Noether's Theorem

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Noether's Theorem for a scalar field

Consider a real scalar field $\phi(x)$, with action

$$S[\phi] = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi, x)$$

We consider a **continuous transformation**:

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu$$

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)$$

Two types of changes occur:

- change of coordinates
- change of field value

Goal: determine consequences of invariance of the action.

Total and Intrinsic Field Variations

Expanding

$$\phi'(x') = \phi'(x) + \delta x^\mu \partial_\mu \phi' = \phi'(x) + \delta x^\mu \partial_\mu \phi + O(\epsilon^2)$$

we

$$\delta\phi(x) = \phi'(x) - \phi(x) = \underbrace{\delta_0\phi}_{\text{intrinsic change}} - \delta x^\mu \partial_\mu \phi$$

where

$$\delta_0\phi = \phi'(x') - \phi(x)\,.$$

Field Transformation Under Infinitesimal Coordinate Shift

Consider an infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} + \delta x^{\mu}.$$

Step 1: Taylor expand the transformed field

$$\phi'(x') = \phi'(x + \delta x) = \phi'(x) + \delta x^{\mu} \partial_{\mu} \phi'(x) + \mathcal{O}(\delta x^2).$$

Step 2: Define the intrinsic field variation

$$\delta\phi(x) = \phi'(x) - \phi(x) \quad \Rightarrow \quad \phi'(x) = \phi(x) + \delta\phi(x).$$

Step 3: Substitute

$$\phi'(x') = \phi(x) + \delta\phi(x) + \delta x^{\mu} \partial_{\mu}[\phi(x) + \delta\phi(x)].$$

Step 4: Keep only first-order terms
Since both δx^{μ} and $\delta\phi$ are infinitesimal,

$$\delta x^{\mu} \partial_{\mu} \delta\phi = \mathcal{O}(\epsilon^2)$$

and can be neglected.

$$\boxed{\phi'(x') = \phi(x) + \delta\phi(x) + \delta x^{\mu} \partial_{\mu} \phi(x)}$$

Variation of the Action

Action:

$$S = \int d^4x \, \mathcal{L}$$

Under coordinate transformation

$$d^4x \to d^4x'$$

Jacobian:

$$d^4x' = d^4x(1 + \partial_\mu \delta x^\mu)$$

Thus

$$\delta S = \int d^4x \, [\delta\mathcal{L} + \mathcal{L}\partial_\mu \delta x^\mu]$$

Goal: compute $\delta\mathcal{L}$.

Transformation of the Measure

Consider an infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} + \delta x^{\mu}(x).$$

The integration measure transforms with the Jacobian:

$$d^4x' = d^4x \ \det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right).$$

Compute the Jacobian matrix:

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \frac{\partial}{\partial x^{\nu}}\left(x^{\mu} + \delta x^{\mu}\right) = \delta^{\mu}_{\ \nu} + \partial_{\nu}\delta x^{\mu}.$$

For an infinitesimal transformation we expand the determinant:

$$\det\left(\delta^{\mu}_{\ \nu} + \partial_{\nu}\delta x^{\mu}\right) = 1 + \mathrm{Tr}(\partial_{\nu}\delta x^{\mu}) = 1 + \partial_{\mu}\delta x^{\mu}.$$

Therefore

$$\boxed{d^4x' = d^4x\left(1 + \partial_{\mu}\delta x^{\mu}\right)}.$$

Variation of the Lagrangian

Using functional dependence

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi, x)$$

Variation:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu$$

Also

$$\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi) - (\partial_\nu \phi)\partial_\mu \delta x^\nu$$

Substitute total field variation

$$\delta\phi = \delta_0\phi - \delta x^\nu \partial_\nu \phi$$

Variation of the Derivative of a Field

Infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu$$

Field variation

$$\delta\phi(x) = \phi'(x) - \phi(x)$$

Step 1: write the variation

$$\delta(\partial_\mu\phi) = \partial'_\mu\phi'(x') - \partial_\mu\phi(x)$$

Step 2: transform the derivative

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu = (\delta^\nu_\mu - \partial_\mu \delta x^\nu)\partial_\nu$$

Step 3: keep first-order terms

$$\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi) - (\partial_\nu\phi)\partial_\mu\delta x^\nu$$

$$\boxed{\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi) - (\partial_\nu\phi)\partial_\mu\delta x^\nu}$$

Rewriting the Action Variation

After substitution and rearranging:

$$\delta S = \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta_0 \phi \right]$$

$$+ \int d^4x \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_0 \phi + \mathcal{L} \delta x^\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \delta x^\nu \right]$$

First term vanishes using the Euler–Lagrange equation.

Euler–Lagrange Equations

Equation of motion:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

Thus on-shell

$$\delta S = \int d^4 x \, \partial_\mu (\cdots)$$

If the transformation is a symmetry

$$\delta S = 0$$

Therefore

$$\partial_\mu J^\mu = 0$$

where the current J^μ emerges from the boundary term.

General Noether Current

The conserved current is

$$\boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \mathcal{L}\delta x^\mu}$$

Conservation law:

$$\boxed{\partial_\mu J^\mu = 0}$$

This is the **most general Noether current** for transformations involving both

- internal field transformations
- spacetime transformations

Noether's Theorem (Final Statement)

If the action is invariant under a continuous transformation

$$x^\mu \to x^\mu + \delta x^\mu$$

$$\phi(x) \to \phi(x) + \delta\phi(x)$$

then there exists a conserved current

$$\boxed{\partial_\mu J^\mu = 0}$$

and a conserved charge

$$Q = \int d^3x\, J^0 \qquad \frac{dQ}{dt} = 0$$

Continuous symmetry ⇒ conserved quantity. Show that this charge is conserved

Time Evolution of the Charge

Compute the time derivative:

$$\frac{dQ}{dt} = \frac{d}{dt} \int d^3x \, J^0 = \int d^3x \, \partial_0 J^0$$

Using the continuity equation

$$\partial_\mu J^\mu = \partial_0 J^0 + \nabla \cdot \mathrm{J} = 0$$

we obtain

$$\partial_0 J^0 = -\nabla \cdot \mathrm{J}$$

Thus

$$\frac{dQ}{dt} = -\int d^3x \, \nabla \cdot \mathrm{J}$$

Using Gauss's Theorem

Apply Gauss's theorem:

$$\int d^3x \, \nabla \cdot \mathsf{J} = \oint d\mathsf{S} \cdot \mathsf{J}$$

Thus

$$\frac{dQ}{dt} = -\oint d\mathsf{S} \cdot \mathsf{J}$$

If the current vanishes sufficiently fast at spatial infinity

$$\mathsf{J} \to 0$$

then the surface term vanishes.

Conserved Charge

Under the assumption that the current vanishes at infinity:

$$\oint d\mathsf{S} \cdot \mathsf{J} = 0$$

Therefore

$$\boxed{\frac{dQ}{dt} = 0}$$

Thus

$$Q = \int d^3x \, J^0$$

is a conserved quantity.

Energy–Momentum Tensor from Noether's Theorem

We apply Noether's theorem to **spacetime translations**.

$$x^\mu \to x'^\mu = x^\mu + a^\mu$$

where a^μ is a constant infinitesimal parameter.

Physical meaning

- Time translation → energy conservation
- Spatial translations → momentum conservation

Goal: derive the conserved current associated with this symmetry.

Field Transformation Under Translations

Under a translation the scalar field satisfies

$$\phi'(x') = \phi(x)$$

At the same spacetime point x:

$$\phi'(x) = \phi(x - a)$$

Expand to first order:

$$\phi'(x) = \phi(x) - a^\nu \partial_\nu \phi(x)$$

Thus the variation is

$$\delta\phi(x) = \phi'(x) - \phi(x) = -a^\nu \partial_\nu \phi$$

The coordinate variation is

$$\delta x^\mu = a^\mu$$

General Noether Current

For a transformation

$$x^\mu \to x^\mu + \delta x^\mu$$

$$\phi \to \phi + \delta\phi$$

Noether's theorem gives

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \mathcal{L}\delta x^\mu$$

Insert the translation variations:

$$\delta\phi = -a^\nu \partial_\nu \phi$$

$$\delta x^\mu = a^\mu$$

Derivation of the Energy–Momentum Tensor

Substitute into the current:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(-a^\nu \partial_\nu \phi) + \mathcal{L} a^\mu$$

Factor out a^ν:

$$J^\mu = a^\nu \left(-\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \delta^\mu{}_\nu \mathcal{L} \right)$$

Define the tensor

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L}$$

Thus

$$J^\mu = a^\nu T^\mu_\nu$$

Conservation Law

Noether's theorem states

$$\partial_\mu J^\mu = 0$$

Using

$$J^\mu = a^\nu T^\mu_{\ \nu}$$

and noting that a^ν is constant:

$$a^\nu \partial_\mu T^\mu_{\ \nu} = 0$$

Since this holds for arbitrary a^ν,

$$\boxed{\partial_\mu T^\mu_{\ \nu} = 0}$$

This expresses conservation of energy and momentum.

Canonical Energy–Momentum Tensor

The tensor obtained is the **canonical energy–momentum tensor**

$$\boxed{T^{\mu}_{\ \nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \delta^{\mu}_{\ \nu}\mathcal{L}}$$

Components:

$$T^{00} = \text{energy density}$$

$$T^{0i} = \text{momentum density}$$

$$T^{ij} = \text{stress tensor}$$

Conserved quantities:

$$P_\nu = \int d^3x \, T^{0}_{\ \nu}$$

Example: Klein–Gordon Field

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

Compute

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$$

Thus

$$T^\mu_{\ \nu} = \partial^\mu\phi\,\partial_\nu\phi - \delta^\mu_{\ \nu}\mathcal{L}$$

This tensor satisfies

$$\partial_\mu T^\mu_{\ \nu} = 0$$

which expresses energy–momentum conservation.

Example: $U(1)$ Symmetry of a Complex Scalar Field

Consider a complex scalar field

$$\phi(x), \qquad \phi^*(x)$$

Lagrangian density:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

This theory is invariant under the **global** $U(1)$ **transformation**

$$\phi \to e^{i\alpha}\phi \qquad \phi^* \to e^{-i\alpha}\phi^*$$

where α is a constant parameter.

Derive the conserved Noether current.

$$J^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i$$

Infinitesimal Transformation

For small α

$$e^{i\alpha} \approx 1 + i\alpha$$

Thus

$$\delta\phi = i\alpha\phi$$

$$\delta\phi^* = -i\alpha\phi^*$$

Since the symmetry is internal

$$\delta x^\mu = 0$$

Therefore we can use the internal symmetry form of the Noether current.

Noether Current Formula

For multiple fields the Noether current is

$$J^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i$$

Here the independent fields are

$$\phi, \qquad \phi^*$$

Thus

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \delta\phi^*$$

Compute Functional Derivatives

From the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

we obtain

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^*$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi$$

Substitute into the current.

Deriving the Current

Substitute the variations:

$$J^\mu = (\partial^\mu \phi^*)(i\alpha\phi) + (\partial^\mu \phi)(-i\alpha\phi^*)$$

Factor out $i\alpha$:

$$J^\mu = i\alpha\left(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right)$$

Since α is arbitrary we define

$$\boxed{J^\mu = i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)}$$

Current Conservation

Using the equations of motion show that the current is conserved

$$(\Box + m^2)\phi = 0$$

$$(\Box + m^2)\phi^* = 0$$

one finds

$$\partial_\mu J^\mu = 0$$

Thus the current is conserved.

Conserved Charge

Define the conserved charge

$$Q = \int d^3x \, J^0$$

Using

$$J^0 = i \left(\phi^* \dot{\phi} - \phi \dot{\phi}^* \right)$$

we obtain

$$\boxed{Q = \int d^3x \, i(\phi^* \dot{\phi} - \phi \dot{\phi}^*)}$$

This charge generates the $U(1)$ symmetry.

Physical Interpretation

The $U(1)$ symmetry corresponds to conservation of a global charge. Examples in physics:

- electric charge (in scalar QED)
- particle number symmetry in many field theories
- phase symmetry of complex scalar fields

Thus

$$U(1) \text{ symmetry} \quad \Rightarrow \quad \text{charge conservation}$$

From Global to Local $U(1)$ Symmetry

Start with the complex scalar field Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

This theory is invariant under the **global** $U(1)$ **symmetry**

$$\phi \to e^{i\alpha} \phi$$

$$\phi^* \to e^{-i\alpha} \phi^*$$

where α is constant.

Question:

Can we promote the symmetry to a **local symmetry**?

$$\alpha \to \alpha(x)$$

Local $U(1)$ Transformation

Under a local transformation

$$\phi(x) \to e^{i\alpha(x)}\phi(x)$$

$$\phi^*(x) \to e^{-i\alpha(x)}\phi^*(x)$$

Compute the derivative transformation:

$$\partial_\mu\phi \to \partial_\mu(e^{i\alpha(x)}\phi)$$

$$= e^{i\alpha(x)}(\partial_\mu\phi + i(\partial_\mu\alpha)\phi)$$

Thus

$$\partial_\mu\phi \not\to e^{i\alpha(x)}\partial_\mu\phi$$

Therefore the kinetic term is **not invariant**.

Problem with Local Symmetry

The transformed kinetic term becomes

$$|\partial_\mu \phi|^2 \to |\partial_\mu \phi + i(\partial_\mu \alpha)\phi|^2$$

Extra terms proportional to

$$\partial_\mu \alpha$$

appear.
Thus

$$\boxed{\text{The Lagrangian is no longer invariant}}$$

To restore invariance we must modify the derivative.

Introducing the Covariant Derivative

Define the **covariant derivative**

$$D_\mu = \partial_\mu + ieA_\mu$$

where

$$A_\mu(x)$$

is a new field.
We demand that

$$D_\mu\phi \to e^{i\alpha(x)} D_\mu\phi$$

under the local $U(1)$ transformation.

Gauge Field Transformation

For covariance to hold,

$$A_\mu$$

must transform as

$$\boxed{A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha}$$

Then

$$D_\mu \phi \to e^{i\alpha(x)} D_\mu \phi$$

Thus the derivative transforms in the same way as the field.

Gauge-Invariant Lagrangian

Replace ordinary derivatives with covariant derivatives:

$$\partial_\mu \to D_\mu$$

The Lagrangian becomes

$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi$$

This is invariant under local $U(1)$ transformations.
However, we must also give dynamics to the gauge field.

Field Strength Tensor

Define the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This tensor is gauge invariant.
Add the kinetic term

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

to the Lagrangian.

Scalar QED Lagrangian

The full gauge-invariant Lagrangian becomes

$$\boxed{\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

where

$$D_\mu = \partial_\mu + ieA_\mu$$

This describes

- a complex scalar field
- interacting with a gauge field
- via a local $U(1)$ symmetry

Physical Interpretation

The field

$$A_\mu$$

is identified with the **electromagnetic gauge field**.
The interaction term arises automatically:

$$ieA_\mu(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

Thus

$$\boxed{\text{Gauge symmetry} \ \Rightarrow \ \text{electromagnetic interaction}}$$

Local symmetry determines the form of the interaction.

Dirac Lagrangian

The Dirac Lagrangian for a free fermion field is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

or equivalently

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

We look for continuous symmetries of the action in order to apply **Noether's theorem**.

Global $U(1)$ Symmetry

The Lagrangian is invariant under the global phase transformation

$$\psi \to e^{i\alpha}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{-i\alpha}$$

For an infinitesimal parameter α:

$$\delta\psi = i\alpha\psi$$

$$\delta\bar{\psi} = -i\alpha\bar{\psi}$$

Since α is constant, the Lagrangian is unchanged:

$$\delta\mathcal{L} = 0$$

Thus a conserved current must exist.

Noether Current

Noether's theorem states that for a symmetry transformation

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi + \delta\bar{\psi}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}$$

For the Dirac Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$

we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu$$

and

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0$$

Fermionic Noether Current

Using

$$\delta\psi = i\alpha\psi$$

we obtain

$$j^\mu = (i\bar{\psi}\gamma^\mu)(i\alpha\psi)$$

Ignoring the constant α gives the conserved current

$$\boxed{j^\mu = \bar{\psi}\gamma^\mu\psi}$$

The corresponding conserved charge is

$$Q = \int d^3x\, j^0$$

$$Q = \int d^3x\, \psi^\dagger\psi$$

Current Conservation

Compute the divergence:

$$\partial_\mu j^\mu = \partial_\mu(\bar{\psi}\gamma^\mu\psi)$$

Using the product rule:

$$= (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)$$

Using the Dirac equation and its adjoint, the terms cancel:

$$\boxed{\partial_\mu j^\mu = 0}$$

Thus the fermionic current is conserved.

Gauge Field and Covariant Derivative

Introduce:

$$D_\mu = \partial_\mu + ieA_\mu$$

Gauge transformations:

$$\psi \to e^{i\alpha(x)}\psi$$

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha$$

QED Lagrangian

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Final Perspective

- Symmetries dictate interactions
- Gauge invariance forces the existence of forces
- QED is the simplest non-trivial gauge theory
- Dynamics follows from an action, not directly from equations
- Lorentz invariance is built into the formalism
- Fields, not particles, are the fundamental objects

Bridge to QFT

Quantization will promote $\phi(x)$ to an operator.

Minimal Coupling for an *SU*(2) Gauge Symmetry

Consider a fermion doublet

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

with global $SU(2)$ symmetry

$$\psi \to U\psi, \qquad U = e^{i\alpha^a \tau^a/2}$$

where τ^a are the Pauli matrices.

Free Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

Under a **local transformation**

$$\psi(x) \to U(x)\psi(x)$$

the derivative produces an extra term:

$$\partial_\mu \psi \to U \partial_\mu \psi + (\partial_\mu U)\psi$$

Introduce a **covariant derivative**

$$D_\mu = \partial_\mu - igA_\mu^a \frac{\tau^a}{2}$$

so that

$$D_\mu \psi \to U(x) D_\mu \psi .$$

Minimal coupling

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

Transformation of the $SU(2)$ Gauge Field

Require the covariant derivative to transform as

$$D_\mu\psi \to U(x)D_\mu\psi$$

under the local transformation

$$\psi(x) \to U(x)\psi(x), \qquad U(x) = e^{i\alpha^a(x)\tau^a/2}.$$

Recall

$$D_\mu = \partial_\mu - igA_\mu, \qquad A_\mu = A_\mu^a \frac{\tau^a}{2}.$$

Apply the transformation:

$$D'_\mu\psi' = (\partial_\mu - igA'_\mu)U\psi$$

$$= (\partial_\mu U)\psi + U\partial_\mu\psi - igA'_\mu U\psi.$$

For covariance we require

$$D'_\mu \psi' = U(\partial_\mu - igA_\mu)\psi.$$

Comparing both expressions gives

$$\boxed{A'_\mu = UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}}$$

Gauge Symmetry vs Global Symmetry

Recall:

Global symmetry

Parameter is constant:

$$\phi \to \phi + \epsilon\,\Delta\phi$$

Noether theorem gives

$$\partial_\mu J^\mu = 0$$

with a conserved charge

$$Q = \int d^3x\, J^0$$

Gauge symmetries are different because the parameter depends on spacetime.

Local Gauge Transformations

Example: QED

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

$$\psi \to e^{ie\alpha(x)}\psi$$

Key feature:

$$\alpha = \alpha(x)$$

The symmetry is local.
This leads to the structure described by Noether's second theorem.

QED Lagrangian

Consider

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

with

$$D_\mu = \partial_\mu + ieA_\mu$$

and infinitesimal transformations:

$$\delta A_\mu = \partial_\mu \alpha$$

$$\delta\psi = ie\alpha\,\psi$$

$$\delta\bar{\psi} = -ie\alpha\,\bar{\psi}$$

Noether Current in QED: Total Expression

From the Noether procedure, the current is:

$$J^{\mu} = e\alpha \, \bar{\psi}\gamma^{\mu}\psi - F^{\mu\nu}\partial_{\nu}\alpha$$

Key features:

- Depends explicitly on the local parameter $\alpha(x)$
- Contains both matter and gauge-field contributions

Rewriting the Current

We can rewrite the current using integration by parts:

$$J^\mu = \alpha \left(e\bar{\psi}\gamma^\mu\psi - \partial_\nu F^{\mu\nu} \right) + \partial_\nu(\alpha F^{\mu\nu})$$

Structure:

- First term: proportional to equations of motion
- Second term: total derivative

Using the Equations of Motion

Maxwell equations:

$$\partial_\nu F^{\mu\nu} = e\bar{\psi}\gamma^\mu\psi$$

Substituting:

$$e\bar{\psi}\gamma^\mu\psi - \partial_\nu F^{\mu\nu} = 0$$

Therefore:

$$J^\mu = \partial_\nu(\alpha F^{\mu\nu})$$

Final Form of the Noether Current

The Noether current becomes:

$$J^\mu = \partial_\nu(\alpha F^{\mu\nu})$$

Properties:

- Pure total derivative
- Automatically conserved:
$$\partial_\mu J^\mu = 0$$
- No local (bulk) conserved charge

Physical Interpretation

This result implies:

- Gauge symmetry ⇒ redundancy of description
- No physical Noether charge from local symmetry
- Current reduces to a boundary term

Only global transformations give physical currents.

Extracting the Physical Current

For constant α:

$$\partial_\nu \alpha = 0$$

Then:

$$J^\mu = e\bar{\psi}\gamma^\mu\psi$$

This is the physical electric current:

- Conserved: $\partial_\mu j^\mu = 0$
- Generates electric charge

Summary

- Full gauge Noether current:

$$J^{\mu} = \partial_{\nu}(\alpha F^{\mu\nu})$$

- Trivial (boundary) current
- Reflects Noether's second theorem
- Physical current arises from global subgroup:

$$j^{\mu} = e\bar{\psi}\gamma^{\mu}\psi$$

Relation to Noether's Second Theorem

Noether's second theorem states:

Local symmetry

Continuous symmetries with arbitrary functions $\alpha(x)$ imply identities among equations of motion.

For gauge theories this identity becomes

$$\partial_\mu \left(\frac{\delta S}{\delta A_\mu} \right) = 0$$

which follows from gauge invariance.

Gauss Law

In electromagnetism the temporal equation of motion gives

$$\nabla \cdot \mathsf{E} = \rho$$

This is Gauss law. It is not a dynamical equation but a constraint. Integrating Gauss law:

$$Q = \int d^3x \, \rho$$

Using

$$\rho = \nabla \cdot \mathsf{E}$$

we obtain

$$Q = \int d^3x \, \nabla \cdot \mathsf{E}$$

Surface Representation of Charge

Applying Gauss theorem:

$$Q = \oint \mathrm{E} \cdot d\mathrm{S}$$

Thus the charge can be written as a boundary integral.
This result reveals an important property:

Gauge charges

Physical charges in gauge theories arise from boundary terms.

The bulk Noether current is trivial, but nontrivial charges arise from surface integrals.

General Yang–Mills Case

For non-Abelian gauge theory A^a_μ with field strength $F^a_{\mu\nu}$ gauge transformations are

$$\delta A^a_\mu = D_\mu \alpha^a$$

with

$$D_\mu = \partial_\mu + g A_\mu$$

The Noether current again has the form

$$J^\mu_a = \partial_\nu K^{\mu\nu}_a$$

with

$$K^{\mu\nu}_a = -F^{\mu\nu}_a \alpha^a$$

Thus the current is again a total derivative.

Summary

Key points:

- Gauge symmetries are local redundancies
- Their Noether currents are total derivatives
- The bulk current vanishes on-shell
- Physical charges arise from boundary terms
- The global subgroup produces observable charges

Gauge theories therefore illustrate the deeper structure of Noether's second theorem.

Noether's Theorem: Two Independent U(1) Symmetries

Consider two complex scalar fields ϕ_1, ϕ_2 with Lagrangian

$$\mathcal{L} = \sum_{i=1}^{2} |\partial_\mu \phi_i|^2 - V(\phi_1, \phi_2)$$

Symmetries:

- $U(1)_1$: $\phi_1 \to e^{i\alpha_1}\phi_1$, ϕ_2 unchanged
- $U(1)_2$: $\phi_2 \to e^{i\alpha_2}\phi_2$, ϕ_1 unchanged

Noether Currents:

$$j_1^\mu = i(\phi_1^* \partial^\mu \phi_1 - \phi_1 \partial^\mu \phi_1^*), \quad j_2^\mu = i(\phi_2^* \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_2^*)$$

Conserved Charges: $Q_1 = \int d^3x\, j_1^0$, $Q_2 = \int d^3x\, j_2^0$

Exercises

- Two scalars ϕ_1, ϕ_2 with independent $U(1)$: two conserved charges Q_1, Q_2
- Interaction $|\phi_1|^2|\phi_2|^2$ preserves both symmetries
- Mixing $\kappa(\phi_1^*\phi_2 + \phi_2^*\phi_1)$ breaks $U(1)_1 \times U(1)_2 \to U(1)_{\text{diag}}$
- Conserved current is $j^\mu_{\text{diag}} = j^\mu_1 + j^\mu_2$
- Exercises:
  1. Compute currents for arbitrary interactions
  2. Diagonalize mass matrix in presence of mixing
  3. Discuss how symmetry breaking affects mass eigenstates

Solutions: Noether Currents for Two Scalars

Exercise 1: Currents for independent $U(1)_1$ and $U(1)_2$

Step 1: Identify infinitesimal transformations

$$\phi_1 \to \phi_1 + i\alpha_1\phi_1, \quad \phi_2 \to \phi_2 + i\alpha_2\phi_2$$

Step 2: Variation of Lagrangian

For a generic Lagrangian

$$\mathcal{L} = |\partial_\mu\phi_1|^2 + |\partial_\mu\phi_2|^2 - V(\phi_1, \phi_2)$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta(\partial_\mu\phi_i) + \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \text{c.c.}$$

Since $V(\phi_1, \phi_2)$ is invariant under phase rotations, only the kinetic term contributes:

$$\delta\mathcal{L} = (\partial^\mu\phi_1^*)(i\alpha_1\partial_\mu\phi_1) + (\partial^\mu\phi_1)(-i\alpha_1\partial_\mu\phi_1^*) + \text{same for } \phi_2$$

Step 3: Identify Noether currents

$$j_1^\mu = i(\phi_1^* \partial^\mu \phi_1 - \phi_1 \partial^\mu \phi_1^*), \quad j_2^\mu = i(\phi_2^* \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_2^*)$$

Step 4: Conserved charges

$$Q_1 = \int d^3x \, j_1^0, \quad Q_2 = \int d^3x \, j_2^0$$

Observation: Interaction term $|\phi_1|^2|\phi_2|^2$ does not affect these currents; Q_1, Q_2 remain conserved.

Solutions: Mixing Term and Mass Eigenstates

Exercise 2: Diagonalizing mass matrix with mixing

Step 1: Lagrangian with mixing

$$\mathcal{L} = |\partial_\mu \phi_1|^2 + |\partial_\mu \phi_2|^2 - m_1^2 |\phi_1|^2 - m_2^2 |\phi_2|^2 - \kappa(\phi_1^* \phi_2 + \phi_2^* \phi_1)$$

Step 2: Mass matrix in basis (ϕ_1, ϕ_2)

$$M^2 = \begin{pmatrix} m_1^2 & \kappa \\ \kappa & m_2^2 \end{pmatrix}$$

Step 3: Eigenvalues (mass squared)

$$m_\pm^2 = \frac{m_1^2 + m_2^2}{2} \pm \sqrt{\left(\frac{m_1^2 - m_2^2}{2}\right)^2 + \kappa^2}$$

Step 4: Eigenvectors (mass eigenstates)

$$\phi_+ = \cos\theta\,\phi_1 + \sin\theta\,\phi_2, \quad \phi_- = -\sin\theta\,\phi_1 + \cos\theta\,\phi_2$$

with mixing angle θ given by

$$\tan 2\theta = \frac{2\kappa}{m_1^2 - m_2^2}$$

Exercise 3: Solution – Conserved Charges with Mixing

Step 1: Infinitesimal U(1) transformations
Let us consider the original independent U(1) transformations:

$$\phi_1 \to \phi_1 + i\alpha_1\phi_1, \quad \phi_2 \to \phi_2 + i\alpha_2\phi_2$$

where α_1 and α_2 are infinitesimal real parameters.

Step 2: Variation of the mixing term

$$\delta\big[-\kappa(\phi_1^*\phi_2 + \phi_2^*\phi_1)\big] = -\kappa\big[(\delta\phi_1^*)\phi_2 + \phi_1^*(\delta\phi_2) + (\delta\phi_2^*)\phi_1 + \phi_2^*(\delta\phi_1)\big]$$

Substitute $\delta\phi_i = i\alpha_i\phi_i$, $\delta\phi_i^* = -i\alpha_i\phi_i^*$:

$$\delta\mathcal{L}_{\text{mix}} = -\kappa\big[(-i\alpha_1\phi_1^*)\phi_2 + \phi_1^*(i\alpha_2\phi_2) + (-i\alpha_2\phi_2^*)\phi_1 + \phi_2^*(i\alpha_1\phi_1)\big]$$

$$\delta\mathcal{L}_{\text{mix}} = i\kappa(\alpha_1 - \alpha_2)(\phi_1^*\phi_2 - \phi_2^*\phi_1)$$

Step 3: Condition for conserved current
Noether's theorem requires $\delta\mathcal{L} = 0$ for the symmetry to be exact.

$$\delta\mathcal{L}_{\text{mix}} = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2 \equiv \alpha$$

⇒ Only the **diagonal U(1) transformation** is conserved:

$$\phi_1 \to e^{i\alpha}\phi_1, \quad \phi_2 \to e^{i\alpha}\phi_2$$

Step 4: Diagonal conserved current

$$j^\mu_{\text{diag}} = j^\mu_1 + j^\mu_2 = i(\phi_1^*\partial^\mu\phi_1 - \phi_1\partial^\mu\phi_1^*) + i(\phi_2^*\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_2^*)$$

$$Q_{\text{diag}} = \int d^3x\, j^0_{\text{diag}} \quad \text{is conserved.}$$

Step 5: Non-conserved combination

The orthogonal combination

$$Q_1 - Q_2 = \int d^3x\,(j^0_1 - j^0_2)$$

is no longer conserved because $\delta\mathcal{L}_{\text{mix}} \neq 0$ for $\alpha_1 = -\alpha_2$.

Why do we need covariant derivatives?

Symmetries require derivatives that transform properly.

General Relativity
Under coordinate transformations

$$x^{\mu} \to x'^{\mu}(x),$$

ordinary derivatives do not transform tensorially. We introduce the covariant derivative

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\rho} V^{\rho}.$$

$U(1)$ **Gauge Theory**
Fields transform under a local phase rotation

$$\psi(x) \to e^{i\alpha(x)} \psi(x).$$

Ordinary derivatives break the symmetry, so we define

$$D_{\mu}\psi = (\partial_{\mu} + ieA_{\mu})\psi.$$

Analogy between GR and Gauge Theory

Both theories modify the derivative by adding a **connection**.

	General Relativity	$U(1)$ Gauge Theory
Symmetry	Coordinate invariance	Local phase symmetry
Field	V^μ	ψ
Covariant derivative	$\nabla_\mu = \partial_\mu + \Gamma$	$D_\mu = \partial_\mu + ieA_\mu$
Connection	$\Gamma^\rho_{\mu\nu}$	A_μ
Interpretation	Spacetime geometry	Phase bundle geometry

Both define **parallel transport**. Both describe how fields rotate when transported through space.

Curvature from commutators

Curvature arises from the commutator of covariant derivatives.

$U(1)$ **gauge theory**

$$[D_\mu, D_\nu]\psi = ieF_{\mu\nu}\psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

General Relativity

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma.$$

The "internal space" is the space of field values, not the space of physical locations.