Spin 1

Rui Santos

FCUL & CFTC

2026

Maxwell Equations

$$\nabla \cdot \vec{E} = \rho \qquad \text{(Gauss' law)}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} \qquad \text{(Ampère–Maxwell law)}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \text{(Faraday's law)}$$

Four-Potential

Define the electromagnetic four-potential:

$$A^\mu = (A^0, \vec{A}) = (V, \vec{A})$$

Lowered index:

$$A_\mu = (A_0, -\vec{A})$$

- A^0 is the scalar potential
- $\vec{A}$ is the vector potential

The electromagnetic field tensor is defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Antisymmetric tensor:

$$F_{\mu\nu} = -F_{\nu\mu}$$

- Contains both $\vec{E}$ and $\vec{B}$

Raising and Lowering Field Components

Using the metric:

$$E^i = E_i, \qquad B^i = B_i$$

But:

$$E^i = -E_i \quad \text{(when appearing as } F^{0i}\text{)}$$

Electric field example:

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 = -\frac{\partial A^1}{\partial t} - \frac{\partial A_0}{\partial x_1} = E^1$$

Magnetic field example:

$$F_{23} = \partial_2 A_3 - \partial_3 A_2 = -B^1$$

Fields from the Vector Potential

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla A_0 - \frac{\partial \vec{A}}{\partial t}$$

- Definitions consistent with $F_{\mu\nu}$
- Gauge invariant expressions

Field Tensor Matrix

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

The dual tensor is defined as:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \qquad \tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu}$$

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}$$

- Electric and magnetic fields exchange roles

Lorentz Invariants

Two independent Lorentz scalars can be constructed:

$$F_{\mu\nu}F^{\mu\nu} = -2\left(\vec{E}^2 - \vec{B}^2\right)$$

$$\tilde{F}_{\mu\nu}F^{\mu\nu} = -4\vec{E}\cdot\vec{B}$$

These combinations are invariant under Lorentz transformations.

Deriving Maxwell's Equations from the Classical Lagrangian

Step 1: Classical Lagrangian for Electromagnetism

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

where the **field strength tensor** is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and $A_\mu = (\phi, \mathrm{A})$ is the 4-vector potential, $j_\mu = (\rho, \mathrm{J})$ the 4-current.

Step 2: Euler-Lagrange Equations for Fields

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

Compute derivatives:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}, \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = -j^\nu$$

Step 3: Euler-Lagrange gives Maxwell equations

$$\partial_\mu F^{\mu\nu} = j^\nu$$

This is the **inhomogeneous Maxwell equations**:

- $\nu = 0 \Rightarrow \nabla \cdot \mathrm{E} = \rho$
- $\nu = i \Rightarrow \nabla \times \mathrm{B} - \frac{\partial \mathrm{E}}{\partial t} = \mathrm{J}$

Step 4: Homogeneous Maxwell Equations from $F_{\mu\nu}$

Definition of field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Antisymmetric: $F_{\mu\nu} = -F_{\nu\mu}$ - Encodes E and B:

$$E_i = F_{0i}, \quad B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$$

Step 4a: Bianchi identity (geometric identity)

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

- Automatically satisfied for any $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ - Independent of charges or currents

Step 4b: Extract the homogeneous Maxwell equations

1. $\nabla \cdot \mathrm{B} = 0$

Take $\mu = i, \nu = j, \lambda = k$ (spatial indices). Using $B_i = \frac{1}{2}\epsilon_{ijk} F_{jk}$:

$$\epsilon_{ijk} \partial_i F_{jk} = 2\nabla \cdot \mathrm{B} = 0$$

2. $\nabla \times \mathrm{E} + \partial_t \mathrm{B} = 0$

Take $\mu = 0, \nu = i, \lambda = j$:

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0$$

Substitute $F_{0i} = E_i$, $F_{ij} = -\epsilon_{ijk} B_k$:

$$\partial_t B_k + (\nabla \times \mathrm{E})_k = 0 \quad \Rightarrow \quad \nabla \times \mathrm{E} + \frac{\partial \mathrm{B}}{\partial t} = 0$$

The homogeneous equations arise automatically from the geometric structure of $F_{\mu\nu}$, independent of the dynamics (no sources needed).

Geometric Interpretation of the Bianchi Identity

Electromagnetic field as a curl of a potential:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \Longleftrightarrow \quad F = dA$$

Bianchi Identity:

$$\boxed{dF = 0 \quad \text{or equivalently} \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0}$$

Geometric meaning:

- The electromagnetic field F is a "curl" in spacetime.
- Magnetic field lines are continuous loops $\nabla \cdot \mathrm{B} = 0$
- Time-varying magnetic fields induce electric fields $\nabla \times \mathrm{E} + \partial_t \mathrm{B} = 0$
- Locally: there are no monopoles or "holes" in the field because F is exact ($F = dA$)

- Magnetic field lines form closed loops. - Induced electric field circulates around changing magnetic flux. - This is the physical content of the Bianchi identity.

Step 5: Summary

$$\begin{cases} \nabla \cdot \mathsf{E} = \rho \\ \nabla \times \mathsf{B} - \partial_t \mathsf{E} = \mathsf{J} \\ \nabla \cdot \mathsf{B} = 0 \\ \nabla \times \mathsf{E} + \partial_t \mathsf{B} = 0 \end{cases}$$

All Maxwell equations derived from the classical Lagrangian.

Including Sources

Adding interaction with a conserved current:

$$\mathcal{L}_{\text{int}} = -J^\mu A_\mu$$

Total Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu$$

Show that

$$\Rightarrow \quad \partial_\mu F^{\mu\nu} = J^\nu$$

applying the equations of motion.

Current Conservation

Taking the divergence:

$$\partial_\alpha \partial_\mu F^{\mu\alpha} = \partial_\alpha J^\alpha$$

Antisymmetry

$$F^{\mu\alpha} = -F^{\alpha\mu} \quad \Rightarrow \quad \partial_\alpha \partial_\mu F^{\mu\alpha} = 0$$

$$\Rightarrow \quad \partial_\alpha J^\alpha = 0$$

- Charge conservation emerges automatically
- Compatibility condition for Maxwell equations

Four-Current

The conserved four-current is:

$$J^\mu = (\rho, \vec{J})$$

- Time component: charge density
- Spatial components: current density

$$\partial_\mu J^\mu = 0 \quad \Longleftrightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

Gauge Invariance

Consider the gauge transformation:

$$A^\mu \longrightarrow A^\mu + \partial^\mu \phi$$

- $\phi(x)$ is an arbitrary scalar function
- Physical observables must be gauge invariant

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu} \longrightarrow (\partial_\mu A_\nu + \partial_\mu \partial_\nu \phi) - (\partial_\nu A_\mu + \partial_\nu \partial_\mu \phi) = F_{\mu\nu}$$

Physical Interpretation

- Different potentials A^μ describe the same physics
- Only gauge-invariant quantities are observable
- Gauge symmetry is a redundancy of descriptio
- Maxwell equations follow from a variational principle
- Charge conservation is automatic

- Gauge invariance underlies electromagnetism
- This structure generalizes to non-Abelian gauge theories

This is the prototype of all gauge theories in particle physics.

Connections and Curvature

In modern physics many interactions are described by a **connection**.

General Relativity
Spacetime derivative of a vector:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho$$

Connection: Christoffel symbols $\Gamma^\nu_{\mu\rho}$
Curvature:

$$R^\rho{}_{\sigma\mu\nu}$$

Electromagnetism
Gauge-covariant derivative:

$$D_\mu = \partial_\mu + ieA_\mu$$

Connection: electromagnetic potential A_μ
Field strength:

$$F_{\mu\nu}$$

Curvature from Commutators

Curvature appears when covariant derivatives do not commute.

Electromagnetism

$$[D_\mu, D_\nu]\psi = ieF_{\mu\nu}\psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

General Relativity

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma$$

Both theories interpret physical fields as **curvature of a connection**.

Bianchi Identities

Curvature tensors satisfy geometric identities.

Electromagnetism

$$\partial_{[\lambda} F_{\mu\nu]} = 0$$

This gives the homogeneous Maxwell equations:

$$\nabla \cdot \mathrm{B} = 0 \qquad \nabla \times \mathrm{E} + \partial_t \mathrm{B} = 0$$

General Relativity

$$\nabla_{[\lambda} R_{\mu\nu]\rho\sigma} = 0$$

Contracting indices gives

$$\nabla_\mu G^{\mu\nu} = 0$$

which implies conservation of energy–momentum.

Problems 1 and 2

Problem 1: Field Strength Tensor
The electromagnetic tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

1. Show that $F_{\mu\nu}$ is antisymmetric.
2. Write the matrix form of $F_{\mu\nu}$ in terms of E and B.
3. How many independent components does $F_{\mu\nu}$ have?

Problem 2: Maxwell Equations from Tensor Form
Starting from

$$\partial_\mu F^{\mu\nu} = j^\nu$$

1. Show that $\nu = 0$ gives Gauss's law:
$$\nabla \cdot \mathrm{E} = \rho$$
2. Show that $\nu = i$ gives the Ampère–Maxwell law:
$$\nabla \times \mathrm{B} - \frac{\partial \mathrm{E}}{\partial t} = \mathrm{J}$$

Problems 3 and 4

Problem 3: Homogeneous Maxwell Equations
Consider the identity

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

1. Show that it follows from the definition of $F_{\mu\nu}$.
2. Show that it implies
$$\nabla \cdot \mathrm{B} = 0$$
3. Show that it implies
$$\nabla \times \mathrm{E} + \frac{\partial \mathrm{B}}{\partial t} = 0$$

Problem 4: Dual Field Tensor
Define

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

1. Express $\tilde{F}^{\mu\nu}$ in terms of E and B.
2. Show that the homogeneous Maxwell equations become
$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

Problems 5 and 6

Problem 5: Lorentz Invariants
Show that the following are Lorentz scalars:

$$F_{\mu\nu}F^{\mu\nu}, \qquad \tilde{F}_{\mu\nu}F^{\mu\nu}$$

Express them in terms of E and B.

Problem 6: Current Conservation
Starting from

$$\partial_\mu F^{\mu\nu} = j^\nu$$

1. Show that
$$\partial_\nu j^\nu = 0$$
2. Interpret this equation physically.

Problems 7 and 8

Problem 7: Maxwell Equations from the Lagrangian
Consider

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

1. Apply the Euler–Lagrange equations to A_μ.
2. Show that the equations of motion are

$$\partial_\mu F^{\mu\nu} = j^\nu$$

Problem 8: Gauge Transformations
Consider

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

1. Show that $F_{\mu\nu}$ is invariant.
2. Show that the Lagrangian changes by a total derivative.

Problems 9 and 10

Problem 9: Plane Wave Solutions
In vacuum ($j^\mu = 0$) consider

$$A^\mu(x) = \epsilon^\mu e^{-ik\cdot x}$$

1. Insert into Maxwell equations and find the condition on k^μ.
2. Show that
$$k_\mu \epsilon^\mu = 0$$

Problem 10: Energy-Momentum Tensor
The electromagnetic tensor is

$$T^{\mu\nu} = F^{\mu\alpha} F^\nu_{\ \alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

1. Show that it is conserved in vacuum.
2. Compute T^{00} and interpret the result.

Solutions

Problem 1

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Interchanging indices:

$$F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu = -F_{\mu\nu}$$

Thus $F_{\mu\nu}$ is antisymmetric.
Matrix form:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Since the tensor is antisymmetric in 4 dimensions:

$$N = \frac{4(4-1)}{2} = 6$$

independent components.

Problem 2
Starting from

$$\partial_\mu F^{\mu\nu} = j^\nu$$

For $\nu = 0$:

$$\partial_i F^{i0} = \rho$$

Using $F^{i0} = E^i$:

$$\nabla \cdot \mathrm{E} = \rho$$

For $\nu = i$:

$$\partial_0 F^{0i} + \partial_j F^{ji} = J^i$$

Using $F^{0i} = -E^i$ and $F^{ij} = -\epsilon^{ijk} B_k$:

$$\nabla \times \mathrm{B} - \frac{\partial \mathrm{E}}{\partial t} = \mathrm{J}$$

Solutions

Problem 3
Using

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

compute

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}$$

Substituting and using commutativity of derivatives ($\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$) all terms cancel. Thus

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

From spatial components:

$$\nabla \cdot \mathrm{B} = 0$$

and from mixed components:

$$\nabla \times \mathrm{E} + \frac{\partial \mathrm{B}}{\partial t} = 0$$

Problem 4
Dual tensor:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

Components:

$$\tilde{F}^{0i} = B^i$$

$$\tilde{F}^{ij} = \epsilon^{ijk}E_k$$

Then

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

reproduces

$$\nabla \cdot \mathrm{B} = 0$$

and

$$\nabla \times \mathrm{E} + \partial_t \mathrm{B} = 0$$

Solutions

Problem 5
First invariant:

$$F_{\mu\nu}F^{\mu\nu}$$

Substituting components:

$$F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2)$$

Second invariant:

$$\tilde{F}_{\mu\nu}F^{\mu\nu}$$

which gives

$$\tilde{F}_{\mu\nu}F^{\mu\nu} = -4\mathrm{E}\cdot\mathrm{B}$$

Both are Lorentz scalars.

Problem 6
Take divergence of Maxwell equation:

$$\partial_\nu(\partial_\mu F^{\mu\nu}) = \partial_\nu j^\nu$$

Left side vanishes because

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0$$

(due to antisymmetry of $F^{\mu\nu}$).
Thus

$$\partial_\nu j^\nu = 0$$

which is the **continuity equation** expressing conservation of electric charge.

Solutions

Problem 7 Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

Euler–Lagrange equation:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

Compute derivatives:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = -j^\nu$$

Thus

$$\partial_\mu F^{\mu\nu} = j^\nu$$

Problem 8
Gauge transformation:

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

Then

$$F_{\mu\nu} \to \partial_\mu(A_\nu + \partial_\nu \Lambda) - \partial_\nu(A_\mu + \partial_\mu \Lambda)$$

$$= F_{\mu\nu}$$

Thus $F_{\mu\nu}$ is gauge invariant.
The Lagrangian changes by

$$\delta\mathcal{L} = -j_\mu \partial^\mu \Lambda$$

which is a total derivative.

Solutions

Problem 9
Assume

$$A^{\mu} = \epsilon^{\mu} e^{-ik\cdot x}$$

Insert into vacuum equation:

$$\partial_{\mu} F^{\mu\nu} = 0$$

giving

$$k^2 = 0$$

Thus electromagnetic waves travel at the speed of light.
Gauge condition:

$$k_{\mu} \epsilon^{\mu} = 0$$

meaning the polarization is transverse.

Problem 10
Energy–momentum tensor:

$$T^{\mu\nu} = F^{\mu\alpha} F^{\nu}_{\ \alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

Using Maxwell equations one finds

$$\partial_\mu T^{\mu\nu} = 0$$

so energy–momentum is conserved.
For $\nu = 0$:

$$T^{00} = \frac{1}{2}(E^2 + B^2)$$

which is the electromagnetic energy density.