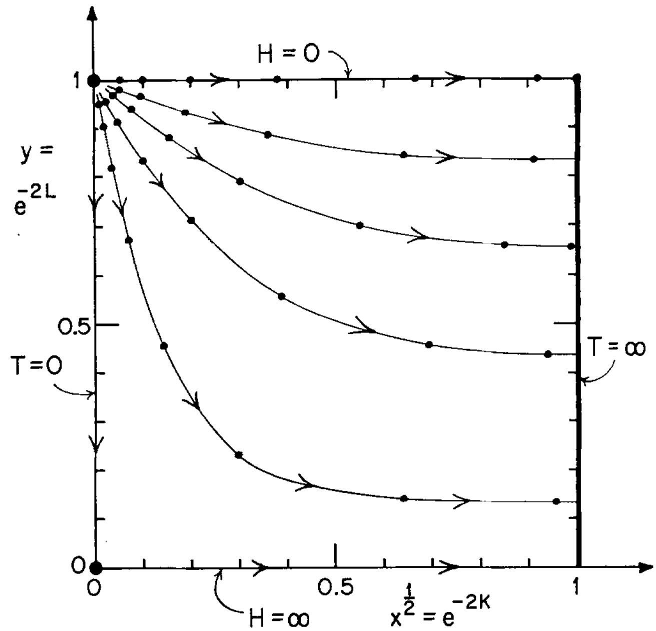

FIG. 2. Trajectories and fixed points for the dedecoration renormalization group applied to the linear Ising chain. Isolated fixed points occur at $(x^*, y^*) = (0, 0)$ and $(0, 1)$, while a line of fixed points appears at $x = 1$. The dots represent sucessive applications of the $b = 2$ transformation. Different initial conditions would give different sets of dots, all of which would fall on the continuous curves sketched here. The trajectories are the same for the $b = 3$ dedecoration group: see Section 3.5.

These relations are already diagonal so we find the eigenvalues

$$\Lambda_x = 4, \qquad \Lambda_y = 2, \tag{3.14}$$

$$\lambda_x = 2, \qquad \lambda_y = 1, \tag{3.15}$$

so that both critical fields x and Δy are relevant.

Formally from (3.9) the ferromagnetic fixed point value of w is $w^* = \infty$: this indicates that a change of variable is appropriate. If one multiplies (3.10) and (3.11) together one obtains

$$(wx)' = (wx)^2 y^2/(x + y)^2 (1 + xy)^2, \tag{3.16}$$

which near the fixed point reduces to $(wx)' \approx (wx)^2$. The appropriate fixed point is $(wx)^* = 1$ and the eigenvalue is $\Lambda_{wx} = 2$. It is convenient to interpret this result by imagining the temperature T is unaffected by the transformation. When one is near the fixed point, so that x_0 and Δy_0 are small, one may take the logarithm of (3.16) to find

$$(E + J)' \approx 2(E + J), \tag{3.17}$$

so that $\Lambda_{E+J} = 2$ or $\lambda_{E+J} = 1 = d$; this confirms the general observation about the eigenvalue of the constant term made in Section 2.2. Furthermore if one chooses $E_0 = -J$, the fixed point value, one has $E_l = -J_l$ throughout the linear region. This choice of E corresponds to setting the ground state energy of the system equal to zero.

It should be noted that the recursion relations (3.10) and (3.11) are invariant under the replacement $y \Rightarrow y^{-1}$ as expected by the symmetry of the initial Hamiltonian. (However, we will examine a renormalization group which does not respect this symmetry.) If, however, one starts with $x_0 > 1$ (corresponding to antiferromagnetic coupling $J < 0$) one finds that x_0 is immediately mapped into a value $x_1 < 1$. This is because the effective interaction between alternate spins in a antiferromagnetic chain is, indeed, ferromagnetic. The effective interaction remains ferromagnetic under further iterations. Thus no specifically antiferromagnetic fixed point appears. Furthermore, the uniform magnetic field H is expected to be a thermodynamically irrelevant variable near an antiferromagnetic critical point (i.e., to have a fixed point eigenvalue Λ, less than unity). After one iteration step, however, the uniform field is transformed close to the ferromagnetic fixed point ($x^* = 0$, $y^* = 1$), where it is apparently thermodynamically relevant. Thus an irrelevant variable is projected onto a relevant one. The resolution of this paradox in treating antiferromagnetic coupling is presented in Section 3.7.

3.3. *Scaling Relations for $b = 2$ Dedecoration Group*

The transformation of the reduced free energy

$$f(C, K, L) \equiv f(w, x, y) = \lim_{N \to \infty} N^{-1} \ln Z_N(C, K, L) \tag{3.18}$$

of the chain follows from (3.7) as

$$f(w, x, y) = \tfrac{1}{2} f(w', x', y'). \tag{3.19}$$

If we choose $E_0 = -J$ and then neglect the w- or E-dependence (as is valid in the linear region) we obtain on iteration l times

$$f(x, \Delta y) = 2^{-l} f(x_l, \Delta y_l). \tag{3.20}$$

But from the linearized Eqs. (3.13) one has

$$x_l = 4^l x \qquad \text{and} \qquad \Delta y_l = 2^l \Delta y. \tag{3.21}$$

We now choose l so that $2^{2l}x = k < 1$ for x close to the critical value $x_c = 0$, where k is small enough that the linearized recursion relations are valid. On substituting into (3.20) and using the approximation $\Delta y \approx 2L$ valid near the critical value $L_c = 0$, we obtain the scaling prediction

$$f(x, y) = f(T, H) \approx e^{-2K} Y(Le^{2K}), \tag{3.22}$$

where, as before, $L = H/k_B T$ and $K = J/k_B T$, while the scaling function is given by

$$Y(v) = k^{-1} f(k, 2kv). \tag{3.23}$$

In principle $x = e^{-4K}$ can take only the values $k/2^{-2l}$ for l integral so that $k_B T/J$ takes only values of the form $1/(c_0 l + c_1)$, but since these values become closely spaced near the critical point this is not a serious drawback. However, further investigation is needed to check that no "ripple" is, in fact, generated in the thermodynamic functions. The analysis of Section 6, however, shows how to construct an analytic continuation in b.

The low temperature properties of the Ising chain follow immediately from (3.22). Thus, the zero-field energy approaches its "critical" value $U_c = 0$ as e^{-2K}. By differentiating twice with respect to L the zero-field susceptibility, $\chi = (\partial M/\partial H)_T \sim \partial\langle s_0\rangle/\partial L$, is seen to diverge exponentially as e^{+2K} as $K^{-1} = k_B T/J \to 0$.

Now the exact free energy of an Ising chain with $E_0 = -J < 0$ is well known [25] to be

$$f(K, L) = -K + \ln[e^K \cosh L + (e^{2K} \sinh^2 L + e^{-2K})^{1/2}], \tag{3.24}$$

which near $L = 0$, $K^{-1} = 0$ reduces to

$$f(K, L) = e^{-2K}(1 + L^2 e^{4K})^{1/2} + O(L^2, e^{-4K}). \tag{3.25}$$

From this we may identify the scaling function (3.23) as

$$Y(v) = (1 + v^2)^{1/2}. \tag{3.26}$$

The spin-spin correlation function

$$G(R, T, H) = \langle s_0 s_R\rangle = Z_N^{-1} \sum_{\{s_i=\pm 1\}} s_0 s_R e^{\mathscr{H}\{s_i\}}, \tag{3.27}$$

may be analyzed similarly. (We measure R in units of a lattice spacing.) If one takes s_0 and s_R as spins which are not removed by decoration, one finds the exact recursion relation

$$G(R, x, \Delta y) = G(\tfrac{1}{2}R, x', \Delta y'), \tag{3.28}$$

as is intuitively obvious from the nature of the dedecoration transformation. Evidently the dedecoration renormalization group is "linear" in the sense explained in Section 2. Furthermore the spin rescaling factor, c, is simply constant and equal to unity. [See Eq. (2.9).] On iteration within the linear regime, where (3.13) apply, we obtain

$$G(R, x, \Delta y) \approx G(2^{-l}R, 2^{2l}x, 2^{l}\,\Delta y), \tag{3.29}$$

which leads to the scaling expression

$$G(R, x, \Delta y) \approx D(Re^{-2K}, Le^{2K}), \tag{3.30}$$

where, with $k \ll 1$,

$$D(u, v) = G(u/k, k, 2kv). \tag{3.31}$$

Identification of the scaling combination $u = Re^{-2K}$ as $R/\xi(T)$, shows that the zero-field correlation length $\xi(T)$, diverges as e^{2K} when $T \to 0$. The correctness of this result is easily checked from the exact expression [25] $G(R, T, 0) = (\tanh K)^R$.

The complete exact expression for the correlation function may be obtained via the transfer matrix approach [25] which yields

$$G(R, K, L) = [\sinh^2 L + (\lambda_+/\lambda_-)^R]/(1 + e^{4K}\sinh^2 L), \tag{3.32}$$

where

$$\lambda_\pm = e^K \cosh L \pm (e^{2K}\sinh^2 L + e^{-2K})^{1/2}. \tag{3.33}$$

The constant term in (3.32) merely represents $\langle s_0\rangle^2$, the square of the magnetization. On extracting the behavior for L, $K^{-1} \to 0$ the scaling form (3.30) is confirmed with

$$D(u, v) = (1 + v^2)^{-1}\{v^2 + \exp[-2u(1 + v^2)^{1/2}]\}. \tag{3.34}$$

3.4. *Critical Exponents in One Dimension*

The zero-field critical exponents α, β, γ, and ν are normally defined in terms of the reduced temperature variable $t = (T - T_c)/T_c$ via

$$\begin{aligned} \Delta f = f - f_c \sim t^{2-\alpha}, \qquad \langle s_0\rangle \sim t^{\beta}, \\ \chi \sim t^{-\gamma}, \qquad \xi \sim t^{-\nu}. \end{aligned} \tag{3.35}$$

In the present case, where $T_c = 0$, these definitions clearly fail. Alternatively if one replaces t by $\Delta T = T$ one would conclude that $2 - \alpha$, γ, and ν are all equal to $+\infty$. However, one can define reduced critical exponents in terms of the correlation length by writing

$$\Delta f \sim \xi^{-(2-\alpha)/\nu}, \qquad \langle s_0\rangle \sim \xi^{-\beta/\nu}, \qquad \chi \sim \xi^{\gamma/\nu}. \tag{3.36}$$

The results $\Delta f \sim e^{-2K}$, and $\chi \sim \xi \sim e^{2K}$ then imply

$$\gamma = \nu = 2 - \alpha. \tag{3.36}$$

The last equality here is consistent with the hyperscaling relation [22] $d\nu = 2 - \alpha$ for $d = 1$. The first equality combined with the scaling relation $\gamma = (2 - \eta)\nu$ implies

$$\eta = 1. \tag{3.37}$$

However, this can be checked directly through the usual definition

$$G_c(R) \sim 1/R^{d-2+\eta} \sim R^{1-\eta}, \tag{3.38}$$

where the last relation applies for $d = 1$. Since $G(R) = (\tanh K)^R$ in zero field [25], one has $G_c(R) \equiv 1$ which confirms (3.37). This value of η is also consistent with the spin rescaling factor $c^* = c = 1$ noted above, as follows from (2.29).

The hyperscaling relation $d(\delta - 1)/(\delta + 1) = 2 - \eta$ now leads to the prediction $\delta = \infty$. This result makes good sense if one examines the magnetization isotherms which follow from (3.24) or (3.22) and (3.26), namely,

$$M(T, H) = \langle s_0 \rangle = \sinh L/(e^{4K} + \sinh^2 L)^{1/2} \approx L/(x + L^2)^{1/2}. \tag{3.39}$$

For x or $T > 0$ this describes a normal, analytic paramagnetic magnetization curve saturating at $M(T, \pm\infty) = \pm 1$. For $x = T = 0$, however, one obtains $M(0, H) = \operatorname{sgn}\{H\}$; this discontinuous critical point variation is well described by $\delta = \infty$.

Finally we note that $M(T, H)$ always vanishes as $H \to 0$ for $T \neq 0$, even if $x < 0$; thus there is no spontaneous magnetization. However, following (3.36) one may define β through the scaling form [22]

$$M \approx \xi^{-\beta/\nu} W(L\xi^{\Delta/\nu}). \tag{3.40}$$

Comparison with (3.39) yields

$$\beta = 0 \qquad \text{and} \qquad \Delta = \nu. \tag{3.41}$$

These results are consistent with (3.36) and the usual exponent relations $\alpha + 2\beta + \gamma = 2$ and $\Delta = \beta + \gamma$.

3.5. *Dedecoration Group with $b = 3$*

We now discuss an alternative dedecoration renormalization group with $b = 3$ generated simply by removing the first two spins out of every triple of spins along the chain. Since the renormalized interactions between the remaining spins retains the sign of the original nearest neighbor coupling, this group provides a natural framework for discussing antiferromagnetic as well as ferromagnetic chains. Thus we include a staggered field

$$H^{\dagger} = k_B T G \tag{3.41}$$

and take the reduced Hamiltonian to be

$$\mathscr{H} = K \sum_{j=1}^{N} s_j s_{j+1} + L \sum_{j=1}^{N} s_j + G \sum_{j=1}^{N} (-)^j s_j + NC(K, L, G). \tag{3.42}$$

The object to be removed by dedecoration is now a block of two spins with conditional partition function

$$\psi(s_1, s_2) = 2e^K \cosh[K(s_1 + s_2) + 2L] + 2e^{-K} \cosh[K(s_1 - s_2) + 2G]. \tag{3.43}$$

Insertion of this into (3.1) to (3.3) yields the recursion relations

$$w' = w^3 x^3 y^3 z^3 / \psi_1 \psi_2 \psi_3 \psi_4, \tag{3.44}$$

$$x' = x\psi_1\psi_2/\psi_3\psi_4, \qquad y' = y\psi_4/\psi_3, \qquad z' = z\psi_1/\psi_2, \tag{3.45}$$

where

$$z = e^{-2G} = \exp(-2H^{\dagger}/k_B T) \tag{3.46}$$

and

$$\begin{aligned} \psi_1 &= y(1 + xz^2) + z(1 + y^2), & \psi_2 &= y(x + z^2) + z(1 + y^2), \\ \psi_3 &= xy(1 + z^2) + z(1 + xy^2), & \psi_4 &= xy(1 + z^2) + z(x + y^2). \end{aligned} \tag{3.47}$$

The three recursion relations (3.45) are independent of w and determine the fixed points. In the cube $0 \leqslant x, y, z \leqslant 1$, which describes ferromagnetic interactions, we find a plane of paramagnetic, infinite-temperature fixed points at $x^* = 1$, $0 \leqslant y^*, z^* \leqslant 1$, as shown in Fig. 3. There is also a line $x^* = 0$, $y^* = 0$, $0 \leqslant z^* \leqslant 1$ of frozen, infinite-field fixed points and, finally, an isolated ferromagnetic fixed point at $x^* = 0$, $y^* = z^* = 1$ (see Fig. 3). Linearization about this ferromagnetic fixed point with $\Delta y = y - y^*$ and $\Delta z = z - z^*$ yields

$$\Delta x' = 9\Delta x, \qquad \Delta y' = 3\Delta y, \qquad \Delta z' = \tfrac{1}{3}\Delta z. \tag{3.48}$$

Thus the eigenvalues are

$$\Lambda_x = 9, \qquad \Lambda_y = 3, \qquad \Lambda_z = \tfrac{1}{3}, \tag{3.49}$$

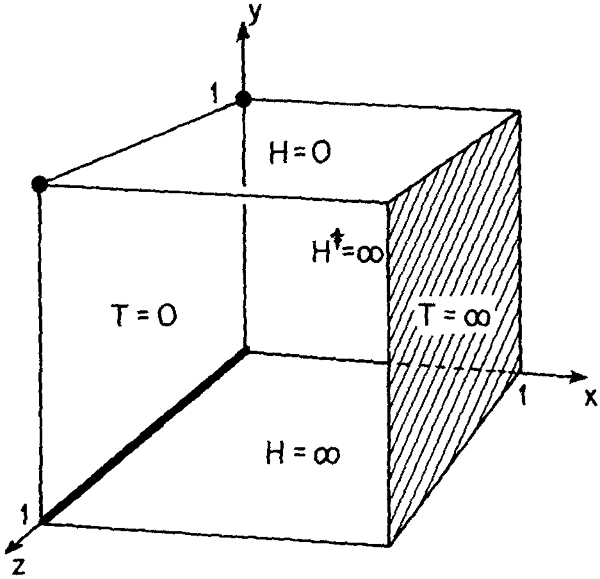

FIG. 3. Fixed points for the $b = 3$ dedecoration group applied to the *linear* Ising chain in both staggered and uniform magnetic fields. (The front face of the cube with $H^\dagger = 0$ corresponds to Fig. 2). Isolated fixed points occur at (0, 1, 1) and (0, 1, 0), a *line* of fixed points at ($x = 0$, $y = 0$), and *plane* of fixed points appears at $x = 1$.

or recalling that $b = 3$,

$$\lambda_x = 2, \qquad \lambda_y = 1, \qquad \lambda_z = -1. \tag{3.50}$$

Evidently, as anticipated on physical grounds, the staggered field variable z is irrelevant at the ferromagnetic fixed point. More strikingly the eigenvalue exponents λ_x and λ_y are precisely the *same* as in the original $b = 2$ dedecoration group (and correspond to the same critical variables or eigenoperators). This confirms the expected independence of the critical exponents on the particular renormalization group and partially checks the semigroup property of the dedecoration class of groups asymptotically in the linear region.

We may note in passing that a dedecoration type of renormalization group for a one-dimensional, continuous spin Gaussian model [26] yields a finite T_c, and the eigenvalues $\lambda_x = 2$, $\lambda_y = \frac{3}{2}$ and $\lambda_z = -\frac{1}{2}$ with exponents $\gamma = 2\nu = 2(2 - \alpha) = 1$ and $\eta = 0$.

It is easy to check that the recursion relations (3.44) to (3.47) are invariant under $y \Rightarrow y^{-1}$ and $z \Rightarrow z^{-1}$ as is to be expected. More importantly, if we let $x \Rightarrow x^{-1}$, corresponding to the change of ferromagnetic into antiferromagnetic coupling, the recursion relations are reproduced except that the roles of y and z

are interchanged. Thus the analysis of the antiferromagnetic chain exactly parallels that for the ferromagnetic chain. In particular there is an antiferromagnetic fixed point at $T = 0$, $H^\dagger = 0$, at which the parallel field H (or y) is an irrelevant variable with eigenvalue $\lambda_y = -1$, while the staggered field becomes relevant with exponent $\lambda_z = 1$.

3.6. *Scaling with an Irrelevant Variable*

The spin-independent term w can be handled as before. If the initial zero-spin energy is taken as $E_0 = -J$ then $(E + J)_l \approx 3^l(E + J)_0$ remains zero throughout the linear region. If attention is concentrated on the equation of state $M(T, H, H^\dagger)$ rather than the free energy one can, in fact, avoid all consideration of the spin-independent term. This is because the recursion relation for the magnetization is simply

$$M(x, \Delta y, \Delta z) = M(x', \Delta y' \Delta z') \tag{3.51}$$

independent of w. Of course this is just another reflection of the simple linearity of the dedecoration groups with $c \equiv 1$ already noted in Section 3.3 in connection with the correlation functions; it shows that the renormalization group trajectories form curves of constant magnetization. On using (3.48) for the linear region this yields the scaling expression

$$M(x, \Delta y, \Delta z) \approx B(Le^{2K}, Ge^{-2K}). \tag{3.52}$$

This result may be compared with the exact result

$$M = e^{2K} \sinh 2L(2 + 2 \cosh 2L \cosh 2G + e^{4K} \sinh^2 2L + e^{-4K} \sinh^2 2G)^{-1/2}. \tag{3.53}$$

Comparison with (3.52), near the fixed point $x = 0$, $L = G = 0$, shows that the scaling function is

$$B(v, v^\dagger) = v/(1 + v^2 + v^{\dagger 2})^{1/2}. \tag{3.54}$$

Since G is an irrelevant variable $v^\dagger = e^{-2K}G \to 0$ as $T \to 0$ and so asymptotic scaling is obtained by replacing $B(v, v^\dagger)$ by $B(v, 0)$. By expanding B in powers of $v^\dagger$ one is tempted to conclude [following the argument sketched after Eq. (2.26)] that the leading corrections to the asymptotic scaling form due to the irrelevant variable G should be represented by

$$M \approx \frac{L/x^{1/2}}{[1 + (L^2/x)]^{1/2}} \left\{1 - \frac{G^2 x}{2[1 + (L^2/x)]} + O(G^4 x^2)\right\}. \tag{3.55}$$

However, expansion of the exact result (3.53), neglecting corrections of order L^2 (arising from the sinh $2L$ and cosh $2L$ terms), yields a true correction factor

$$\left\{1 - \frac{G^2(1 + x)}{2[1 + (L^2/x)]} + O(G^4)\right\}, \tag{3.56}$$

which in order G^2 is larger by a factor of about $x^{-1} = e^{4K}$, which actually *diverges* as $T \to 0$. At first sight this severe discrepancy is quite puzzling. However, in deriving the scaling relation (3.52) from the exact recursion relation (3.51) we used only the *linearized* recursion relations (3.48). We conclude that the most significant corrections are "nonlinear" corrections rather then "irrelevancy" corrections. We will discuss such nonlinear corrections in Section 6. For the moment, we merely observe that replacement of the scaling field x by

$$\tilde{x} = x \cosh^2 G \approx x(1 + G^2) \tag{3.57}$$

enables one to use the *reduced* or *asymptotic* scaling form

$$M(T, H, H^\dagger) \approx B(L\tilde{x}^{1/2}, 0) = [1 + (\tilde{x}/L^2)]^{-1/2}, \tag{3.58}$$

and reproduce the leading part of the correction (3.56) correctly.

It is clear that for an antiferromagnet we merely have to interchange H and $H^\dagger$, or L and G, in the foregoing analysis.

3.7. *Pseudorelevant Variables: Antiferromagnetism with the $b = 2$ Dedecoration Group*

The renormalization group treatment of the antiferromagnetic chain led to the difficulties noted in Section 3.2. Specifically if one starts with

$$x_0^{-1} = \bar{x}_0 = e^{-4|K|} < 1 \tag{3.59}$$

as appropriate to antiferromagnetism, then one iteration of the recursion relation (3.11) yields

$$x_1 = \frac{\bar{x}_0(1 + y_0)^2}{(1 + \bar{x}_0 y_0)(\bar{x}_0 + y_0)} = 4\bar{x}_0[1 - 2\bar{x}_0 + \cdots], \tag{3.60}$$

which represents a *ferro*magnetic coupling which remains so after further iteration. On the other hand the initial parallel field variable $y_0 = 1 + \Delta y_0$, which should represent an *irrelevant* variable at the antiferromagnetic critical point, $\bar{x}_0 = 0$, is by the recursion relation (3.12), transformed again into a parallel field. However, a parallel magnetic field is a *relevant* variable at the ferromagnetic fixed point and we accordingly expect the scaling relation (3.22) to hold with K merely replaced by $|K|$. Two derivatives with respect to L would then indicate a susceptibility diverging strongly at $T \to 0$.

This conclusion must be false! To find the gap in the reasoning let us rewrite the recursion relation for y in terms of the critical point deviation $\Delta y = y - 1$; this yields

$$\Delta y' = \frac{2\Delta y(1 + \frac{1}{2}\Delta y)}{1 + x + x\,\Delta y} = 2\Delta y[1 - x + \tfrac{1}{2}\Delta y + \cdots], \tag{3.61a}$$

so that in the ferromagnetic case $\Delta y_1 \approx 2\Delta y_0$ near the critical point, as expected. However, using (3.59) for antiferromagnetic coupling yields

$$\Delta y_1 = \frac{2\bar{x}_0\,\Delta y_0(1 + \frac{1}{2}\Delta y_0)}{1 + \bar{x}_0 + \Delta y_0} = 2\bar{x}_0\,\Delta y_0[1 - \bar{x}_0 - \tfrac{1}{2}\Delta y_0 + \cdots]. \tag{3.61b}$$

It is now clear what happens: after one iteration the field $L_0 \approx \frac{1}{2}\Delta y_0$ is replaced by $L_1 \approx \frac{1}{2}\Delta y_1 \approx \bar{x}_0\,\Delta y_0 \approx 2e^{-4|K|}L_0$. We may then use the scaling relation (3.22) with K replaced by $|K|$ *and* L replaced by $\frac{1}{2}L_1$ which yields

$$f(K, L) \approx e^{-2|K|}Y(L_0 e^{2|K|-4|K|}) = e^{-2|K|}Y(L_0 e^{-2|K|}). \tag{3.62}$$

As $T \to 0$ the argument $v = L_0 e^{-2|K|}$ vanishes for any fixed L_0. Thus the variable L_0 is only a *pseudorelevant* variable. One step of the iteration has multiplied it by the factor $\bar{x}_0 = e^{-4|K|}$ which, "by accident," vanishes strongly at the critical point and so converts L_0 from a relevant into an effectively irrelevant operator.

This analysis indicates the potential importance of the first few iteration steps in obscuring the true nature of the critical behavior. It illustrates that for a straightforward interpretation one must always attempt to chose a renormalization group which focuses on the critical behavior of interest and represents it by a fixed point of corresponding character.

3.8. *Attracting Fixed Points and the Free Energy*

If all its critical variables are irrelevant a fixed point may be termed an *attracting fixed point* since all trajectories flow into it while none leave it. It is evident from Figs. 2 and 3 that the paramagnetic high-temperature fixed points form such an attracting set. Since it has no relevant variables, an attracting fixed point cannot describe genuine critical behavior. However, since all, or a large class, of trajectories flow into an attracting fixed point one can use the basic recursion relation [(2.7) or (3.19), etc.] to find an expression for the total free energy.

We will illustrate the procedure by analyzing the simplest, zero-field situation with $H = H^{\dagger} = 0$ or $y = z = 1$. The attracting fixed point is then $x^* = 1$ or $K = 0$. After iteration the basic relation reads

$$f(w_0, x_0) = 2^{-l} f(w_l, x_l). \tag{3.63}$$

As $l \to \infty$ the attracting fixed point is approached. The partition function near the fixed point is easily found to be

$$Z_N(C_l, K_l) \approx 2^N \exp[NC_l + \tfrac{1}{2}NK_l^2 + O(NK_l^4)], \tag{3.64}$$

so that, as $l \to \infty$,

$$\begin{aligned} f(w_l, x_l) &\approx \ln 2 + C_l + \tfrac{1}{2}K_l^2 + \cdots, \\ &\approx -\ln(\tfrac{1}{2}w_l^{1/4}) + O(\Delta x^2), \end{aligned} \tag{3.65}$$

where $\Delta x = x - 1 \approx -4K$. By combining with (3.63) we conclude that

$$f(w_0, x_0) = -\lim_{l\to\infty} 2^{-l} \ln(\tfrac{1}{2}w_l^{1/4}). \tag{3.66}$$

This demonstrates explicitly what is, in fact, a general result [1], namely, that the free energy can be expressed wholly in terms of the development of the spin-independent term w_l (or C_l).

To evaluate (3.66) we introduce the variables

$$u = \tfrac{1}{2}(wx)^{1/4} \qquad \text{and} \qquad v = \tanh K = \frac{1 - x^{1/2}}{1 + x^{1/2}}, \tag{3.67}$$

in place of w and x. The recursion relations (3.10) and (3.11) [with $y = 1$] then become

$$u' = u^2(1 + v)^2/(1 + v^2), \tag{3.68}$$

$$v' = v^2. \tag{3.69}$$

The simplicity of this second recursion relation provides, of course, the justification for this choice of variables. Since $x_l \to 1$ the result (3.66) becomes

$$f(w_0, w_0) = -\lim_{l\to\infty} 2^{-l} \ln u_l. \tag{3.70}$$

Iteration of (3.68) and (3.69) reveals that

$$\ln u_l = 2^l \ln u_0 + 2^l \ln(1 + v_0) - \ln(1 + v_0^{2^l}). \tag{3.71}$$

For $v_0 < 1$ (i.e., $T \neq 0$) substitution in (3.70) yields the final solution

$$f(w_0, x_0) = -\ln[u_0(1 + v_0)] = -\tfrac{1}{4}\ln w_0 + \ln(x_0^{1/4} + x_0^{-1/4})$$
$$= C_0 + \ln(2\cosh K), \tag{3.72}$$

which is the known exact answer [25].

It is clear that the same procedure will work more generally although the difficulty of solving the recursion relations analytically may be insurmountable [27]. In principle, however, a numerical solution could still be used to explore the critical region along these lines.

4. Further Linear Models

4.1. *Alternating Coupling Strengths*

By enlarging the space of Hamiltonians to allow for nearest neighbor coupling constants which alternate along the linear Ising chain one discovers examples of a fixed point with a *marginal* operator. (Examples of relevant and irrelevant operators were, of course, discussed in Section 3.) As explained in Section 2, a marginal operator is characterized by a renormalization group eigenvalue $\Lambda = 1$ or $\lambda = 0$.

We will utilize the $b = 3$ dedecoration group in which two out or every three spins are removed. The recursion relations may be derived quite straightforwardly as explained in Section 3.1. If K_a and K_b are the alternating coupling constants, the recursion relations in zero field are most simply expressed in terms of the variables

$$v_a = \tanh K_a \qquad \text{and} \qquad v_b = \tanh K_b \tag{4.1}$$

as introduced in (3.67). The recursion relations are then

$$v_a' = v_a^2 v_b, \qquad v_b' = v_b^2 v_a. \tag{4.2}$$

Linearization of the equations about the ferromagnetic fixed point $v_a^* = v_b^* = 1$ yields

$$\Delta v_a' = 2\Delta v_a + \Delta v_b + O(\Delta v^2),$$
$$\Delta v_b' = \Delta v_a + 2\Delta v_b + O(\Delta v^2). \tag{4.3}$$

By diagonalizing the linear part of these equations the critical fields are seen to be

$$h_1 = \Delta v_a + \Delta v_b \qquad \text{and} \qquad h_2 = \Delta v_a - \Delta v_b \tag{4.4}$$

with eigenvalues $\Lambda_1 = 3$ and $\Lambda_2 = 1$ or

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_2 = 0. \tag{4.5}$$

Thus $h_2 \sim K_a - K_b$ is a marginal field; the corresponding eigenoperator is essentially

$$Q_2 = \sum_j s_{2j}(s_{2j+1} - s_{2j-1}). \tag{4.6}$$

(The value $\lambda_1 = 1$ is consistent with the earlier result $\lambda_x = 2$ if we note that $\Delta v \simeq -2x^{1/2}$.)

The marginality of h_2 can be understood by noting that there is actually a *line* (or curve) of critical fixed points given by $v_a^* = 1/v_b^*$, which just touches the physical region $|v_a|, |v_b| \leqslant 1$ at $v_a^* = v_b^* = 1$. This is illustrated in Fig. 4

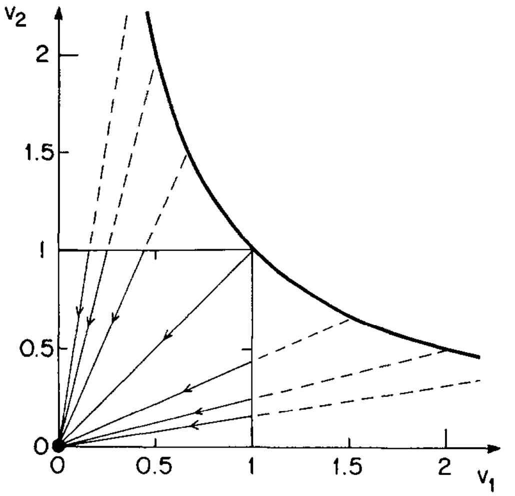

FIG. 4. Trajectories and fixed points for a linear Ising chain with alternating coupling strengths J_1 and J_2. A line of fixed points is determined by $v_1 v_2 = 1$ [with $v_i = \tanh(J_i/k_B T)$], and an isolated attracting fixed point occurs at (0, 0).

which also shows the trajectories. The fact that the other fixed points lie outside the physical region, means, correctly, that one cannot achieve criticality and long range correlations unless Δv_a and $\Delta v_b \to 0$, i.e., unless both K_a and $K_b \to \infty$; but one can still have $J_a \neq J_b$ as $k_B T \to 0$. Nevertheless the eigenvalues at the other fixed points are the same as about the physical point. This can be seen most easily by defining new variables $\bar{v}_\mu = v_\mu / v_\mu^*$ for $\mu = a$ or b. The recursion relations (4.2) are unchanged in terms of the $\bar{v}_\mu$ so that the subsequent analysis, including (4.5), remains valid with v_μ replaced by $\bar{v}_\mu$.

The invariance of the eigenvalues and, hence, of the critical exponents along the line of fixed points is in contradistinction to what is presumed to happen in the two-dimensional eight-vertex or Baxter model [28, 29] where the eigenvalues vary along the line of fixed points so that the critical exponents vary continuously with the marginal field [28].

4.2. *Second Neighbor Interactions*

The Hamiltonian of the linear chain with nearest neighbor interactions J_1 and second neighbor interactions J_2 in zero field is

$$\mathscr{H} = -J_1 \sum_{i=1}^{N} s_i s_{i+1} - J_2 \sum_{i=1}^{N} s_i s_{i+2} . \tag{4.7}$$

If we rewrite $s_i s_{i+2}$ as $(s_i s_{i+1})(s_{i+1} s_{i+2})$ and introduce new spin variables

$$\sigma_i = s_i s_{i+1} , \tag{4.8}$$

the Hamiltonian assumes the form

$$\mathscr{H} = -J_1 \sum_{i=1}^{N} \sigma_i - J_2 \sum_{i=1}^{N} \sigma_i \sigma_{i+1} . \tag{4.9}$$

Furthermore for an open chain the σ_i take the values ± 1 quite independently. Thus the problem of second neighbor interactions in zero field has been reduced to that of first neighbor interactions of strength $J = J_2$ in a field $H = J_1$.

We can thus take over the analysis of Section 3.2 for the $b = 2$ dedecoration group removing alternate σ spins. In particular the recursion relations (3.10) to (3.12) apply but with

$$x = e^{-4K_2} = \exp(-4J_2/k_B T) \qquad \text{and} \qquad y = e^{-2K_1} = \exp(-2J_1/k_B T). \tag{4.10}$$

The original ferromagnetic fixed point $x^* = 0$, $y^* = 1$, now corresponds to zero first-neighbor coupling (so that the chain decomposes into two disconnected second-neighbor chains with $J_2 > 0$). But this fixed point is unstable to perturbations with $\Delta y = y - y^* \neq 0$ (or $J_1 \neq 0$) and so the system crosses over to the fixed point at $x^* = 0, y^* = 0$ (originally termed the "frozen" fixed point). However one cannot directly linearize in terms of x and y about this fixed point owing to the factor $x/(x + y)$ in (3.11). This difficulty can be circumvented by eliminating x in favor of

$$z = xy = e^{-4K_2 - 2K_1}, \tag{4.11}$$

which leads to the new recursion relations

$$y' = (y^2 + z)/(1 + z), \tag{4.12}$$

$$z' = z(1 + y)^2/(1 + z)^2. \tag{4.13}$$

(Note that we use z here with a different meaning than in Section 3.5). These equations are easily linearized to yield

$$y = z + O(y^2, z^2), \tag{4.14}$$

$$z' = z + O(y^2, z^2), \tag{4.15}$$

from which critical fields are

$$h_1 = z \quad \text{and} \quad h_2 = y - z = e^{-2K_1}(1 - e^{-4K_2}), \tag{4.16}$$

with eigenvalues $\Lambda_1 = 1$ ($\lambda_1 = 0$) and $\Lambda_2 = 0$ ($\lambda_2 = -\infty$). Thus the first variable is marginal while the second might be termed totally *irrelevant*! Evidently h_2 vanishes faster than any exponential, which suggests that K_2 converges extremely rapidly to zero. To check this we form the recursion relation for the variable

$$u = h_2/z = (y/z) - 1 = e^{4K_2} - 1 \tag{4.17}$$

and find

$$u' = u^2 z/(1 + z + uz)^2, \tag{4.18}$$

from which one can show that u' is always less than u. Furthermore once $u^2 z < 1$ or $K_2 < \frac{1}{2}K_1$, the value of u goes very rapidly to zero, which implies that K_2 also vanishes rapidly (even though K_1 may be large). The fixed point is thus described by only nearest neighbor interactions.

Since the remaining variable $h_1 = z$ is only marginal rather than relevant, as might have been expected, we learn nothing immediately from the general linear analysis of the renormalization group given in Section 2. As a matter of fact z is *weakly relevant* (for $J_2 > 0$) since its recursion relation can be written

$$z'/z = \left(1 + \frac{h_2}{1 + z}\right)^2 = \left(1 + \frac{uz}{1 + z}\right)^2 > 1, \tag{4.19}$$

for $z \neq 0$; however, this variation does not contribute significantly since $h_2' \approx h_2^2$ so that $h_2^{(l)}$ goes rapidly to zero.

Nonetheless we can obtain information if we recall the recursion relation (3.10) for the constant term $w = e^{-4C}$. This becomes

$$w' = \frac{w^2 y^2 z}{(1 + y)^2 (z + y^2)(1 + z)} \approx \frac{w^2 z^2}{(1 + z)^4}, \tag{4.20}$$

where the second part follows since $h_2 = y - z$ is so strongly irrelevant. Iteration, using the fact that z is marginal, so that $z^{(l)} \approx z$, then yields

$$w^{(l)} \approx [wz^2/(1+z)^4]^{2^l} z^{-2}(1+z)^4, \tag{4.21}$$

while the free energy renormalization relation gives

$$f(w, y, z) \approx 2^{-l} f(w^{(l)}, z, z). \tag{4.22}$$

Now we choose l so that $w^{(l)} = k$ and so obtain

$$\begin{aligned} f(w, y, z) &\approx [\ln wz^2 - 4\ln(1+z)] f(k, z, z)/\ln[kz^2/(1+z)^4] \\ &\approx -4[C + 2K_2 + K_1 + z + O(z^2)]\, \Phi(z), \end{aligned} \tag{4.23}$$

where we have used (3.9) for w, and (4.11) for z, while $\Phi(z)$ stands for the residual function of z in the first line. However, we know on general grounds that the constant term C can enter into f *only* linearly and with coefficient unity! It follows that $-4\Phi(z) = 1$ (at least for $z \ll 1$) and so we finally conclude

$$f(w, y, z) \approx C + 2K_2 + K_1 + z + O(yz, z^2), \tag{4.24}$$

where the correction of order yz is anticipated directly from (4.19). Recalling the identifications $K = K_2$ and $L = K_1$, obtained by going to the σ variables, this expression agrees precisely with that following from the exact result (3.24) in the limit $T \to 0$ with $J = J_2$ and $H = J_1$ fixed. We conclude that even though the standard linearized renormalization group analysis of Section 2 fails, the general formalism can still be used to derive the asymptotic free energy from behavior near an appropriate fixed point.

4.3. *Braced Ladder*

To illustrate a more complex but still exactly realizable renormalization group involving many-spin interactions we consider the "braced ladder" of spins illustrated in Fig. 5. This consists of two parallel one-dimensional chains of spins with interactions along the nearest neighbor bonds (sides and rungs) and second nearest neighbor bonds (braces) plus a four-spin interaction of strength $J_4 = k_B T K_4$ around each braced square (as indicated in the figure by the dotted loops). With the notation of the figure, and supposing all bonds to be ferromagnetic ($J_i > 0$), we may define

$$x_i = e^{-4K_i} = \exp(-4J_i/k_B T). \tag{4.25}$$

If we now remove alternate pairs of spins (which form a ring) as indicated in the figure, we generate a $b = 2$ dedecoration group as before. Even if J_4 was

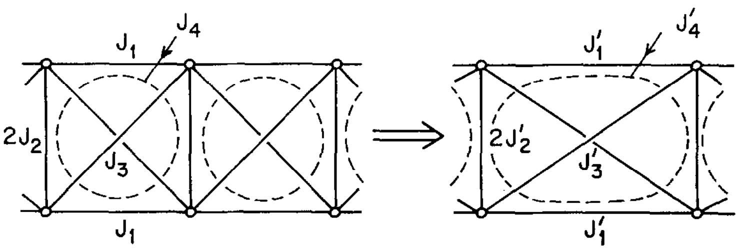

FIG. 5. Dedecoration transformation for a braced ladder of spins; J_4 denotes a four-spin interaction.

initially zero we would be forced to introduce a four-spin coupling at this stage. This effect of "induced interactions" is, of course, a very general feature of renormalization groups which, indeed, normally prevents their construction in explicit closed form. If we define

$$\psi_1 = \tfrac{1}{2}(1 + x_1^2x_3^2) + x_1x_2x_3x_4 , \tag{4.26}$$

$$\psi_2 = \tfrac{1}{2}(x_1^2 + x_3^2)\, x_2^3 + x_1x_2^2x_3x_4 , \tag{4.27}$$

$$\psi_3 = x_1x_2^2x_3(x_2 + x_4), \tag{4.28}$$

$$\psi_4 = x_1x_3(1 + x_2x_4), \tag{4.29}$$

$$\psi_5 = \tfrac{1}{4}x_1x_2^2x_3x_4[1 + x_1x_2 + x_1x_3 + x_2x_3]^2, \tag{4.30}$$

the required recursion relations become

$$x_1'^2 = \psi_3\psi_4/\psi_1\psi_2 , \qquad x_2'^2 = \psi_2\psi_3/x_2^2\psi_1\psi_4 , \tag{4.31}$$

$$x_3'^2 = \psi_2\psi_4/\psi_1\psi_3 , \qquad x_4'^2 = \psi_5^2/\psi_1\psi_2\psi_3\psi_4 . \tag{4.32}$$

These formulae define trajectories for the Hamiltonian in the four-dimensional space (x_1 , x_2 , x_3 , x_4), which eventually terminate in a line of attracting paramagnetic fixed points given by $x_1 = x_3 = x_4 = 1$, $0 \leqslant x_2 \leqslant 1$.

For special parameter values, specifically $x_2 = x_3 = x_4 = 1$ and $x_1 = x_2 = x_4 = 1$, the equations degenerate into versions of the simple linear chain dedecoration group discussed in Section 3. However, at the interesting ferromagnetic fixed point $x_1 = x_2 = x_3 = x_4 = 0$ the equations are nonanalytic as in the linear second neighbor case discussed above.

Investigation indicates that after a few iterations x_1 becomes closely equal to x_3. In fact the condition $x_1 = x_3$ or $J_1 = J_3$ is preserved by the recursion relations, as must be so on the grounds of symmetry. We conclude that $\Delta J_{13} = J_1 - J_3$ is an irrelevant variable and confine ourselves to the case $x_1 = x_3$. However, the recursion relations are still nonanalytic about the ferromagnetic fixed point.

Accordingly we work with x_1 and the variables

$$y = x_2/x_1 = e^{-2(K_2-K_1)}, \tag{4.33}$$

$$z = x_2 x_4 = e^{-2(K_2+K_4)}, \tag{4.34}$$

and obtain the new recursion relations

$$x_1' = \sqrt{2}\, x_1(1 + z)^{1/2}/(1 + 2x_1^2 z + x_1^4)^{1/2}, \tag{4.35}$$

$$y' = (x_1 y + z)/(1 + z), \tag{4.36}$$

$$z' = \tfrac{1}{2} z(1 + x_1^2 + 2x_1^2 y)/(1 + z)(1 + 2x_1^2 z + x_1^4). \tag{4.37}$$

From these it is evident that x_1 is a relevant variable with $\lambda_1 = \frac{1}{2}$, while w is an irrelevant variable with $\lambda_z = -1$. Linearization of (4.36) with (4.37) shows that $w = y - 2z$ is a totally irrelevant variable with $\lambda_w = -\infty$. What happens therefore is that the difference between y and $2z$ goes rapidly to zero so that J_4 is forced to equal $\frac{1}{2}(\ln 2)\, k_B T - J_2$ which, for small T involves a change of sign of the four-spin interaction.

The significance of the eigenvalue $\lambda_1 = \frac{1}{2}$ can be understood in terms of the linear Ising chain if we study the correlation length,

$$\xi \sim t^{-\nu}, \tag{4.38}$$

where $\nu = 1/\lambda_1$ and t represents the temperature-like variable e^{-4K_1}. For the simple chain, $\lambda_1 = 2$ and we predict $\xi \sim t^{-\nu} \approx e^{2K_1}$. The recursion relations (4.35)–(4.37) indicate that K_2 quickly diverges whereupon the sides of the ladder become "locked" together. The four-spin term K_4 now only contributes a constant to the free energy, and the ladder reduces to an Ising chain with an *effective* nearest neighbor coupling $4J_1$ (assuming $J_1 = J_3$). This is precisely the information conveyed if we apply (4.38) with $\lambda_1 = \frac{1}{2}$:

$$\xi \sim t^{-\nu} \approx e^{+8K_1}. \tag{4.39}$$

As in Section 4.2 we could go on to analyze the free energy but this does not seem worthwhile.

5. Truncated Tetrahedron Model

The truncated tetrahedon model is a planar Ising model of coordination number three defined through the decoration and star-triangle transformations [17, 18]. It is simply related to another planar model that is four-coordinated. To describe the truncated tetrahedron "lattice" we start in zero order with a tetrahedron of

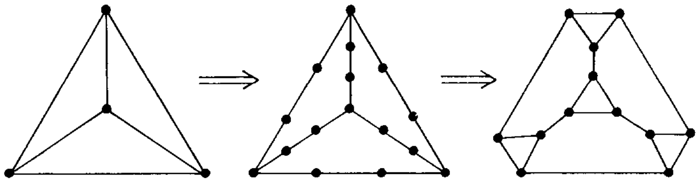

FIG. 6. Conversion of a tetrahedron of Ising spins into a truncated tetrahedron lattice of order $n = 1$ by successive application of decoration and star-triangle transformations.

four Ising spins. This system is then decorated by putting *two* spins on every bond as indicated in Fig. 6. To complete the transition to the first order lattice, a star-triangle transformation is made in order to remove each of the original four vertex spins. We are finally left with the first order "truncated tetrahedron" shown in the last part of Fig. 6.

The next order lattice in the hierarchy is obtained in a precisely similar fashion. To generate a lattice of order $n + 1$, we decorate every bond in a lattice of order n with two spins and then make a star-triangle transformation to remove all spins at three-coordinated vertices. If we define the order of the initial tetrahedron of spins to be $n = 0$, then a lattice of order n is three-coordinated, and contains

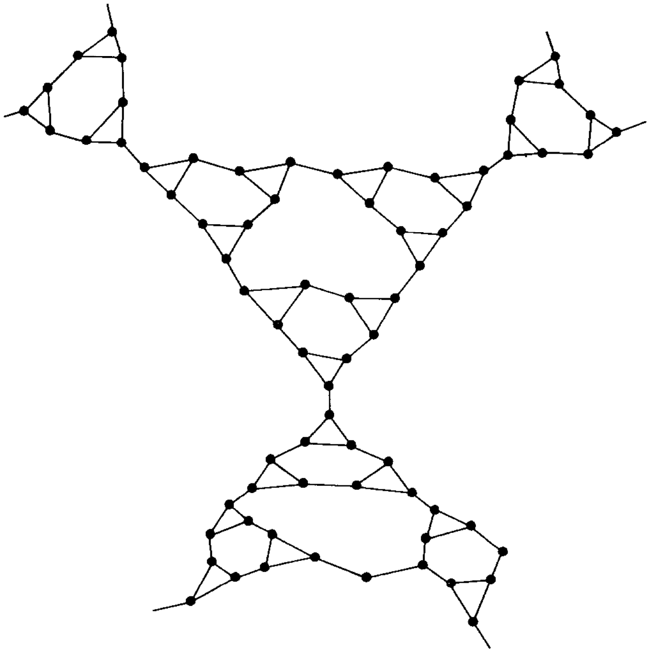

FIG. 7. Section of an infinite truncated tetrahedron lattice.

$N = 4 \times 3^n$ spins and 6×3^n bonds. We pass to the thermodynamic limit by letting $n \to \infty$. A section of an infinite truncated tetrahedron lattice is shown in Fig. 7. A lattice of order n will contain polygons of sizes 3, 6, 12,..., 3×2^n, but will always have a connection number [19] of only 3. Thus the circumference [19] of a lattice of order $n + 1$ will be twice that of a lattice of order n; but a lattice of any order can still be cut into two pieces of arbitrary size by cutting three bonds. For this reason we may guess that the transition temperature will still be at $T_c = 0$. However, the critical behavior should be quite distinct from the linear chains.

The lattice described above is related to a similar lattice of coordination number 4. If we take a truncated tetrahedron lattice of order n and decorate every bond with a *single* spin, and then make a star-triangle transformation at every three-coordinated vertex, we obtain a four-coordinated lattice. The sequence of transformations used to arrive at this four coordinated "fully" truncated lattice from the truncated tetrahedron lattice is illustrated for a lattice of order 1 in Fig. 8.

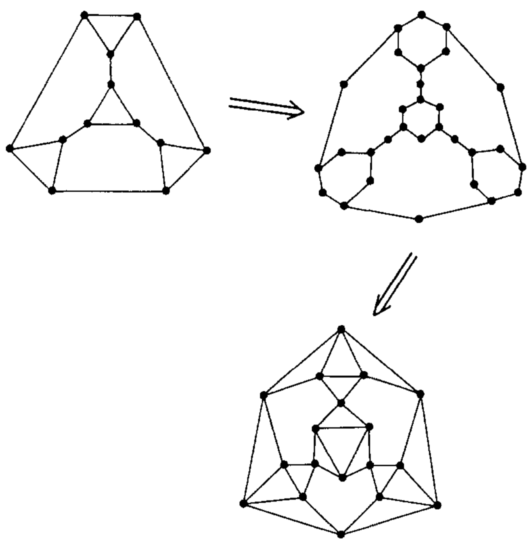

FIG. 8. Conversion of a truncated tetrahedron lattice (of order $n = 1$) into a "fully truncated" lattice of coordination number four by applying a decoration and star-triangle transformation. Note that only a single spin is inserted by the decoration while in Fig. 6 two spins are inserted.

Now the decoration and star-triangle transformations yield an explicit relation between a lattice of order n and a lattice of order $n + 1$. However, because of the properties of the star-triangle transformation, this relation can be obtained only for zero magnetic field. Let Z_n be the partition function of a truncated tetrahedron lattice of order n with zero constant term E_0 (there being $N = 4 \times 3^n$ spins in this lattice). We assume there is a uniform spin–spin coupling of strength J_n and introduce the variable,

$$v_n = \tanh(J_n/k_B T), \tag{5.1}$$

as before. Then, we find

$$Z_n(v_n) = [g(v_n)]^{3^{n-1}} Z_{n-1}(v_{n-1}) \tag{5.2}$$

with

$$g(v) = 256(1 - v + 2v^2)/(1 + v)^3 (1 - v)^6 (1 + v^3)^2 \tag{5.3}$$

and, with v replacing v_n and v' replacing v_{n-1},

$$v' = v^2/(1 - v + v^2). \tag{5.4}$$

If we take the logarithm of (5.2), divide by $N = 4 \times 3^n$, and take the thermodynamic limit $n \to \infty$, we obtain for the free energy the relation

$$f(v) = 3^{-1} f(v') + \tfrac{1}{12} \ln g(v). \tag{5.5}$$

Evidently we have constructed a renormalization group. The second term, depending on $g(v)$, arises because we defined Z_n, and hence f, with zero constant term, E_0. If we regard the dimensionality of the truncated tetrahedron as $d = 1$, on the grounds that it can be cut into indefinitely large pieces by only three cuts, then we should take $b = 3$. On the other hand, if we measure distances along the bonds, as is quite natural, we find that each step of the iteration corresponds to a length rescaling of only $b = 2$. This can be seen from Fig. 7 where the number of points along the "side" of a basic triangular figure (on "tetrahedral face") goes up as 2^n while the number of points increases as 3^n. By this argument the dimensionality of the lattice is $d = \log_2 3 \simeq 1.5850$; however, although the dimensionality is, by this measure, larger than unity the critical point will still be at $T_c = 0$. To see this, we examine the recursion relation (5.4). This relation is compared in Fig. 9 with the analogous relations for the linear Ising chain dedecoration groups $b = 2$ and $b = 3$ which are $v' = v^2$ [by (3.69)] and $v' = v^3$ [say, from (3.45) with $y = z = 1$]. All formulae have a infinite temperature attracting fixed point $v = 0$, and an unstable low temperature, ferromagnetic fixed point $v = 1$ corresponding to $T_c = 0$. However, since the slope of the graph for the truncated tetrahedron approaches unity as $v \to 1$ we are dealing, as in Section 4.1, with a marginal operator, rather than a relevant one.

If we set

$$v = 1 - \bar{v}, \qquad \bar{v} \approx 2e^{-2K} - 2e^{-4K} + \cdots, \tag{5.6}$$

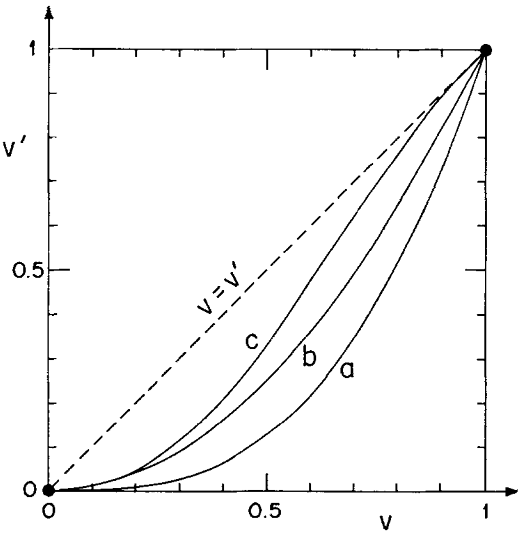

FIG. 9. Recursion relations for two different models expressed in terms of $v = \tanh(J/k_BT)$. Curves (a) and (b) represent the $b = 3$ and $b = 2$ dedecoration groups for the Ising chain, while (c) is the corresponding curve for the truncated tetrahedron model. Both models have fixed points at $v = 0$ and $v = 1$.

we obtain the recursion relation

$$\bar{v}' - \bar{v} = \bar{v}^2[1 - \bar{v}^2(1 - \bar{v} + \bar{v}^2)^{-1}]. \tag{5.7}$$

When $\bar{v}$ is small, i.e., near its critical value, we may approximate this relation by the differential equation

$$d\bar{v}/dl \approx \bar{v}^2, \tag{5.8}$$

(see the Appendix for a justification of this step), where $\bar{v}(l)$ is the result of l iterations. This equation has the solution

$$\bar{v}(l) = \bar{v}_0/(1 - l\bar{v}_0). \tag{5.9}$$

Now, since $b = 2$, if we measure distances along the bonds, the correlation length transforms as

$$\xi[\bar{v}(l)] = 2^{-l}\xi(\bar{v}_0). \tag{5.10}$$

On writing $\bar{v} = \bar{v}(l)$ and eliminating l with (5.9) one finds

$$\xi(\bar{v}) \approx 2^{\bar{v}^{-1} - \bar{v}_0^{-1}}\xi(\bar{v}_0), \tag{5.11}$$

so that the correlation length diverges as the exponential of an exponential when $T \to T_c$, namely as

$$\xi(T) \sim \exp[\tfrac{1}{2}(\ln 2) \exp(2J/k_B T)]. \tag{5.12}$$

This indicates the strong degree of cooperativity in the model.

By casting the recursion relation (5.4) into an integral equation, we can formally extract the l dependence. Repeated iteration of (5.4) gives

$$\bar{v}_l = \bar{v}_0 + \sum_{l'=0}^{l-1} (\bar{v}_{l'}^2 - \bar{v}_{l'}^3)/(1 - \bar{v}_{l'} + \bar{v}_{l'}^2). \tag{5.13}$$

Solving this discrete analogue of an integral equation for $\bar{v}_l$ iteratively, we obtain

$$\bar{v}_l = \bar{v}_0 + l\bar{v}_0^2 + l(l-1)\,\bar{v}_0^3 + O(\bar{v}_0^4). \tag{5.14}$$

The thermodynamic behavior near $T_c = 0$ can be found directly from the basic relation (5.5). Expanding $\ln g(\bar{v})$, we obtain

$$\ln g(\bar{v}) = 4 \ln 2 - 6 \ln \bar{v} + 3\bar{v} + \bar{v}^2 + O(\bar{v}^3), \tag{5.15}$$

which leads us to form the expansion

$$f(\bar{v}) = A \ln \bar{v} + B + C\bar{v} + D\bar{v} + O(\bar{v}^3). \tag{5.16}$$

Substituting this expression into (5.5), we obtain

$$f(\bar{v}) = -\tfrac{3}{4} \ln \bar{v} + \tfrac{1}{2} \ln 2 + \tfrac{9}{32}\bar{v}^2 + O(\bar{v}^3), \tag{5.17}$$

from which we conclude that the energy varies as

$$U(T) \approx -\tfrac{3}{2}J + \tfrac{3}{2}Je^{-2K} + 3Je^{-4K} \tag{5.18}$$

as $T \to 0$.

Although this behavior does not look particularly anomalous one should recall that the lowest order temperature dependence of a two or three-dimensional lattice of coordination number q would be $z = \exp(-2qJ/k_B T)$, as follows easily by overturning spins from the fully ordered state [25]. This would yield a leading variation of $z = e^{-6K}$, whereas (5.18) yields $\Delta U \sim z^{1/3}$; this makes the anomalous behavior evident (a similar analysis of the linear chain yields $\Delta U \sim z^{1/2}$, again indicating "critical" behavior).

6. Corrections to Scaling and Nonlinear Fields

6.1. *Introduction*

The problem of corrections to asymptotic scaling has been treated by Wegner [3] in terms of nonlinear scaling fields. We will apply his ideas to some of our simple, exactly soluble models. It will be convenient to work with the variable $x = e^{-4K}$ as before and the complementary variables

$$\bar{y} = 1 - y = 1 - e^{-2L}, \qquad \bar{z} = 1 - z = 1 - e^{-2G}. \tag{6.1}$$

[See (3.9) and (3.45).]

Consider first the case of a linear Ising chain in a uniform magnetic field ($\bar{z} = 0$). In terms of the variables x and $\bar{y}$, the recursion relations (3.11) and (3.12) for the $b = 2$ dedecoration group have the Taylor series expansions:

$$\begin{aligned} x' &= 4x - 8x^2 + 12x^3 + x\bar{y}^2 + \cdots, \\ \bar{y}' &= 2\bar{y} - \bar{y}^2 - 2\bar{y}x + 3x\bar{y}^2 + 2\bar{y}x^2 + \cdots. \end{aligned} \tag{6.2}$$

For the case $b = 3$ [see (3.45)], the expansions are

$$\begin{aligned} x' &= 9x - 48x^2 + 208x^3 + 6\bar{y}^2x + \cdots, \\ \bar{y}' &= 3\bar{y} - 3\bar{y}^2 - 8\bar{y}x + 20x\bar{y}^2 + 24x^2\bar{y} + \bar{y}^3 + \cdots. \end{aligned} \tag{6.3}$$

The coefficients of these Taylor series are obviously b-dependent, and the leading coefficients have the form $b^{\lambda_x}, b^{\lambda_y}$, with $\lambda_x = 2$, and $\lambda_y = 1$ (as before). This form is expected from the semigroup property of the renormalization group. We will present a procedure for determining the b-dependence of the higher order terms in the recursion relations, given the recursion relations generally for a particular b.

The scaling prediction of the renormalization group for the one-dimensional Ising model is that, in the linearized region about the fixed point, the magnetization is a function only of the ratio $\bar{y}^2/x$. (We consider the magnetization rather than the free energy to avoid worrying about the constant term for the moment.) Following Wegner, we try to find nonlinear scaling fields $g_x(x, \bar{y})$ and $g_y(x, \bar{y})$ which correspond to "exact eigenfunctions" of the renormalization group operator $\mathbb{R}_b$. These nonlinear fields are (i) to reduce to x and $\bar{y}$ near the fixed point and (ii) to transform simply under the renormalization group according to

$$\mathbb{R}_b \begin{pmatrix} g_x \\ g_y \end{pmatrix} = \begin{pmatrix} g_x' \\ g_y' \end{pmatrix} = \begin{pmatrix} b^{\lambda_x} g_x \\ b^{\lambda_y} g_y \end{pmatrix}. \tag{6.4}$$

The result of iterating these recursion relations is obvious. The scaling prediction is then that the magnetization is a function only of the ratio g_y^2/g_x. This prediction should hold *exactly* for all values of g_x and g_y, not just in the linearized region about the fixed point $(g_x^*, g_y^*) = (0, 0)$.

Wegner [3, 30] has presented a procedure for finding the expansions of g_x and g_y in powers of x and $\bar{y}$ given the recursion relations in *differential* form. We review this briefly. Suppose we know the derivatives with respect to b evaluated at $b = 1$, of the recursion relations for a set of fields $\{h_i\}$, namely,

$$\left(\frac{\partial h_i}{\partial b}\right)_{b=1} = \lambda_i h_i + \frac{1}{2} \sum_{jk} \left(\frac{\partial a_{ijk}}{\partial b}\right)_{b=1} h_j h_k + \cdots . \tag{6.5}$$

The $a_{ijk}(b)$ are b-dependent coefficients appearing in the original recursion relations for $b > 1$, and the λ_i are the eigenvalues of the renormalization group. We assume we have chosen the h_i so that the recursion relations are diagonal to first order, as indicated in (6.5), i.e., the h_i are linear scaling fields as in Section 2. The problem is to find nonlinear scaling fields g_i which behave in the simple manner indicated in (6.4) under the action of the renormalization group. Wegner [3, 30] assumes that the h_i can be expanded in a power series in the g_i, as

$$h_i = g_i + \tfrac{1}{2} \sum_{jk} b_{ijk} g_j g_k + \cdots, \tag{6.6}$$

and then finds that the desired coefficients b_{ijk} are given by the set of equations

$$(\lambda_j + \lambda_k - \lambda_l)\, b_{ijk} = (\partial a_{ijk}/\partial b)_{b=1} . \tag{6.7}$$

More complicated expressions are found [3] for the higher order coefficients in (6.6). For the special case where $\lambda_j + \lambda_k - \lambda_l = 0$, Wegner finds that *logarithmic* corrections must be introduced [3, 30].

All the renormalization groups we have discussed involve only discrete values of b. Thus, it is impossible to determine required partial derivatives like $(\partial a_{ijk}/\partial b)_{b=1}$. Accordingly we will develop a method for determining the nonlinear scaling fields g_i when only a discrete renormalization group is given. The method follows Wegner's general approach.

6.2. *Nonlinear Scaling Fields for Discrete Renormalization Groups*

We assume, as in (6.5), that the recursion relations for a set of fields have already been diagonalized to first order, so that we may take

$$h_j' = b^{\lambda_j} h_j + \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{I(n)} a_{jI(n)} h_{I(n)} . \tag{6.8}$$

Here, $I(n)$ is a multiindex which represents $i_1 i_2 \cdots i_n$, and

$$h_{I(n)} \equiv h_{i_1} h_{i_2} \cdots h_{i_n}. \tag{6.9}$$

The fixed point is given by $h_i = 0$ (all i). The coefficients $a_{jI(n)}$ are in general functions of b, but we suppose that they are given for a particular $b > 1$. The nonlinear scaling fields g_j are, as before, to satisfy

$$g_j' = b^{\lambda_j} g_j. \tag{6.10}$$

Since the g_i should reduce to the h_i in first order, we try to express them as

$$g_j = h_j + \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{I(n)} c_{jI(n)} h_{I(n)}. \tag{6.11}$$

The $c_{jI(n)}$ are to be determined, and are expected to be independent of the scale factor b.

We will calculate the g_j by obtaining a sequence $g_j^{(2)}(h_i)$, $g_j^{(3)}(h_i)$,... such that $g_j^{(n)}(h_i)$ satisfies (6.10) to nth order. The function $g_j^{\,n}(h_i)$ is an nth order polynomial which agrees with $g_j^{n-1}(h_i)$ to $O(h_i^{n-1})$. To calculate $g^{(2)}(h_i)$ we form the product

$$h'_{I(2)} = b^{\lambda_{I(2)}} h_{I(2)} + O(h_{I(3)}), \tag{6.12}$$

where we use the notation

$$\lambda_{I(n)} = \lambda_{i_1} + \cdots + \lambda_{i_n}. \tag{6.13}$$

Forming the linear combination

$$\tfrac{1}{2} \sum_{I(2)} c_{jI(2)} h'_{I(2)} = \tfrac{1}{2} \sum_{I(2)} b^{\lambda_{I(2)}} c_{jI(2)} h_{I(2)} + O(h_{I(3)}), \tag{6.14}$$

and adding this to (6.8) we obtain

$$\begin{aligned} h_j' &+ \tfrac{1}{2} \sum_{I(2)} c_{jI(2)} h'_{I(2)} \\ &= b^{\lambda_j} \Big[h_j + \tfrac{1}{2} \sum_{I(2)} \big(b^{-\lambda_j} a_{jI(2)} + b^{\lambda_{I(2)} - \lambda_j} c_{jI(2)}\big)\, h_{I(2)} \Big] + \sum_{n=3}^{\infty} \frac{1}{n!} \sum_{I(2)} a_{jI(n)}^{(3)} h_{I(n)}. \end{aligned} \tag{6.15}$$

The quantity

$$g_j^{(2)}(h_i) = h_j + \tfrac{1}{2} \sum_{I(2)} c_{jI(2)} h_{I(2)} \tag{6.16}$$

will satisfy (6.10) to second order provided

$$c_{jI(2)} = a_{jI(2)}/(b^{\lambda_j} - b^{\lambda_{I(2)}}). \tag{6.17}$$

The $a_{jI(2)}^{(3)}$ result from adding the higher order terms in (6.14) to the higher order terms in (6.8).

Suppose in general that we have recursion relations of the form

$$g_j'^{(m)}(h_i) = b^{\lambda_j} g_j^{(m)}(h_i) + \sum_{n=m+1}^{\infty} \frac{1}{n!} \sum_{I(n)} a_{j(In)}^{(m+1)} h_{I(n)} . \tag{6.18}$$

Repeating the procedure indicated above we can derive the fields

$$g_j^{(m+1)}(h_i) = g_j^{(m)}(h_i) + \frac{1}{(m+1)!} \sum_{I(m+1)} c_{jI(m+1)} h_{I(m+1)} , \tag{6.19}$$

which are correct to order $m + 1$ provided

$$c_{jI(m+1)} = a_{jI(m+1)}^{(m+1)}/(b^{\lambda_j} - b^{\lambda_{I(m+1)}}). \tag{6.20}$$

Thus, we have obtained the desired sequence of functions $g_j^{(m+1)}(h_i)$ which should converge to the scaling fields g_j. The method breaks down if, at some stage, $\lambda_{I(n)} = \lambda_j$, which is precisely the case where Wegner [3, 30] finds the need for logarithmic corrections.

We note finally that in order for the $c_{jI(2)}$ to be independent of b as required, the relation (6.17) dictates the b-dependence of the recursion relation coefficient $a_{jI(2)}(b)$. Once we have evaluated the number $c_{jI(2)}$ for, say, $b = 2$, we simply obtain $a_{jI(2)}(b) = c_{jI(2)}(b^{\lambda_j} - b^{\lambda_{I(2)}})$. The appropriate b-dependence of higher order coefficients can be found in a similar fashion. We will in fact check explicitly for the dedecoration groups that the same results for the g_j follows for both $b = 2$ and $b = 3$.

6.3. *Application to the Linear Ising Chain*

Applying the method outlined in the previous subsection to the recursion relations (6.3) and (6.4) we can deduce the nonlinear scaling fields for the linear Ising chain. We find

$$\begin{aligned} g_x(x, \bar{y}) &= x + \frac{2}{3} x^2 - \frac{1}{12} x\bar{y}^2 + \frac{23}{45} x^3 + \cdots , \\ g_y(x, \bar{y}) &= \bar{y} + \frac{1}{2} \bar{y}^2 + \frac{1}{3} x\bar{y} + \frac{1}{6} x\bar{y}^2 + \frac{1}{5} x^2\bar{y} + \frac{1}{3} \bar{y}^3 + \cdots . \end{aligned} \tag{6.21}$$

The same results are found starting with either the recursion relations for $b = 2$ or those for $b = 3$. Thus g_x and g_y are indeed independent of b, as desired.

The expression for the magnetization obtained from the exact solution is, when expressed in terms of x and $\bar{y}$,

$$M = [1 + 4(x/\bar{y}^2)(1 - \bar{y})]^{-1/2}. \tag{6.22}$$

If we calculate g_x/g_y^2 from (6.21) we obtain

$$g_x/g_y^2 = (x/\bar{y}^2)[1 - \bar{y} + O(x^3, x^2y, xy^2, y^3)]. \tag{6.23}$$

Thus, terms proportional to x, $x\bar{y}$, x^2, and $\bar{y}^2$ cancel exactly, and the scaling property of the nonlinear fields is verified explicitly to this order. Expansion of the nonlinear fields after substitution in the scaling function (3.54) (with $v^\dagger = 0$) thus reproduces all the corrections to asymptotic scaling.

We can also treat the case when both uniform and staggered magnetic fields act. The Taylor series expansions of the recursion formulae (3.45) are

$$x' = 9x - 48x^2 + 208x^3 + 2x\bar{z}^2 + 6x\bar{y}^2 + \cdots, \tag{6.24}$$

$$\bar{y}' = 3\bar{y} - 3\bar{y}^2 - 8x\bar{y} + 20x\bar{y}^2 + 24x^2\bar{y} + \bar{y}^3 + \cdots, \tag{6.25}$$

$$\bar{z}' = \frac{1}{3}\bar{z} + \frac{1}{9}\bar{z}^2 + \frac{8}{9}x\bar{z} + \frac{1}{27}\bar{z}^3 - \frac{8}{27}x^2\bar{z} + \frac{8}{27}x^2\bar{z} + \frac{4}{27}x\bar{z}^2 + \frac{2}{g}\bar{y}^2\bar{z} + \cdots. \tag{6.26}$$

Using the methods developed, one finds the nonlinear scaling fields are

$$\begin{aligned} g_x(x, \bar{y}, \bar{z}) &= x + \frac{2}{3}x^2 + \frac{1}{4}x\bar{z}^2 - \frac{1}{12}x\bar{y}^2 + \frac{23}{45}x^3 + \cdots, \\ g_y(x, \bar{y}, \bar{z}) &= \bar{y} + \frac{1}{2}\bar{y}^2 + \frac{1}{3}x\bar{y} + \frac{1}{6}x\bar{y}^2 + \frac{1}{5}x^2\bar{y} + \frac{1}{3}\bar{y}^3 + \cdots, \\ g_3(x, \bar{y}, \bar{z}) &= \bar{z} + \frac{1}{2}\bar{z}^2 - \frac{1}{3}x\bar{z} - \frac{1}{6}x\bar{z}^2 - \frac{4}{45}x^2\bar{z} + \frac{1}{4}\bar{z}^3 - \frac{1}{12}y^2\bar{z} + \cdots. \end{aligned} \tag{6.27}$$

Again, one can take the appropriate ratios of these scaling fields and verify that they agree with the exact expression (3.53) found for the magnetization, to the order to which we have calculated.

Note that, although these nonlinear scaling fields are unique with respect to the particular renormalization group we are using, they are certainly not unique as regards their ability to represent the exact-solution in a complete scaling form. For example, let $\beta(x, \bar{y})$ be an arbitrary analytic function subject only to $\beta(0, 0) = 1$. Then taking the case of the linear Ising chain in a uniform field, we could replace

g_x by $\hat{g}_x = \beta^2(x, \bar{y})\, g_x$, and g_y by $\hat{g}_y = \beta(x, \bar{y})\, g_y$, and the ratio $\hat{g}_x/\hat{g}_y{}^2$ would have the same value as $g_x/g_y{}^2$ and so reproduce the correct magnetization. These new "nonlinear fields" would equally diagonalize the linearized recursion relations (although they need not come from the solution of a full set of recursion relations). We will actually show in Section 7 that a continuous range of nonlinear scaling fields exists for a range of distinct renormalization groups.

A natural question to ask is whether one can identify in closed form the functions expanded in (6.27). This we have not been able to do in general, but the following special cases can be found

$$g_x(x, 0, 0) = \frac{1}{4} \ln^2 \left(\frac{1 - x^{1/2}}{1 + x^{1/2}}\right),$$
$$g_{\bar{y}}(0, \bar{y}, 0) = -\ln(1 - \bar{y}). \tag{6.28}$$

More generally, however, the convergence of these series (in some appropriate sense) is an open question.

6.4. *Corrections to Scaling and the Spin Independent Term*

So far we have discussed corrections to asymptotic scaling only in terms of the magnetization. This was done for reasons of convenience; consideration of the free energy is complicated by the effects of the constant or spin-independent term. For completeness, we will show generally how the effects of the spin-independent term together with those of irrelevant variables can be taken into account; detailed calculations for the one-dimensional Ising model will not be carried out.

Denote the spin-independent part of the Hamiltonian by h_0. If g_x, g_y, g_z are the nonlinear scaling fields discussed previously, then we have

$$f(g_x, g_y, g_z, h_0) = b^{-1} f(b^2 g_x, b g_y, b^{-1} g_z, 0) + h_0(b), \tag{6.29}$$

and choosing $b = b^* = 1/g_x^{1/2}$ we find

$$f(g_x, g_y, g_z, h_0) = g_x^{1/2} Y(g_y/g_x^{1/2}, g_z g_x^{1/2}) + h_0(b^*). \tag{6.30}$$

To find the b-dependence of $h(b)$, we follow the procedure developed in 6.2, applying it to the linear Ising chain with $z = 1$.

It was convenient to express the recursion relation for the spin independent term [see (3.16)] as

$$(wx)' = (wx)^2\, y^2/(x + y)^2\, (1 + yx)^2, \tag{6.31}$$

where $h_0 = -\frac{1}{4} \ln w$. The fixed point is given by $(xw)^* = 1$. On defining $u = 1 - xw$, this may be written as

$$u' = 2u + 4x + O(u^2, x^2, x\bar{y}) \qquad (b = 2). \tag{6.32}$$

To determine $h_0(b^*)$, it suffices to find the b-dependence of this recursion relation. The corresponding scaling field g_u is found to be

$$g_u = u - 2x + O(u^2, x^2, x\bar{y}). \tag{6.33}$$

Application of the method explained after Eq. (6.21) yields

$$u'(b) = bu + 2(b^2 - b)x + O(u^2, x^2, x\bar{y}). \tag{6.34}$$

Expressing x and $\bar{y}$ in terms of g_x and g_y, we can find $h_0(b^*)$ in terms of the scaling fields, and determine the additional contribution to the free energy on the right-hand side of (6.30).

The division of the free energy given by (6.30) into a piece which scales and an extra term, is a general feature of the renormalization group [3, 32]. One does not expect such a decomposition to be unique; the nonlinear scaling fields will, in fact, be seen to be nonunique in Section 7. (There is a trivial nonuniqueness of scaling fields, due to the possibility of constructing them around the different fixed points of a renormalization group problem, but we will refer in Section 7 to a nonuniqueness associated with different Hamiltonian *flows*.)

7. Spin Rescaling Renormalization Groups

In the dedecoration renormalization groups discussed so far, the spin rescaling factor $c[\mathscr{H}]$ (see Section 2) has been fixed at unity, consistent with the fact that $\eta = 1$ in one dimension. There is, however, a different approach due to Wilson [20], which involves a variable spin rescaling factor. Of course, to find a low temperature ferromagnetic fixed point for our one-dimensional models the spin rescaling factor must approach unity as $T \to 0$ in zero field. Using Wilson's approach, we will generate a range of distinct, new renormalization groups for the Ising chain.

7.1. *Generalized Renormalization Group Transformation*

The idea of a renormalization group transformation is to eliminate a specified fraction of the degrees of freedom of a system by performing some sort of partial trace. The degrees of freedom eliminated could be alternate spins in an Ising chain, or the high momentum modes of a Brillouin zone [1]. If the transformation is chosen sensibly, fixed points describing critical behavior and the various critical exponents can be found.

One way of writing a general renormalization group transformation is to define a new reduced Hamiltonian $\mathscr{H}'[s']$ by

$$\exp(\mathscr{H}'[s']) = \mathrm{Tr}\{\mathscr{P}[s', s] \exp(\mathscr{H}[s])\}, \tag{7.1}$$

where

$$\mathrm{Tr}\{\ \} = \sum_{s_1=\pm1} \cdots \sum_{s_N=\pm1} \{\ \} \tag{7.2}$$

for discrete Ising spins. The spins s' replace the old spins s, and are a "thinned out" set of degrees of freedom which describe the renormalized Hamiltonian $\mathscr{H}'$. A minimal requirement that (7.1) describe a renormalization group transformation is that the transformation preserve the partition function,

$$\mathrm{Tr}'\{\exp(\mathscr{H}'[s'])\} = \mathrm{Tr}\{\exp(\mathscr{H}[s])\}, \tag{7.3}$$

This leads to a condition on $\mathscr{P}[s', s]$, namely,

$$\mathrm{Tr}'\{\mathscr{P}[s', s]\} = 1. \tag{7.4}$$

The Wilson approach embraces the Kadanoff idea [25] of block spin variables directly. With a block of two adjacent spins s_{2k} and s_{2k+1} in the linear chain, we

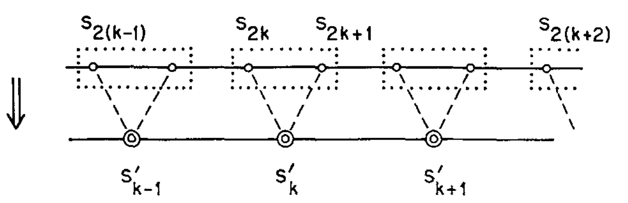

FIG. 10. A generalized renormalization group transformation.

associate a new spin variable s_k' (see Fig. 10) which can assume the usual values $s_k' = \pm 1$. We write the function $\mathscr{P}[s', s]$ in the *pair* factored form

$$\mathscr{P}[s', s] = \prod_{k=1}^{N/2} P(s_k'; s_{2k}, s_{2k+1}) \tag{7.5}$$

with the condition

$$P(1; s_{2k}, s_{2k+1}) + P(-1; s_{2k}, s_{2k+1}) = 1 \tag{7.6}$$

to insure that (7.4) holds. Since $P(s'; s_1, s_2)$ is defined only on a space of eight states, we may write it generally as a product of exponentials containing eight independent constants [18],

$$P(s'; s_1, s_2) \propto \exp(ps' + q_1 s' s_1 + q_2 s' s_2 + r s' s_1 s_2 + \bar{p} + \bar{q}_1 s_1 + \bar{q}_2 s_2 + \bar{r} s_1 s_2). \tag{7.7}$$

The normalization condition eliminates four of these constants, and forces P to take the form

$$P(s'; s_1, s_2) = \exp(ps' + q_1 s' s_1 + q_2 s' s_2 + r s' s_1 s_2)/2 \cosh(p + q_1 s_1 + q_2 s_2 + r s_1 s_2). \tag{7.8}$$

The term $\exp(rs's_1s_2)$ in the numerator leads to a new Hamiltonian not in the space of those with nearest neighbor interactions and magnetic fields, so we set $r = 0$ to obtain finally

$$P(s'; s_1, s_2) = \exp(ps' + q_1 s' s_1 + q_2 s' s_2)/2 \cosh(p + q_1 s_1 + q_2 s_2). \tag{7.9}$$

7.2. *Realization of Transformations in Closed Form*

In order to do the sum (7.1) we write the denominator of $P(s'; s_1, s_2)$ as

$$\cosh(p + q_1 s_1 + q_2 s_2) \equiv g \exp[K s_1 s_2 + \delta L_1 s_1 + \delta L_2 s_2], \tag{7.10}$$

where g, K, δL_1, and δL_2 are given by the formula (3.3) with $\psi(s_1, s_2) = \cosh(p + q_1, s_1 + q_2 s_2)$. We can now express (7.1) as

$$\exp(\mathscr{H}'[s']) = \operatorname{Tr} \left\{ \prod_{k=1}^{N/2} g^{-1} \exp[(K - \bar{K})\, s_{2k} s_{2k+1} + (L - \delta L_1)\, s_{2k} + (L - \delta L_2)\, s_{2k+1} + K s_{2k} s_{2k-1} + p s_k' + q_1 s_k' s_{2k} + q_2 s_k' s_{2k+1}] \right\} \tag{7.11}$$

For most values of p, q_1, and q_2, this transformation removes the Hamiltonian from the original parameter space of nearest neighbor couplings and uniform fields. If, however, we require

$$e^{4K} = [\cosh 2p + \cosh 2(q_1 + q_2)]/[\cosh 2p + \cosh 2(q_1 - q_2)], \tag{7.12}$$

then $\bar{K}$ equals K, the summations can be performed analytically, and we stay within the original parameter space. The recursion relations for K and L are then

$$e^{4K'} = \psi_1 \psi_2 / \psi_3 \psi_4, \qquad e^{2L'} = e^{2p} \psi_1 / \psi_2, \tag{7.13}$$

where

$$\begin{aligned}
\psi_1 &= e^K \cosh(q_1 + q_2 - \delta L_1 - \delta L_2 + 2L) + e^{-K} \cosh(q_1 - q_2 + h_1 - h_2),\\
\psi_2 &= e^K \cosh(q_1 + q_2 + \delta L_1 + \delta L_2 - 2L) + e^{-K} \cosh(q_1 - q_2 - \delta L_1 + \delta L_2),\\
\psi_3 &= e^K \cosh(q_1 - q_2 - \delta L_1 - \delta L_2 + 2L) + e^{-K} \cosh(q_1 + q_2 + \delta L_1 - \delta L_2),\\
\psi_4 &= e^K \cosh(q_1 - q_2 + \delta L_1 + \delta L_2 - 2L) + e^{-K} \cosh(q_1 + q_2 - \delta L_1 + \delta L_2).
\end{aligned} \tag{7.14}$$

The constraint (7.12) determines a two-dimensional surface of allowable renormalization groups in the space indexed by p, q_1, and q_2.

When the parameter p is nonzero, we obtain an example of a nonlinear renormalization group [24] (see Section 2). This is seen by examining the effect of the transformation on the magnetization

$$\langle s_k' \rangle' = \langle \tanh(p + q_1 s_{2k} + q_2 s_{2k+1}) \rangle \tag{7.15}$$

and the spin–spin correction function

$$\langle s_k' s_0' \rangle' = \langle \tanh(p + q_1 s_{2k} + q_2 s_{2k+1}) \tanh(p + q_1 s_0 + q_2 s_1) \rangle. \tag{7.16}$$

Here $\langle \cdot \rangle'$ denotes an expectation taken with respect to the primed spins. It is easy to show that (7.15) and (7.16) do not have the simple scaling properties associated with linear renormalization groups and discussed in Section 2. In fact, a nonzero value of p breaks the symmetry of a zero field nearest neighbor Hamiltonian after one renormalization group iteration, and artificially shifts the magnetization as indicated by (7.15). Because of these features, the parameter p appears to be the Ising spin analogue of the spin "shift" used in calculations with continuous spins below T_c [8]. A linear Ising chain is always above $T_c = 0$, so it is doubtful if useful renormalization groups can be obtained from (7.13) with $p \neq 0$.

7.3. *A Continuum of Renormalization Groups*

With $p = 0$ we can readily analyze a continuum of renormalization groups dependent on a single parameter. The condition (7.12) now reduces to

$$e^{2K} = \cosh(q_1 + q_2)/\cosh(q_1 - q_2), \tag{7.15}$$

and we imagine $\cosh(q_1 + q_2)$ is chosen to fulfill this requirement with $\cosh(q_1 - q_2)$ left as a free parameter. Using the usual variables $x = e^{-4K}$, $y = e^{-2L}$, we obtain the θ-dependent recursion relations

$$x' = \frac{x\theta^2(1+y)^4 + (1-\theta^2)\,4y(1+y)^2}{(1+y^2)^2 + 4x(y^3+y) + 4x^2y^2 - (1-\theta^2 x)(1-y^2)^2}, \tag{7.16}$$

$$y' = \frac{1 + y^2 + 2yx - (1-\theta^2 x)^{1/2}(1-y^2)}{1 + y^2 + 2yx + (1-\theta^2 x)^{1/2}(1-y^2)}, \tag{7.17}$$

where

$$\theta = 1/\cosh(q_1 - q_2), \qquad 0 \leqslant \theta \leqslant 1. \tag{7.18}$$

For $\theta = 0$, we recover the original dedecoration recursion relations (3.11) and (3.12). The flows and fixed points for $\theta = 1$ ($q_1 = q_2$) are shown in Fig. 11. There

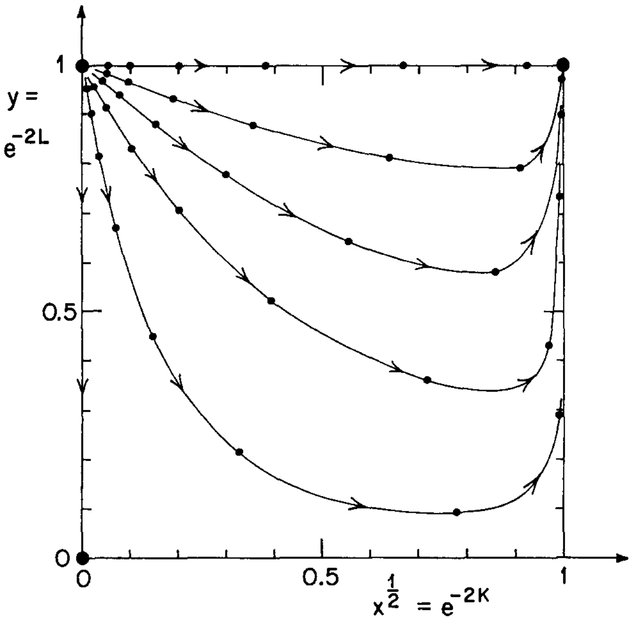

FIG. 11. Trajectories and fixed points describing the linear Ising chain under the action of a spin rescaling renormalization group (with $\theta = 1$). There are fixed points at (0, 1), (1, 1), and (0, 0).

are fixed points located at $(x^*, y^*) = (0, 1)$, $(1, 1)$ and $(0, 0)$. The trajectories differ markedly from those for the dedecoration group (see Fig. 2) at large values of x. In particular, the "paramagnetic line" of fixed points at $x = 1$ has vanished. All groups with $0 < \theta \leqslant 1$ have the same general structure as depicted in Fig. 11.

Linearization of (7.16) and (7.17) about the ferromagnetic fixed point at (0, 1) yields

$$\Lambda_x = 4, \qquad \Lambda_y = 2, \tag{7.19}$$

$$\lambda_x = 2, \qquad \lambda_y = 1, \tag{7.20}$$

independently of θ. Thus the eigenvalues and ferromagnetic fixed point are independent of the particular group chosen to describe the physics. This invariance of physically significant eigenvalues is, of course expected on general grounds [33].

Although both the s_k' and the s_k attain only the values ± 1, by examining the way the magnetization transforms (an equivalent result derives from treating the spin–spin correlation function) we can see that there is an effective spin rescaling. Analyzing (7.15) with $p = 0$ we find

$$\langle s_k' \rangle = (1 - x\theta^2)^{1/2} \langle s_{2k} \rangle. \tag{7.21}$$

The spin rescaling factor (discussed in Section 2) is thus

$$c[\mathscr{H}] = (1 - x\theta^2)^{1/2}, \tag{7.22}$$

and it changes with each iteration step. However, for any value of θ, c approaches unity as the ferromagnetic fixed point is approached, which is consistent with $\eta = 1$.

Just as in Section 6 one can now calculate θ-dependent nonlinear scaling fields. The results are

$$g_x(\theta; x, \bar{y}) = x + \frac{2}{3}x^2 - \left(\frac{1}{12} + \frac{1}{12}\theta^2\right)x\bar{y}^2 + \frac{23}{45}x^3 + \cdots, \tag{7.21}$$

$$g_y(\theta; x, \bar{y}) = y + \frac{1}{2}\bar{y}^2 + \left(\frac{1}{3} + \frac{1}{6}\theta^2\right)x\bar{y} + \left(\frac{1}{6} + \frac{1}{12}\theta^2\right)x\bar{y}^2$$
$$+ \left(\frac{1}{5} + \frac{13}{90}\theta^2 + \frac{11}{360}\theta^4\right)x^2\bar{y} + \frac{1}{3}\bar{y}^3 + \cdots. \tag{7.22}$$

These expressions demonstrate explicitly that the nonlinear scaling fields are *nonunique*; they are θ-independent only to order x and $\bar{y}$ so that the linear scaling fields are preserved. [For $\theta = 0$ they reduce to the previous results (6.21) as they should.] Thus although the linear scaling fields have a definite physical significance, the nonlinear fields cannot have a general significance. Indeed, the existence of distinct renormalization groups with differing global Hamiltonian flows necessarily implies distinct nonlinear fields.

APPENDIX: DIFFERENTIAL EQUATIONS FROM DISCRETE RECURSION RELATIONS: TRUNCATED TETRAHEDRON MODEL

It is interesting to determine to what extent the differential approximation (5.8) to the discrete recursion relation (5.7) for the truncated tetrahedron model is valid. Consequently, we present here a systematic procedure for calculating corrections to (5.9) and calculate the first correction term.

The recursion relation (5.7) may be written as

$$\bar{v}' = \bar{v} + \bar{v}^2 - \bar{v}^4 + O(\bar{v}^5). \tag{A1}$$

We will treat only the *truncated* recursion relation

$$\bar{v}(l + I) = \bar{v}(l) + \bar{v}^2(l). \tag{A2}$$

Making the change of variable

$$w = \bar{v}_0/\bar{v}, \qquad \bar{v}_0 = \bar{v}(0), \tag{A3}$$

we obtain

$$w(l+1) = w(l) - \bar{v}_0 + \bar{v}_0^2/[w(l) + \bar{v}_0]. \tag{A4}$$

Approximating the function $w(l)$ by the solution of $dw/dl = -\bar{v}_0$ as was done in Section 5 gives $w(l) \approx 1 - \bar{v}_0 l$. We obtain a correction to this result by substituting the expression

$$w(l) \equiv 1 - \bar{v}_0 l + g(l) \tag{A5}$$

into (A4). The resulting recursion relation for $g(l)$ is

$$g(l+1) = g(l) + \bar{v}_0^2/[1 - \bar{v}_0 l + \bar{v}_0 + g(l)]. \tag{A6}$$

We now approximate $g(l)$ by the solution of $dg/dl = \bar{v}_0^2/(1 - \bar{v}_0 l + \bar{v}_0)$ and obtain

$$w(l) = 1 - \bar{v}_0 l - \bar{v}_0 \ln(1 - \bar{v}_0 l) + O[\bar{v}_0^2 \ln(1 - \bar{v}_0 l)]. \tag{A7}$$

Clearly, one can continue this process of successive approximations indefinitely. However, the next term produced by iteration of the procedure is of the same order as the error introduced by truncating the original recursion relation. Thus, higher order terms in the expansion (A1) have to be taken into account.

The extra term in (A7) shows the correlation length of the truncated tetrahedron model varies as

$$\xi \sim \exp[\tfrac{1}{2}(\ln 2)\exp(2J/k_B T) + (\ln 2)(2J/k_B T)], \tag{A8}$$

which should be compared with (5.12).

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