Outline

Thanks to Ian Blockland and Randy Sobie for these slides

- Lifetimes of Decaying Particles
- Scattering Cross Sections
- Fermi's Golden Rule

Observables

We want to relate experimental measurements to theoretical predictions

- Decay widths and lifetimes
  $\Gamma = h/\tau$ (units of energy)

- Scattering cross-sections
  σ is the total cross section
  $\frac{d\sigma}{d\Omega}$ is the angular distribution
  $\frac{d\sigma}{dE}$ is the energy distribution

Lifetime of an Unstable Particle

- The decay rate, Γ, represents the probability per unit time of the particle decaying:

$$\begin{aligned} dN &= -\Gamma N \, dt \\ \Rightarrow \quad N(t) &= N(0)\, e^{-\Gamma t} \end{aligned}$$

- The decay rate determines the (mean) lifetime of the particle:

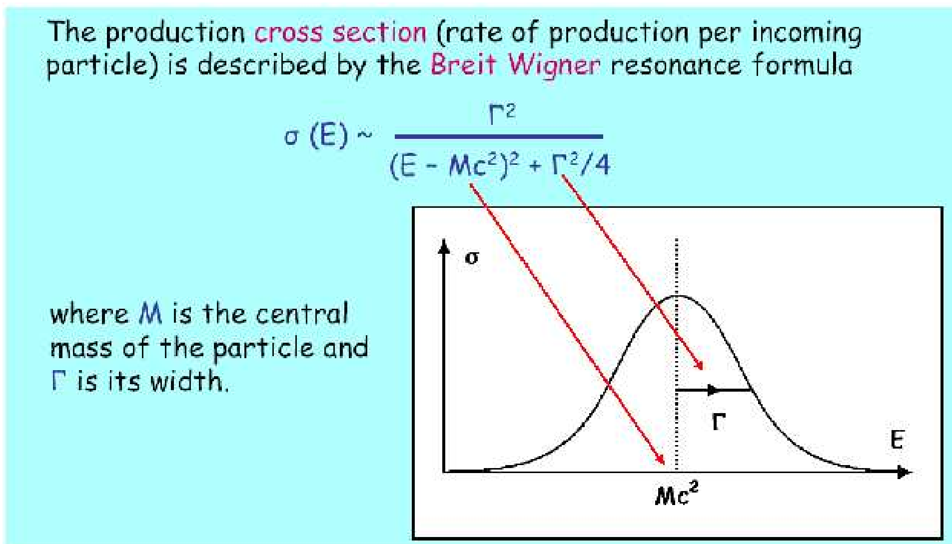
$$\tau = \frac{1}{\Gamma}$$

Breit-Wigner Resonance

Wavefunction for a particle with mass M and width Γ is $\Psi(t) = \Psi(0)e^{-i(M-i\Gamma/2)t}$

Fourier transform $\qquad \chi(E) \propto \frac{1}{(M-E)-i\Gamma/2}$

Breit-Wigner formula : $\qquad |\chi(E)|^2 \propto \frac{1}{(M-E)^2+\Gamma^2/4}$



Luminosity

- We relate cross sections to observed detection rates, per unit time, by
$$dN = \mathcal{L}\, d\sigma$$
- Since N is the number of events observed per unit time, $\mathcal{L}$ has the dimensions of an inverse cross-section per unit time. The PEP2 collider at SLAC has: $\mathcal{L} \simeq 10^{34}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}$
- Only *peak luminosity* is typically quoted in per-unit-time form. Usually we integrate $\mathcal{L}$ over the running time of an experiment in order to determine what sorts of cross sections we are sensitive to. This *integrated luminosity* is measured in, say, pb^{-1}.

Luminosity Example

At PEP2 $L = 10^{34}\ \text{cm}^{-2}\ \text{s}^{-1}$

1 barn = 10^{-24} cm^2 or 1 nb = 10^{-33} cm^2

Hence L = 10 nb^{-1} s^{-1}

The $e^+e^- \to B\overline{B}$ cross section at the operating energy is about 1 nb, so the rate for producing $B\overline{B}$ pairs in BaBar is

$$N_{B\overline{B}} = \sigma_{B\overline{B}} L = 10\ \text{s}^{-1}$$

Branching Ratios

- It is the decay rate that we will be calculating from Feynman diagrams. If a particle can decay via multiple routes, we have

$$\Gamma_{tot} = \sum_i \Gamma_i \qquad \tau = \frac{1}{\Gamma_{tot}}$$

- We define the branching ratio for a particular decay mode as

$$B_i = \frac{\Gamma_i}{\Gamma_{tot}}$$

- This is all just terminology and basic probability...

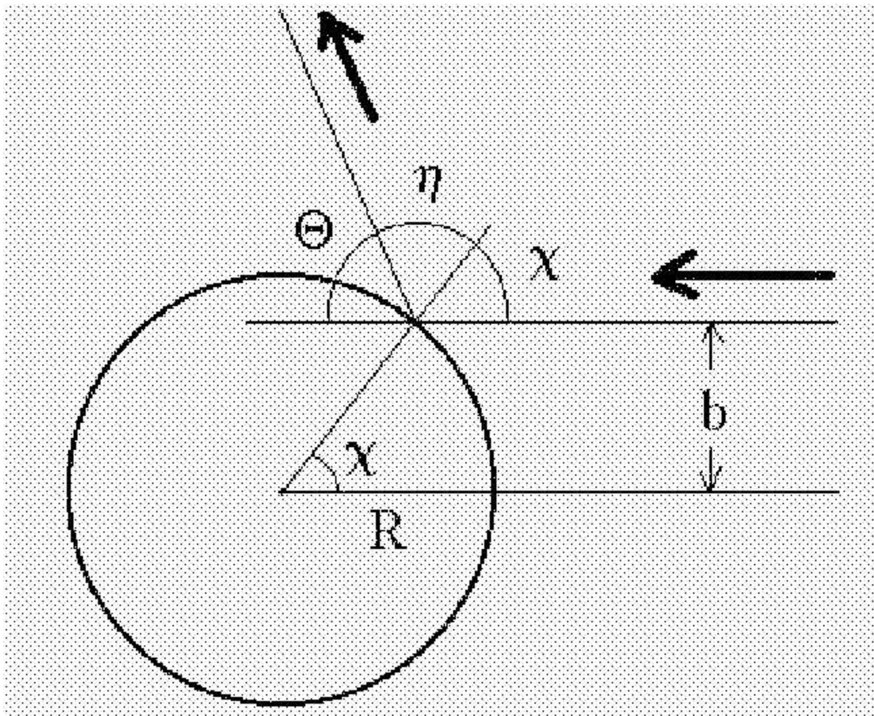
Scattering Cross Sections

- We need to generalize the intuitive notion of the geometrical cross section of a target.
- First, the interaction between the projectile and the target may be a *long-range* one, such that it is not a case of "hit or miss", but rather, "how much deflection"? This might depend on the particle energies.
- Second, the cross section will no longer be the sole property of the target but rather a joint characteristic of both the projectile and the target.
- Third, we need to be able to account for the inelastic processes in which the final-state particles are different from those of the initial state.

Scattering Formalism

- We can interpret classical scattering experiments as prescribing a unique relationship between the impact parameter, b, and the scattering angle, θ.
- Expressing things in terms of $\theta(b)$ already allows us to treat short-range and long-range forces on the same footing. Let's look at an example of each in classical physics.
- Short-range interaction example: Hard-Sphere Scattering
- Long-range interaction example: Rutherford Scattering

Hard-Sphere Scattering



$$\eta = \chi \qquad \chi + \eta + \theta = \pi \quad \Rightarrow \quad \chi = \pi/2 - \theta/2$$

$$b = R\sin\chi = R\sin(\pi/2 - \theta/2) = R\cos(\theta/2)$$

Note that $\theta = 0$ for all $b \geq R$.

Rutherford Scattering

- Coulomb repulsion of a heavy stationary target of charge q_2 and a light incident particle of charge q_1 and kinetic energy E.
- With a great deal of effort, classical mechanics can be used to relate the impact parameter to the scattering angle:

$$b = \frac{q_1 q_2}{2E} \cot(\theta/2)$$

In this case note that $\theta > 0$ for any finite value of b.

The Differential Cross Section...

- ... is written as $d\sigma/d\Omega$
- ... often depends on θ.
- Geometrically, it is easy to see that

$$d\sigma = |\, b\, db\, d\phi \,| \qquad\qquad d\Omega = |\, \sin\theta \; d\theta \; d\phi \,|$$

and so the (classical) differential cross section is

$$\frac{d\sigma}{d\Omega} = \left| \, \frac{b}{\sin\theta} \left(\frac{db}{d\theta} \right) \, \right|$$

Hard-Sphere Scattering

$$b = R\cos(\theta/2) \qquad \Rightarrow \quad \frac{db}{d\theta} = -\,\frac{R}{2}\,\sin(\theta/2)$$

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left| \frac{b}{\sin\theta}\left(\frac{db}{d\theta}\right) \right| \\ &= \frac{R^2\,\sin(\theta/2)\cos(\theta/2)}{2\,\sin\theta} \\ &= R^2/4 \end{aligned}$$

$$\sigma = \int (R^2/4)\, d\Omega = \pi R^2$$

Rutherford Scattering

$$b = \frac{q_1 q_2}{2E}\cot(\theta/2) \qquad \Rightarrow \quad \frac{db}{d\theta} = -\,\frac{q_1 q_2}{4E}\,\csc^2(\theta/2)$$

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left|\,\frac{b}{\sin\theta}\left(\frac{db}{d\theta}\right)\,\right| \\ &= \frac{q_1^2 q_2^2\,\cot(\theta/2)\csc^2(\theta/2)}{8E^2\,\sin\theta} \\ &= \left(\frac{q_1 q_2}{4E\sin^2(\theta/2)}\right)^2\end{aligned}$$

$$\sigma = 2\pi\left(\frac{q_1 q_2}{4E}\right)^2 \int_0^\pi \frac{\sin\theta}{\sin^4(\theta/2)}\,d\theta \;\to \infty$$

Fermi's Golden Rule

$$\text{Transition rate} \sim |\mathcal{M}|^2 \times (\text{Phase space})$$

- The amplitude $\mathcal{M}$ contains the dynamical information about the process. We use Feynman diagrams to calculate this.
- The phase space is a kinematical factor. The bigger the phase space available, the larger the transition rate.
- Alternate terminology:
  Amplitude $\Leftrightarrow$ Matrix Element
  Phase Space $\Leftrightarrow$ Density of Final States

Golden Rule for Decays

- For the decay $1 \longrightarrow 2 + 3 + 4 + \ldots + n$

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2m_1} \left[\left(\frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} \right) \left(\frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3} \right) \cdots \left(\frac{d^3\mathbf{p}_n}{(2\pi)^3 2E_n} \right) \right]$$
$$\times (2\pi)^4 \delta^4 (p_1 - p_2 - p_3 - \ldots - p_n)$$

- S is a symmetry factor of $1/j!$ for every group of j identical particles in the final state.

- $p_i = (E_i, \mathbf{p}_i)$ is the four-momentum of the i-th particle. The volume elements of the final-state particles are (subtly) Lorentz invariant:

$$\int d^4p_i \, \delta \left(p_i^2 - m_i^2 \right) = \int \frac{d^3\mathbf{p}_i}{2E_i}$$

Example: $\pi \to \gamma + \gamma$

- From the general formula, set $S = 1/2$ and gather the factors of 2 and (2π):
$$d\Gamma = \frac{|\mathcal{M}|^2}{(8\pi)^2 m_1} \frac{d^3\mathbf{p}_2}{E_2} \frac{d^3\mathbf{p}_3}{E_3} \, \delta^4(p_1 - p_2 - p_3)$$

- Factor the δ-function into an energy part and a momentum part:
$$\delta^4(p_1 - p_2 - p_3) = \delta(m - E_2 - E_3) \, \delta^3(\mathbf{0} - \mathbf{p}_2 - \mathbf{p}_3)$$

- The momentum part of the δ-function, upon integration over $\mathbf{p}_3$, sets $\mathbf{p}_3 = -\mathbf{p}_2$. Writing E_2 and E_3 in terms of $\mathbf{p}_2$ and $\mathbf{p}_3$, we obtain
$$d\Gamma = \frac{|\mathcal{M}|^2}{(8\pi)^2 m} \frac{d^3\mathbf{p}_2}{|\mathbf{p}_2|^2} \, \delta(m - 2|\mathbf{p}_2|)$$

- The δ-function identity

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \qquad \forall\, i \mid f(x_i) = 0$$

allows us to write the remaining δ-function as

$$\delta(m - 2|\mathbf{p}_2|) = \frac{1}{2}\,\delta\left(|\mathbf{p}_2| - \frac{m}{2}\right)$$

- Next we write the $d^3\mathbf{p}_2$ integration element in spherical coordinates:

$$d^3\mathbf{p}_2 = |\mathbf{p}_2|^2\; d\mathbf{p}_2\;\sin\theta\; d\theta\; d\phi = p^2\; dp\; d\Omega$$

- For a 2-body decay without spin, $|\mathcal{M}|^2$ can only depend on p, therefore we can integrate over $d\Omega \to 4\pi$. In more complicated cases, $|\mathcal{M}|^2$ will also depend on θ, in which case we can only perform the ϕ-integral in advance.

$$d\Gamma = \frac{|\mathcal{M}|^2}{32\pi m} \frac{p^2\, dp}{p^2}\, \delta\left(p - \frac{m}{2}\right)$$

- $d\Gamma$ becomes Γ once we have integrated over every bit of phase space, and even without knowing the specific form of $\mathcal{M}$, we can complete the integration in this case. Restoring the factor of S,

$$\Gamma = \frac{S}{16\pi m}\, |\mathcal{M}|^2$$

where $\mathcal{M}$ is to be evaluated using $\mathbf{p}_3 = -\mathbf{p}_2$ and $|\mathbf{p}_2| = m/2$.

- Remember that Γ has dimensions of energy, therefore $\mathcal{M}$ will also have dimensions of energy for 2-body decays.

Example: $\rho \to \pi + \pi$

- Although the basic procedure remains the same when the final-state particles have mass, the algebra becomes more complicated.
- Starting from the Golden Rule, collecting the constants, and setting $\mathbf{p}_3 = -\mathbf{p}_2$, we have

$$
\begin{aligned}
d\Gamma \quad &= \quad \frac{S|\mathcal{M}|^2}{2(4\pi)^2 m_1} \frac{d^3\mathbf{p}_2}{\sqrt{\mathbf{p}_2^2 + m_2^2}\sqrt{\mathbf{p}_2^2 + m_3^2}} \\
&\quad\quad \times \delta\left(m_1 - \sqrt{\mathbf{p}_2^2 + m_2^2} - \sqrt{\mathbf{p}_2^2 + m_3^2}\right)
\end{aligned}
$$

- Going to spherical coordinates ($p = |\mathbf{p}_2|$) and integrating over the angles,

$$\Gamma = \frac{S}{8\pi m_1} \int \frac{|\mathcal{M}|^2 \; \delta(m_1 - \sqrt{p^2 + m_2^2} - \sqrt{p^2 + m_3^2})}{\sqrt{p^2 + m_2^2}\sqrt{p^2 + m_3^2}} \; p^2 \; dp$$

- In order to make use of the δ-function, we can either solve its argument for p and apply the chain rule δ-function identity (this is not advised) or we can hope to find a change of integration variables within which the δ-function is more manageable.

- The prescient change of variables is to

$$\begin{aligned} E &= \left(\sqrt{p^2 + m_2^2} + \sqrt{p^2 + m_3^2}\right) \\ \Rightarrow \; dE &= \frac{Ep\,dp}{\sqrt{p^2 + m_2^2}\sqrt{p^2 + m_3^2}} \end{aligned}$$

- This simplifies things dramatically, leaving us with

$$\Gamma = \frac{S}{8\pi m_1} \int |\mathcal{M}|^2 \, \frac{p}{E} \, \delta(m_1 - E) \, dE$$

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi m_1^2} \, |\mathcal{M}|^2$$

- $|\mathbf{p}|$ is fixed by energy conservation. With $|\mathbf{p}| = m_1/2$, we recover our previous result.

3-Body Decays

- In the previous two examples, we have shown how the phase space for the decay rate of a 2-body decay can be integrated completely without any information about $\mathcal{M}$.
- For 3-body decays (and beyond), this is no longer possible, as the amplitude will typically depend non-trivially upon several of the phase space integration variables, so that we have to do the integration by hand for each specific $\mathcal{M}$.

Golden Rule for Scattering

- For the process $1 + 2 \longrightarrow 3 + 4 + \ldots + n$

$$
\begin{aligned}
d\sigma \quad = \quad & |\mathcal{M}|^2 \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \\
& \times \left[\left(\frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3} \right) \left(\frac{d^3\mathbf{p}_4}{(2\pi)^3 2E_4} \right) \cdots \left(\frac{d^3\mathbf{p}_n}{(2\pi)^3 2E_n} \right) \right] \\
& \times (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4 - \ldots - p_n)
\end{aligned}
$$

- The factor S contains a $1/j!$ for each group of j identical particles

Example: 2 to 2 Scattering in the CM Frame

- In the CM frame,

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2} = (E_1 + E_2)|\mathbf{p}_1|$$

  Substituting this result into the Golden Rule and collecting the constants, we have

$$d\sigma \quad = \quad \frac{S\,|\mathcal{M}|^2}{(8\pi)^2(E_1+E_2)|\mathbf{p}_1|}\,\frac{d^3\mathbf{p}_3\,d^3\mathbf{p}_4}{E_3E_4}\,\delta^4(p_1+p_2-p_3-p_4)$$

- As usual, we use the momentum part of the δ-function to set $\mathbf{p}_4 = -\mathbf{p}_3$ and we express the energies E_3 and E_4 in terms of the corresponding momenta.

$$
\begin{aligned}
d\sigma &= \frac{S\,|\mathcal{M}|^2}{(8\pi)^2(E_1+E_2)|\mathbf{p}_2|} \\
&\times \frac{\delta(E_1+E_2-\sqrt{\mathbf{p}_3^2+m_3^2}-\sqrt{\mathbf{p}_3^2+m_4^2})}{\sqrt{\mathbf{p}_3^2+m_3^2}\sqrt{\mathbf{p}_3^2+m_4^2}}\ d^3\mathbf{p}_3
\end{aligned}
$$

- Next, we go to spherical coordinates (with $p = |\mathbf{p}_3|$). Unlike the 2-body decays, $\mathcal{M}$ can depend on the scattering angle θ, therefore only the azimuthal integral can be performed for the general case. We choose not to do this, instead moving the entire $d\Omega$ factor to the left-hand side to form a conventional differential cross section:

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{S}{(8\pi)^2\ (E_1+E_2)|\mathbf{p}_1|}\int |\mathcal{M}|^2 \\
&\times \frac{\delta(E_1+E_2-\sqrt{p^2+m_3^2}-\sqrt{p^2+m_4^2})}{\sqrt{p^2+m_3^2}\sqrt{p^2+m_4^2}}\ p^2\ dp
\end{aligned}
$$

- With $(E_1 + E_2)$ taking the place of m_1 in the δ-function, this is precisely the integral we encountered in the previous example of a general 2-body decay. Applying the result we derived there, we obtain the final expression

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \, \frac{S\,|\mathcal{M}|^2}{(E_1 + E_2)^2} \, \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

- We're going to be using the above result quite a bit in the chapters ahead.

Summary

- The statistical mean lifetime of a particle is the inverse of the decay rate.
- The notion of a geometrical cross section can be generalized so as to incorporate a variety of classical scattering results and to carry over to the quantum realm of particle physics.
- Fermi's Golden Rule provides the prescription for combining dynamical information about the amplitude and kinematical information about the phase space in order to obtain observable quantities like decay rates and scattering cross sections.

Feynman rules for QED

- The Feynman Rules for QED
- Setting up Amplitudes
- Casimir's Trick
- Trace Theorems

Slides from Sobie and Blokland

Electrons and positrons

- spinors
  $u^{(s)}$ and $v^{(s)}$ (s = spin) satisfy the Dirac equations $(\gamma^\mu p_\mu - m)u = 0$ and $(\gamma^\mu p_\mu + m)v = 0$
- adjoints
  $\overline{u} = u^\dagger \gamma^0$ and $\overline{v} = v^\dagger \gamma^0$ satisfy $\overline{u}(\gamma^\mu p_\mu - m) = 0$ and $\overline{v}(\gamma^\mu p_\mu + m) = 0$
- orthogonality
  $\overline{u}^{(1)} u^{(2)} = 0$ and $\overline{v}^{(1)} v^{(2)} = 0$
- normalization
  $\overline{u}u = 2m$ and $\overline{v}v = -2m$
- completeness
  $\sum_s u^{(s)} \overline{u}^{(s)} = \gamma^\mu p_\mu + m$ and $\sum_s v^{(s)} \overline{v}^{(s)} = \gamma^\mu p_\mu - m$

Photons

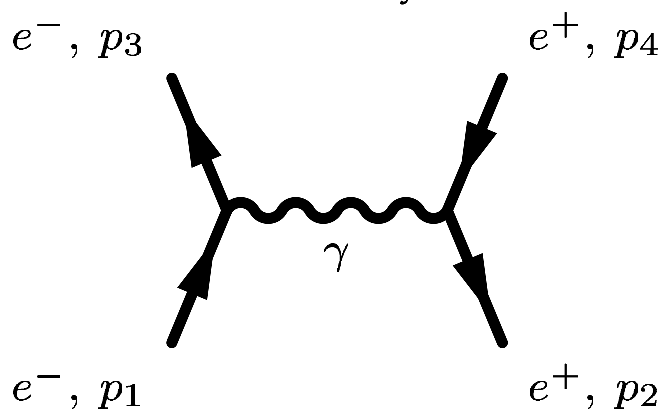
$$A^\mu(x) = ae^{-ip\cdot x}\epsilon^\mu(p)$$

- Lorentz condition
  $\epsilon^\mu p_\mu = 0$
- orthogonality
  $\epsilon^{\mu *}_{(1)}\epsilon_{\mu(2)} = 0$
- normalization
  $\epsilon^{\mu *}\epsilon_\mu = 1$
- Coulomb gauge
  $\epsilon^0 = 0$ and $\epsilon \cdot \mathbf{p} = 0$
- Completeness
  $\sum_s (\epsilon_{(s)})_i(\epsilon^*_{(s)})_j = \delta_{ij} - (p_i p_j)/p^2$

The Feynman Rules for QED

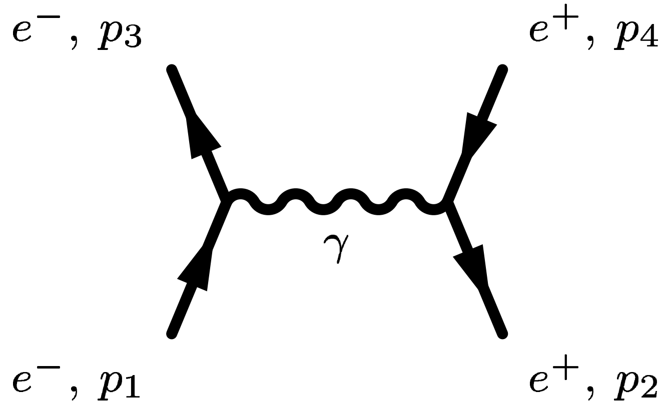
The Feynman rules provide the recipe for constructing an amplitude $\mathcal{M}$ from a Feynman diagram.

- **Step 1:** For a particular process of interest, draw a Feynman diagram with the minimum number of vertices. There may be more than one.



The Feynman Rules for QED

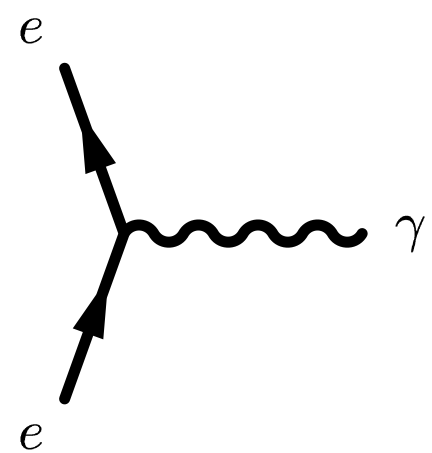
- **Step 2:** For each Feynman diagram, label the four-momentum of each line, enforcing four-momentum conservation at every vertex. Note that arrows are only present on fermion lines and they represent particle flow, not momentum.



- **Step 3:** The amplitude depends on
  1. Vertex factors
  2. Propagators for internal lines
  3. Wavefunctions for external lines

Vertex Factors

- Every QED vertex,



contributes a factor of $ig_e\gamma^\mu$.

- g_e is a dimensionless coupling constant and is related to the fine-structure constant by

$$\alpha = \frac{g_e^2}{4\pi}$$

Propagators

- Each internal photon connects two vertices of the form $ig_e\gamma^\mu$ and $ig_e\gamma^\nu$, so we should expect the photon propagator to contract the indices μ and ν.

$$\text{Photon propagator:} \quad \frac{-ig_{\mu\nu}}{q^2}$$

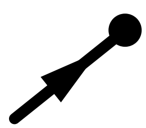
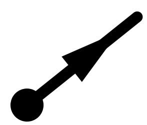
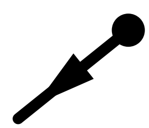
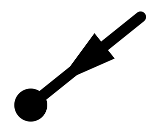
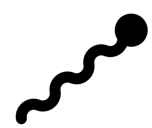
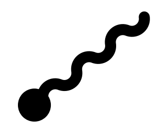
- Internal fermions have a more complicated propagator,

$$\text{Fermion propagator:} \quad \frac{i(\not{q}+m)}{q^2-m^2}$$

The sign of q matters here — we take it to be in the same direction as the fermion arrow.

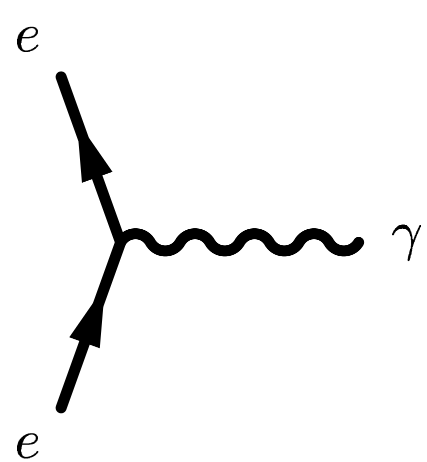
External Lines

- Since both the vertex factor and the fermion propagators involve 4×4 matrices, but the amplitude must be a scalar, the external line factors must sit on the outside.
- Work *backwards* along every fermion line using:

					
e^- in	e^- out	e^+ in	e^+ out	γ in	γ out
u	$\bar{u}$	$\bar{v}$	v	ϵ_μ	ϵ^*_μ

Matrix elements I

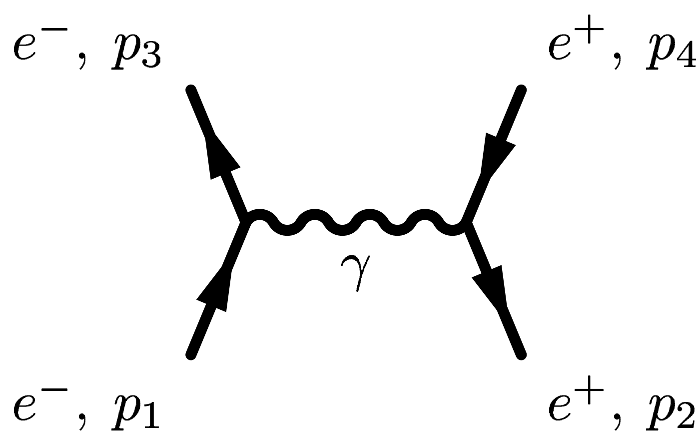
follow fermion lines backward to give $\overline{u}(2) i g \gamma^{\mu} u(1)$



Matrix elements II

The matrix element is proportional to the two currents in the diagram below.

$$[\bar{u}_3(ig_e\gamma^\mu)u_1]\left(\frac{-ig_{\mu\nu}}{(p_1-p_3)^2}\right)[\bar{v}_2(ig_e\gamma^\nu)v_4]$$



And Finally...

- **Step 4:** The overall amplitude is the coherent sum of the individual amplitudes for each diagram:

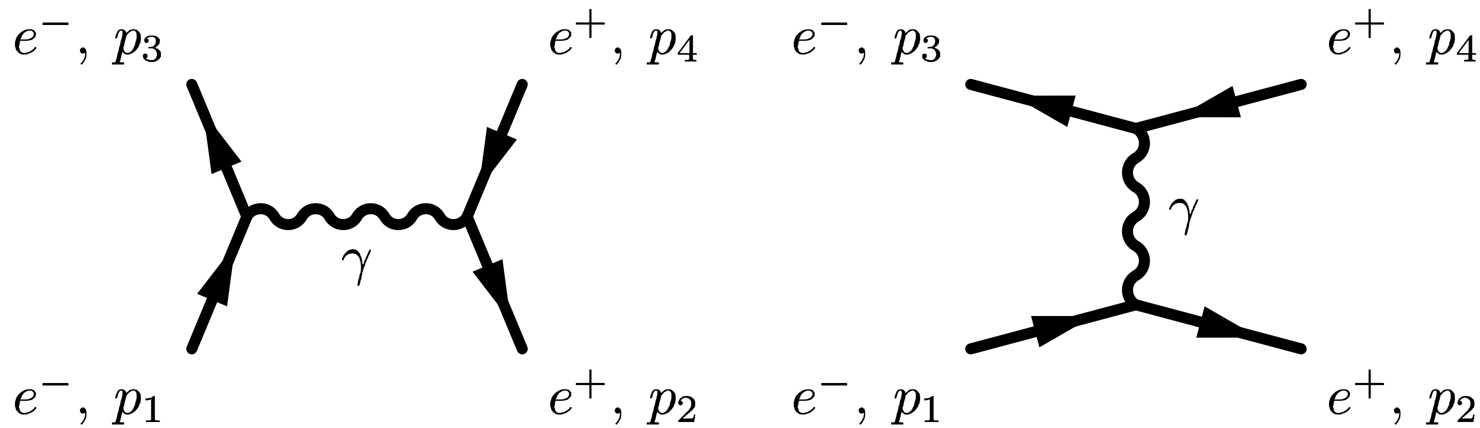
$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 + \mathcal{M}_2 + \dots \\ \Rightarrow \ |\mathcal{M}^2| &= |\mathcal{M}_1 + \mathcal{M}_2 + \dots|^2 \end{aligned}$$

- **Step 4a:** *Antisymmetrization*. Include a minus sign between diagrams that differ only in the exchange of two identical fermions.

Examples

- There are only a handful of ways to make tree-level diagrams in QED.
- Today, we will construct amplitudes for Bhabha scattering $(e^+\, e^- \to e^+\, e^-)$ and Compton scattering $(e\,\gamma \to e\,\gamma)$.
- Next week, we will undertake thorough calculations for Mott scattering $(e\,\ell \to e\,\ell)$, pair annihilation $(e^+\, e^- \to \gamma\,\gamma)$. You will examine fermion pair-production via $(e^+\, e^- \to f\,\bar{f})$ for your assignment.

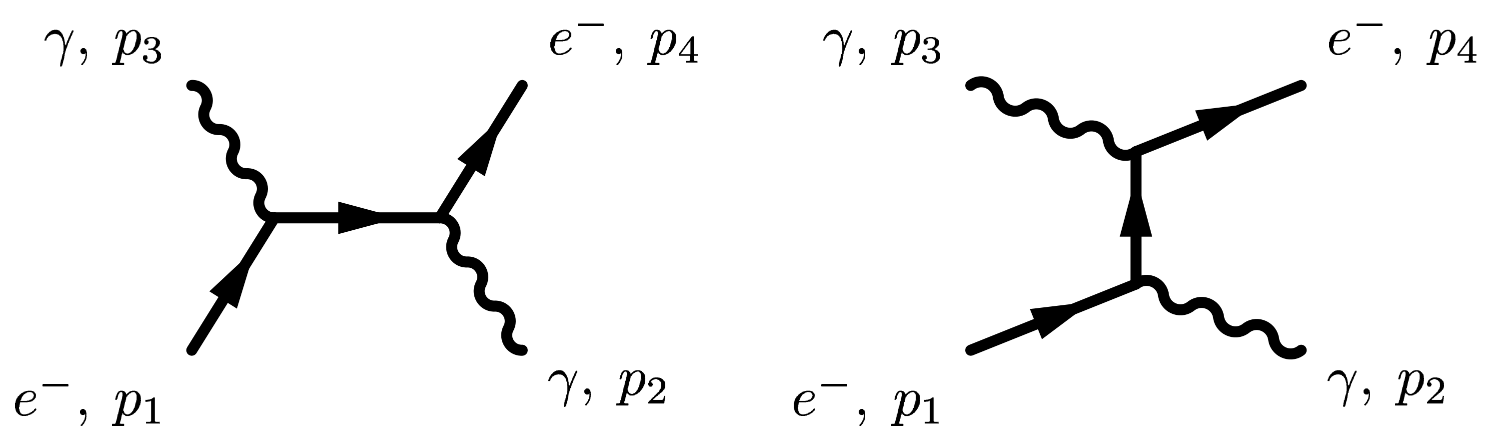
Example: Bhabha Scattering



- Antisymmetrization $\Rightarrow\ \mathcal{M} = \mathcal{M}_t - \mathcal{M}_s$

$$\begin{aligned}
\mathcal{M}_t &= i\left[\bar{u}_3(ig_e\gamma^\mu)u_1\right]\left(\frac{-ig_{\mu\nu}}{(p_1-p_3)^2}\right)\left[\bar{v}_2(ig_e\gamma^\nu)v_4\right] \\
\mathcal{M}_s &= i\left[\bar{u}_3(ig_e\gamma^\mu)v_4\right]\left(\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\right)\left[\bar{v}_2(ig_e\gamma^\nu)u_1\right]
\end{aligned}$$

Example: Compton Scattering



- No antisymmetrization $\Rightarrow\ \mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$

$$
\begin{aligned}
\mathcal{M}_1 &= i\left[\bar{u}_4(ig_e\gamma^\mu)\left(\frac{i(\not{p}_1 - \not{p}_3 + m)}{(p_1 - p_3)^2 - m^2}\right)(ig_e\gamma^\nu)u_1\right]\epsilon^*_{3\nu}\epsilon_{2\mu} \\
\mathcal{M}_2 &= i\left[\bar{u}_4(ig_e\gamma^\mu)\left(\frac{i(\not{p}_1 + \not{p}_2 + m)}{(p_1 + p_2)^2 - m^2}\right)(ig_e\gamma^\nu)u_1\right]\epsilon^*_{3\mu}\epsilon_{2\nu}
\end{aligned}
$$

Polarized Particles

- A typical QED amplitude might look something like

$$\mathcal{M} \sim \left[\bar{u}_1 \Gamma^\mu v_2\right] \epsilon_{3\mu}$$

The Feynman rules won't take us any further, but to get a number for $\mathcal{M}$ we will need to substitute explicit forms for the wavefunctions of the external particles: $\bar{u}_1$, v_2, and $\epsilon_{3\mu}$.

- If all external particles have a known polarization, this might be a reasonable way to calculate things. More often, though, we are interested in unpolarized particles.

Spin-Averaged Amplitudes

- If we do not care about the polarizations of the particles then we need to

  1. *Average* over the polarizations of the initial-state particles
  2. *Sum* over the polarizations of the final-state particles

  in the squared amplitude $|\mathcal{M}|^2$.

- We call this the spin-averaged amplitude and we denote it by

$$\left\langle |\mathcal{M}|^2 \right\rangle$$

- Note that the averaging over initial state polarizations involves summing over all polarizations and then dividing by the number of independent polarizations, so $\left\langle |\mathcal{M}|^2 \right\rangle$ involves a sum over the polarizations of *all* external particles.

Spin Sums

- Let's simplify things even further and suppose that we have

$$\mathcal{M} \sim [\bar{u}_1 \Gamma u_2]$$

Then

$$\begin{aligned}
|\mathcal{M}|^2 &\sim [\bar{u}_1 \Gamma u_2]\,[\bar{u}_1 \Gamma u_2]^* \\
&\sim [\bar{u}_1 \Gamma u_2]\left[u_1^\dagger \gamma^0 \Gamma u_2\right]^\dagger \\
&\sim [\bar{u}_1 \Gamma u_2]\left[u_2^\dagger \Gamma^\dagger \gamma^{0\dagger} u_1\right] \\
&\sim [\bar{u}_1 \Gamma u_2]\left[u_2^\dagger \gamma^0 \gamma^0 \Gamma^\dagger \gamma^0 u_1\right] \\
&\sim [\bar{u}_1 \Gamma u_2]\left[\bar{u}_2 \bar{\Gamma} u_1\right]
\end{aligned}$$

$$|\mathcal{M}|^2 \quad \sim \quad [\bar{u}_1 \Gamma u_2]\,[\bar{u}_2 \bar{\Gamma} u_1]$$

- Applying the completeness relation

$$\sum_{s_i=1,2} u_i^{s_i} \bar{u}_i^{s_i} = (\not{p}_i + m_i)$$

to $u_2 \bar{u}_2$ in the squared-amplitude above (summing over the spins of paticle 2),

$$\begin{aligned} \sum_{s_2} |\mathcal{M}|^2 \quad &\sim \quad [\bar{u}_1 \Gamma (\not{p}_2 + m_2) \bar{\Gamma} u_1] \\ &\sim \quad [\bar{u}_1 Q u_1] \end{aligned}$$

- The right-hand side is just a number, but if we represent the matrix multiplication with summations over indices, we can rewrite it as

$$\begin{aligned}
[\bar{u}_1 Q u_1] &= (\bar{u}_1)_i \, Q_{ij} \, (u_1)_j \\
&= Q_{ij} \, (u_1 \bar{u}_1)_{ji} \\
&= [Q \, (u_1 \bar{u}_1)]_{ii} \\
&= \mathrm{Tr} \, [Q(u_1 \bar{u}_1)]
\end{aligned}$$

- Finally, we apply the completeness relation once again, so that we get

$$\sum_{s_1} |\mathcal{M}|^2 \sim \mathrm{Tr} \, [Q(\not{p}_1 + m_1)]$$

- In total, we have

$$\begin{aligned} \mathcal{M} &\sim [\bar{u}_1 \Gamma u_2] \\ \Rightarrow \left\langle |\mathcal{M}|^2 \right\rangle &\sim \frac{1}{2} \operatorname{Tr} \left[\Gamma(\not{p}_2 + m_2) \bar{\Gamma}(\not{p}_1 + m_1) \right] \end{aligned}$$

The factor of $\frac{1}{2}$ is from the averaging over initial spins, assuming exactly one of u_1 and u_2 corresponds to an initial-state particle. If they are both in the initial state (e.g., pair annihilation), the factor is $\frac{1}{4}$. If neither is in the initial state (e.g., pair production), the factor is 1.

Casimir's Trick

- This procedure of calculating spin-averaged amplitudes in terms of traces is known as Casimir's Trick.

$$\sum_{\text{all spins}} \left[\bar{u}_a \Gamma_1 u_b\right] \left[\bar{u}_a \Gamma_2 u_b\right]^* = \mathrm{Tr}\left[\Gamma_1 (\not{p}_b + m_b) \bar{\Gamma}_2 (\not{p}_a + m_a)\right]$$

- If antiparticle spinors (v) are present in the spin sum, we use the corresponding completeness relation

$$\sum_{s_i=1,2} v_i^{s_i} \bar{v}_i^{s_i} = (\not{p}_i - m_i)$$

Traces

- Because of Casimir's Trick, we're going to find ourselves calculating a lot of traces involving γ-matrices.
- General identities about traces:

$$\begin{aligned} \mathrm{Tr}(A+B) &= \mathrm{Tr}(A) + \mathrm{Tr}(B) \\ \mathrm{Tr}(\alpha A) &= \alpha \mathrm{Tr}(A) \\ \mathrm{Tr}(AB) &= \mathrm{Tr}(BA) \\ \mathrm{Tr}(ABC) &= \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA) \end{aligned}$$

Building Blocks

- The two major identities that we will need in order to build more complicated trace identities are

$$
\begin{aligned}
g_{\mu\nu} g^{\mu\nu} &= 4 \\
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}(\times\mathbf{I})
\end{aligned}
$$

- You can show that $\gamma_\mu \gamma^\mu = 4$ and $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu\lambda}$. In a similar fashion, we find that

$$
\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\mu &= \gamma_\mu \left(2g^{\mu\nu} - \gamma^\mu \gamma^\nu\right) \\
&= 2\gamma^\nu - \gamma_\mu \gamma^\mu \gamma^\nu \\
&= 2\gamma^\nu - 4\gamma^\nu \\
&= -2\gamma^\nu
\end{aligned}
$$

Simple Trace Identities

- The simplest trace identity is: $\mathrm{Tr}(1) = 4$
- The trace of a single γ matrix is zero, as is the trace of *any* odd number of γ-matrices.
- For 2 γ-matrices,
$$\begin{aligned}
\mathrm{Tr}(\gamma^\mu \gamma^\nu) &= \mathrm{Tr}\left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right)/2 \\
&= \mathrm{Tr}(2g^{\mu\nu})/2 \\
&= g^{\mu\nu}\mathrm{Tr}(1) \\
&= 4g^{\mu\nu}
\end{aligned}$$
-

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4\left(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda}\right)$$

Traces With γ^5

- The vertex factor for weak interactions involves γ^5.
- By inspection, $\mathrm{Tr}(\gamma^5) = 0$.
- Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ (an even number of γ-matrices),

$$\begin{aligned} \mathrm{Tr}(\gamma^5\gamma^\mu) &= 0 \\ \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda) &= 0 \end{aligned}$$

- Also,

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$$

The Non-Trivial γ^5 Trace

- Only with 4 (or more) other γ-matrices can we obtain a nonzero trace involving γ^5:

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma}$$

where the totally antisymmetric tensor is defined as

$$\epsilon^{\mu\nu\lambda\sigma} \equiv \begin{cases} -1 & \text{for } \textit{even} \text{ permutations of } 0123 \\ +1 & \text{for } \textit{odd} \text{ permutations of } 0123 \\ 0 & \text{if any 2 indices are the same} \end{cases}$$

Contractions of the ϵ Tensor

- Since $\epsilon^{\mu\nu\lambda\sigma}$ is completely antisymmetric, we will get zero when we contract this with any tensor that is symmetric in 2 indices, such as $g^{\mu\nu}$ or $(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)$.
- Only contractions with another antisymmetric tensor survive:

$$\begin{aligned}
\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\lambda\sigma} &= -24 \\
\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\lambda\tau} &= -6\,\delta^\sigma_\tau \\
\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\theta\tau} &= -2\left(\delta^\lambda_\theta\delta^\sigma_\tau - \delta^\lambda_\tau\delta^\sigma_\theta\right) \\
&\vdots
\end{aligned}$$

Example 1

- One of the traces involved in Bhabha scattering is

$$T = \mathrm{Tr}\left[\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_3 + m)\right]$$

We can expand this out to create 4 terms, but 2 of these terms (the ones linear in m) will involve 3 γ-matrices, and are therefore zero. Thus,

$$\begin{aligned} T &= \mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) + m^2 \mathrm{Tr}(\gamma^\mu \gamma^\nu) \\ &= 4\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu - (p_1 \cdot p_3) g^{\mu\nu}\right) + 4m^2 g^{\mu\nu} \end{aligned}$$

This result will be contracted with another trace that is covariant (i.e., $_{\mu\nu}$ as opposed to contravariant $^{\mu\nu}$) in μ and ν.

Example 2

- It isn't always a joyous task to contract 2 traces together.
- Consider $$\mathcal{A} = \mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2)\, \mathrm{Tr}(\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_2)$$

  Evaluating the traces,

$$\begin{aligned} \mathcal{A} &= 4\left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1 \cdot p_2) g^{\mu\nu}\right] \\ &\quad \times 4\left[p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu} - (p_1 \cdot p_2) g_{\mu\nu}\right] \\ &= 16\left[2 p_1^2 p_2^2 + 2(p_1 \cdot p_2)^2 + 4(p_1 \cdot p_2)^2 - 4(p_1 \cdot p_2)^2\right] \\ &= 32\left[m_1^2 m_2^2 + (p_1 \cdot p_2)^2\right] \end{aligned}$$

Summary

- The Feynman rules for QED provide the recipe for translating Feynman diagrams into mathematical expressions for the amplitude.
- If we are interested in the spin-averaged amplitude $\left\langle |\mathcal{M}|^2 \right\rangle$ then we need not ever use explicit fermion spinors and photon polarization vectors.
- Instead, Casimir's Trick allows us to calculated spin-averaged amplitudes in terms of traces of γ-matrices.
- With practice, γ-matrix traces can be taken quite quickly.

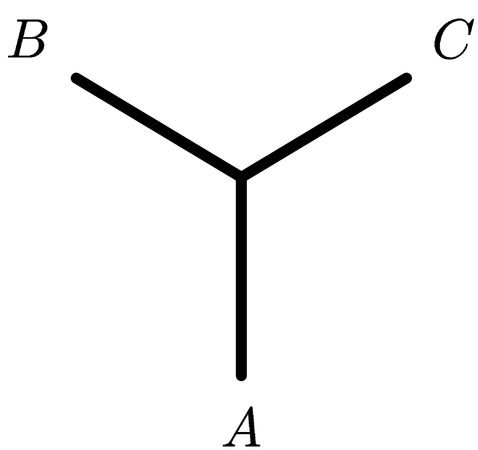
Last class we saw that..

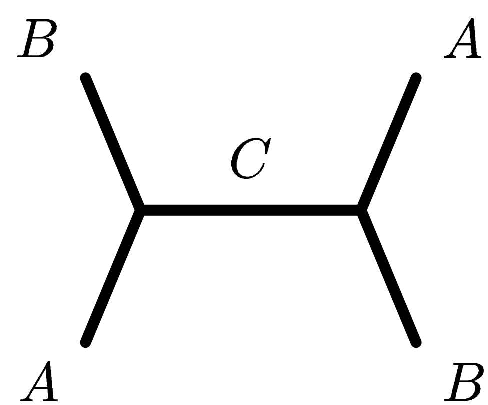
- ... the statistical mean lifetime of a particle is the inverse of the decay rate.
- ... the notion of a geometrical cross section can be generalized so as to incorporate a variety of classical scattering results and to carry over to the quantum realm of particle physics.
- ... Fermi's Golden Rule provides the prescription for combining dynamical information about the amplitude and kinematical information about the phase space in order to obtain observable quantities like decay rates and scattering cross sections.

Thanks to Ian Blockland and Randy Sobie for these slides

Feynman diagrams

- The graphical representation of an interaction has a 1-to-1 correspondence with the mathematical expression describing the amplitude
- The time axis in these diagrams is vertically upward (as in the Griffiths text)
- We'll use arrows to indicate particle/anti-particle later; in today's lecture all particles are their own antiparticles





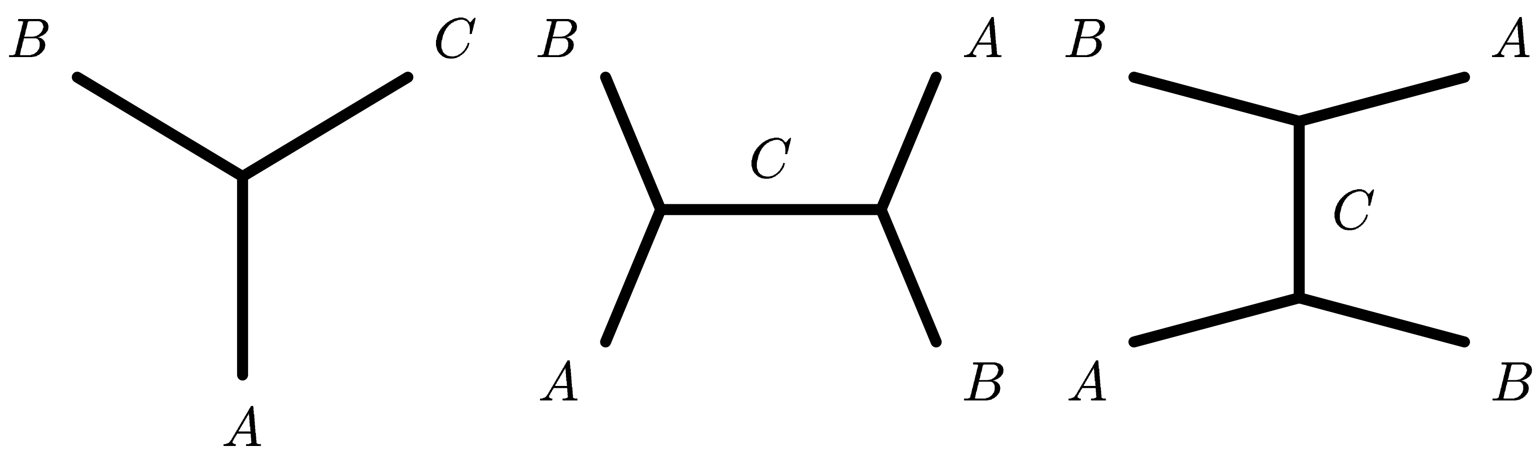
ABC **Theory**

This follows closely the development in Griffiths' text "Introduction to Elementary Particles"

- Feynman Diagrams
- Feynman Rules
- Calculating Decay Rates
- Calculating Cross Sections
- Higher-Order Diagrams

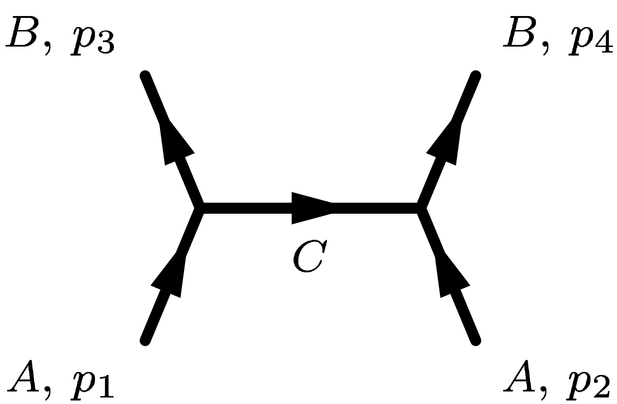
ABC Theory

- 3 spinless particles, A, B, and C, each of which is its own antiparticle.
- One only primitive vertex, with coupling g (dimensions of momentum).
- If $m_A > m_B + m_C$ then A can decay.



The Feynman Rules

- The Feynman rules provide the recipe for constructing an amplitude $\mathcal{M}$ from a Feynman diagram.
- **Step 1:** Draw the Feynman diagram(s) with the minimum number of vertices. There may be more than one.
- **Step 2:** Label the four-momentum of each line (with arrows), enforcing four-momentum conservation at every vertex.



$p_1, p_2, \ldots$ external momenta

$q_1, q_2, \ldots$ internal momenta

arrows indicate positive direction

- **Step 3:** Each vertex contributes a factor of $(-ig)$.
  Each internal line, with mass m and four-momentum q, contributes a propagator of $\frac{i}{q^2-m^2}$
- **Step 4:** Conserve 4-momentum at each vertex

$$(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)$$

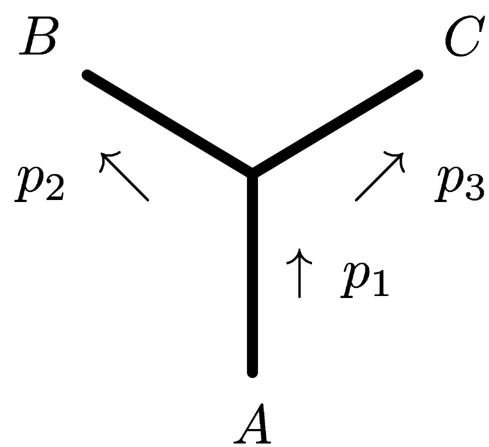
  where k_i are the momenta coming into the vertex.
- **Step 5** Form the amplitude

$$\mathcal{M} = i \text{ (vertex factors)(propagators)(momentum conservation)}$$

- **Step 6:** Integrate over the internal momenta $\frac{1}{(2\pi)^4}d^4q_j$
- **Step 7:** Drop the extra $(2\pi)^4\,\delta^{(4)}$-function and $\mathcal{M}$ remains.

Example: $A \to B + C$

- To lowest order ($\mathcal{O}(g)$), we have just one diagram:



- There is just one vertex and no propagators, therefore

$$\mathcal{M} = i(-ig) = g$$

Lifetime of A

- From Fermi's Golden Rule, the decay rate is given by

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi m_1^2} \, |\mathcal{M}|^2$$

  in the rest frame of A

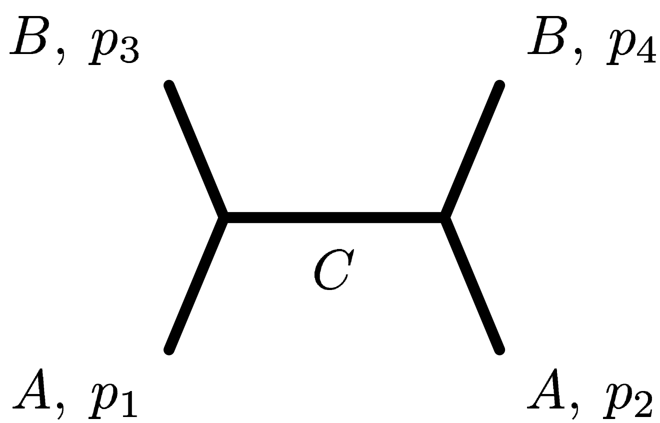
- With $S = 1$, $m_1 = m_A$, $\mathcal{M} = g$, and $|\mathbf{p}|$ representing the magnitude of the spatial momentum of either B or C, we find that

$$\tau_A = \frac{8\pi m_A^2}{g^2|\mathbf{p}|}$$

- Note that g has dimensions of mass.

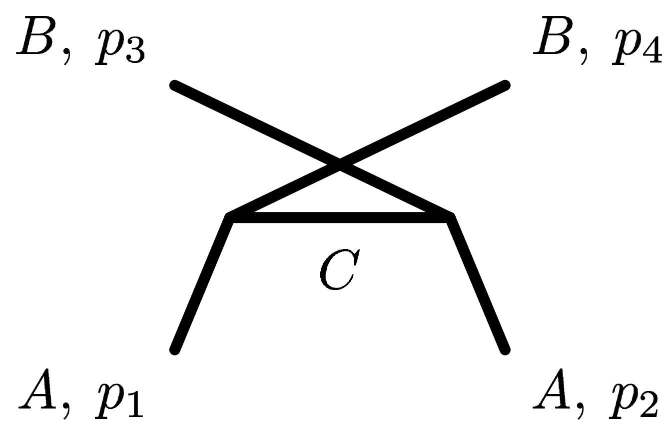
Example: $A + A \to B + B$

- To lowest order ($\mathcal{O}(g^2)$), we have two diagrams:



t-channel

mom. transf. $= t = (p_1 - p_3)^2$



u-channel

mom. transf. $= u = (p_1 - p_4)^2$

$$\mathcal{M}_t = i(-ig)^2 \frac{i}{(p_1 - p_3)^2 - m_C^2} = \frac{g^2}{t - m_C^2}$$

Cross Section for $A + A \to B + B$

$$\mathcal{M} = \frac{g^2}{t - m_C^2} + \frac{g^2}{u - m_C^2}$$

- Notice that $\mathcal{M}$ is Lorentz invariant. This is *always* true.
- To convert $\mathcal{M}$ to a cross section, we use Fermi's Golden Rule. In the CM frame,

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{S\,|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

$S = 1/2$ because we have two identical particles ($B + B$) in the final state. Also, $E_1 = E_2 = E$.

$$\frac{d\sigma}{d\Omega} = \frac{1}{2(16\pi E)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} \left| \frac{g^2}{t - m_C^2} + \frac{g^2}{u - m_C^2} \right|^2$$

- To take this calculation further, let's assume that $m_A = m_B = m$ and $m_C = 0$. Then $|\mathbf{p}_f| = |\mathbf{p}_i|$ and

$$\begin{aligned} t = (p_1 - p_3)^2 &= -2\mathbf{p}^2(1 - \cos\theta) \\ u = (p_1 - p_4)^2 &= -2\mathbf{p}^2(1 + \cos\theta) \end{aligned}$$

$$\Rightarrow \; \frac{d\sigma}{d\Omega} = \frac{1}{2} \left(\frac{g^2}{16\pi E \mathbf{p}^2 \sin^2\theta} \right)^2$$

- Note that $\sigma \to \infty$, just as for Rutherford scattering.

Higher-Order Diagrams

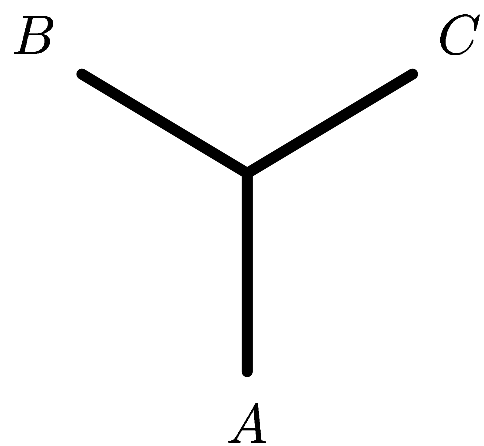
- By considering more complicated Feynman diagrams, we can generate additional contributions to the amplitude:

$$\begin{aligned}\mathcal{M}_{A\to B+C} &= g\mathcal{A}_1 + g^3\mathcal{A}_3 + g^5\mathcal{A}_5 + \ldots \\ \mathcal{M}_{A+A\to B+B} &= g^2\mathcal{A}_2 + g^4\mathcal{A}_4 + g^6\mathcal{A}_6 + \ldots\end{aligned}$$

  If $g \ll 1$ (or, more precisely, $(g/m_A) \ll 1$ in ABC Theory), we can see how each successive term in the *perturbation series* provides smaller and smaller corrections to the amplitude.

Corrections to τ_A

- We have already calculated the lifetime of A due to the simple vertex diagram

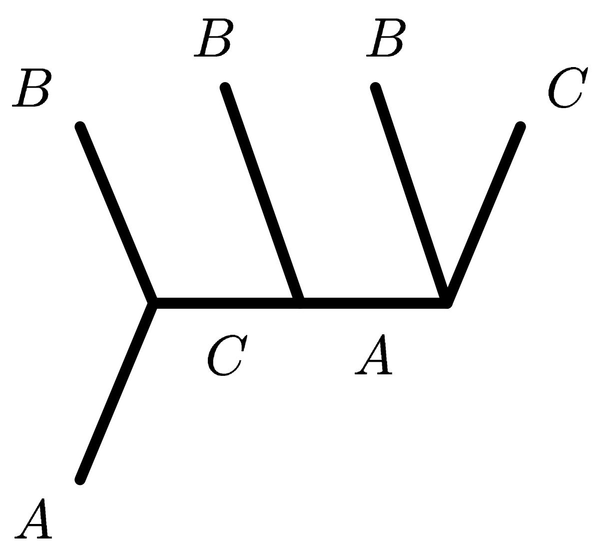


- Since $\mathcal{M} \sim g,\ \Gamma \sim g^2$. The leading corrections to Γ will be $\mathcal{O}(g^4)$ and they will arise from the interference of the $\mathcal{O}(g)$ diagram with a $\mathcal{O}(g^3)$ diagram in the coherent sum:

$$|\mathcal{M}|^2 = \left|g\mathcal{A}_1 + g^3\mathcal{A}_3 + \ldots\right|^2$$

Other Decay Modes

- Note that we are only interested in the $\mathcal{O}(g^3)$ diagrams in which $A \to B + C$. If A is sufficiently heavy, other decay modes such as $A \to 3B + C$ and $A \to B + 3C$ are possible.



Incoherent Sums

- Since $A \to 3B + C$ is a distinct decay mode, we calculate its Γ separately from that of $A \to B + C$. As a result, even though $\mathcal{M} \sim g^3,\ \Gamma \sim g^6$, and so we need not consider these diagrams for a $\Gamma \sim g^4$ calculation.

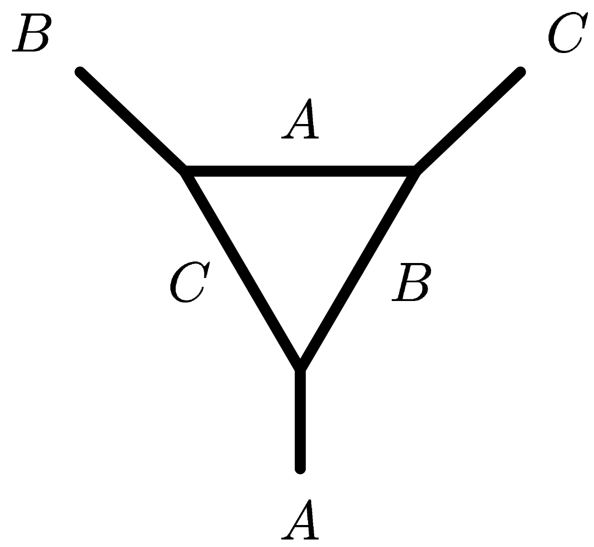
- Terminology: The decay of A involves both *coherent* and *incoherent* sums.

$$\begin{aligned}
\Gamma(A \to anything) &= \Gamma_{A\to BC} + \Gamma_{A\to BBBC} + \Gamma_{A\to BCCC} + \ldots \\
&= C_1 \left|\sum \mathcal{M}_{A\to BC}\right|^2 + C_2 \left|\sum \mathcal{M}_{A\to BBBC}\right|^2 \\
&\quad + C_3 \left|\sum \mathcal{M}_{A\to BCCC}\right|^2 + \ldots
\end{aligned}$$

(C_1, C_2, and C_3 arise from Fermi's Golden Rule.)

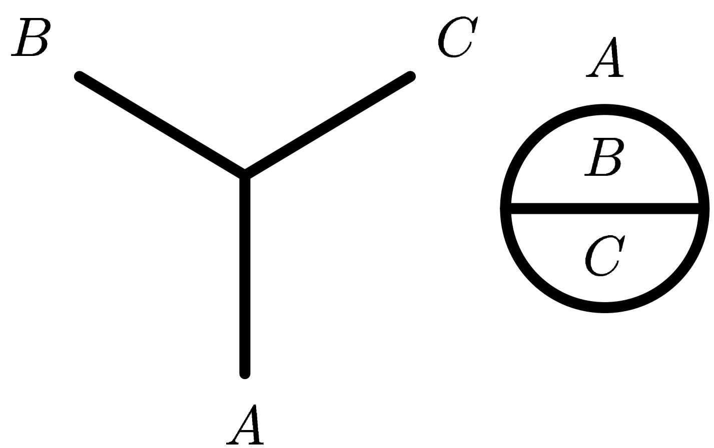
Third-Order $A \to B + C$ Diagrams

- There is one legal third-order diagram to consider:

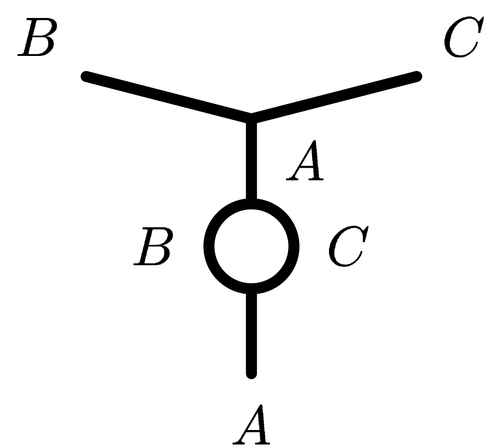


Illegal Loop Diagrams

- There are several other $\mathcal{O}(g^3)$ diagrams that can be drawn for $A \to B + C$, however these are not to be calculated using the Feynman Rules.



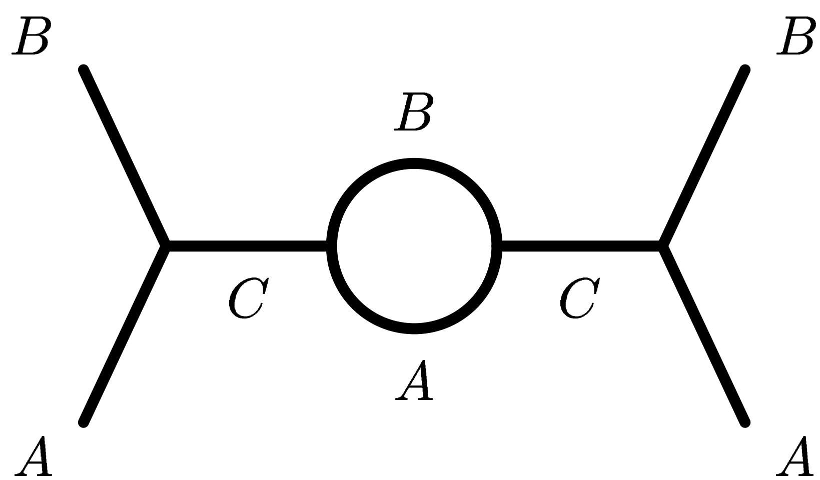
Disconnected



Reducible

Corrections to $A + A \to B + B$

- The interference of the *one-loop* diagrams ($\mathcal{O}(g^4)$) with the tree-level diagram ($\mathcal{O}(g^2)$) provides $\mathcal{O}(g^6)$ corrections to the cross section.



- If you go through this calculation (see Griffiths) you'll find that the amplitude associated with this diagram is *divergent*.

Renormalization

- All divergences in the final physical observables (Γ or σ) seem to be affiliated with the coupling constants and the masses.
- In other words, we can define *renormalized* couplings and masses which absorb the divergences. We then assume that it is these renormalized parameters which we have been measuring all along. This would mean that the *bare* parameters aren't physical.
- Renormalization is a feature of all quantum field theories, including those found outside of particle physics.

Summary

- Feynman diagrams provide a convenient and intuitive representation of particle interactions.
- The Feynman rules allow us to translate Feynman diagrams into mathematical expressions for the amplitudes.
- ABC Theory is a toy theory which makes it easier to learn how to use Feynman rules without having to worry about some of the complexities which accompany more realistic theories.
- Sometimes higher-order Feynman diagrams are needed to improve the precision of a calculation. This is a minefield, but maps are available.