Cosmologia Física

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The Homogeneous Universe

The background dynamics

Newtonian dynamics of the expansion

We saw that it was possible to introduce the concepts of expansion and redshift in a general way, without specifying the theory of gravitation (that drives the expansion) → they are not necessarily a consequence of general relativity.

Let us now try to derive the equations of movement of the gravitational expansion, i.e., the equations for the evolution of the scale factor a(t), or the Hubble function H(t), using Newtonian mechanics.

Is this possible?

Let us consider the homogeneous and isotropic Universe as a sphere of radius r that expands radially and is filled by a homogeneous cosmological fluid with density ρ.

i) Energy conservation (kinetic + potential)

$$E_k = v^2 / 2 \qquad E_V = - G\, M / r$$

The mass relates to the cosmological fluid density:

Mass inside of the sphere of radius $r(t)$ is $M = \frac{4}{3}\pi r^3(t)\, \rho(t)$

Energy conservation + Newtonian gravity yields:

$$\frac{v^2}{2} - \frac{4}{3}\pi G r^2 \rho = c^{te}$$

We can introduce the scale factor by considering this equation in comoving coordinates:

$$\frac{\dot{a}^2 x^2}{2} - \frac{4}{3}\pi G a^2 x^2 \rho = K^{te}$$

$$\dot{a}^2 - \frac{8}{3}\pi G a^2 \rho = \frac{2}{x^2} c^{te}$$

$$\dot{a}^2(t) = \frac{8\pi G}{3}\rho(t)\, a^2(t) - K$$

$$\left[H^2(t) = \frac{8\pi G}{3}\rho(t) - \frac{K}{a^2} \right]$$

Friedmann’s equation

The constant is K = 2E / (x^2)

where E is the total energy of the Universe and x is the comoving coordinate of the surface of the “Newtonian Universe” - the Hubble radius.

So we get Friedmann’s equation, identical to the one derived in General Relativity (although in GR the constant K has a different and well-defined meaning: it is the curvature of space.

ii) To solve Friedmann's equation for a(t) we need to know the source of gravity, i.e., the mass of the Universe, i.e., we need to know ρ(t).

The evolution of ρ(t) is constrained by the conservation of mass (the continuity equation in the Newtonian approach).

For this, let us consider the 1st law of thermodynamics for the expanding cosmologcial fluid:

$$dU = -p\, dV$$

(there is no heat dissipation to the exterior of the expanding sphere that constitutes the whole Universe)

The energy of the Universe is

$$M = \frac{4}{3}\pi r^3 \rho \quad \Rightarrow \quad U = \frac{4}{3}\pi r^3 \rho c^2$$

So, $\frac{4}{3}\pi\, d(r^3 \rho c^2) = -p\,\frac{4}{3}\pi\, d(r^3)$

Comoving coordinates $\Rightarrow$ $d(a^3 x^3 \rho c^2) = -p\, d(a^3 x^3)$

$(v = \dot{a}x,\ r = ax)$

$\dot{\rho} a^3 + \rho\, da^3 = -p\, da^3$

$\frac{d(a^3)}{dt} = 3a^2\dot{a}$

$\Leftrightarrow$ $\dot{\rho} a^3 + \rho\, 3a^2\dot{a} = -p\, 3a^2\dot{a}$

$\Leftrightarrow$ $\dot{\rho} + \frac{3a^2\dot{a}(\rho + p)}{a^3} = 0$

$\Leftrightarrow$ $\left[\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0\right]$ eq

This is identical to the conservation equation derived in GR.

iii) Finally, to find the equation of movement of the expanding Universe, we consider the 2nd law of Newton:

$$\ddot{r} = -\frac{GM}{r^2} = -\frac{G}{r^2}\frac{4}{3}\pi r^3 \rho$$

$$\Leftrightarrow \ddot{r} = -\frac{4\pi G}{3}\rho r$$

$$\Rightarrow \left[\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho \right]$$

This equation is different from its GR counterpart, which also involves pressure (in GR, pressure is source of gravity, while in Newtonian gravity it is not).

However, if we combine the 1st Friedmann equation with the conservation equation that we found, we obtain the following:

(differentiate Friedmann's equation + use conservation equation → eliminate dρ/dt and get an equation for ä :

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - K$$

$$\rightarrow \quad 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left(\dot{\rho} a^2 + \rho\, 2 a \dot{a}\right)$$

$$\Leftrightarrow \quad 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left(-3\frac{\dot{a}}{a}(\rho + p)\, a^2 + \rho\, 2 a \dot{a}\right)$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\dot{a}\, a\left(-3\rho - 3p + 2\rho\right)$$

$$2\cancel{\dot{a}}\ddot{a} = -\frac{8\pi G}{3}\cancel{\dot{a}}\, a\left(\rho + 3p\right)$$

$$\left[\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)\right]$$

This is the 2nd Friedmann equation, also called Raychadhuri equation and now it is identical to the one derived in GR.

How was this possible? From where did the we get pressure in our Newtonian description?

It came from using the first law of thermodynamics to get the continuity equation, i.e., we used a conservation of energy instead of conservation of mass. In other words, we wrote U from ρ, implicitly assuming mass-energy equivalence.

In conclusion: Newtonian gravity does not find the correct evolution equations. We could however find them using *relativistic Newtonian gravity*, i.e., Newtonian gravity + special relativity.

Note that relativistic Newtonian gravity is different from General Relativity. It is just Newtonian physics + the assumption that the energy is source of gravity. It does not include the concept of curvature, which also contributes to gravity.

Einstein tensor

The “equations of GR” are the Einstein equations, which are **a set of constraint differential equations that relate gravity with the sources of gravity.**
For this, gravity is represented by the **Einstein tensor,** and the sources of gravity are encoded in the **energy-momentum tensor**. This tensor is the left-side of the **Einstein equations**: $G_{ab} = 8\pi G\, T_{ab}$

In GR, gravity arises from the curvature of space-time.

Given that the curvature of a manifold in any number of dimensions is described by the **Riemann tensor**, the Einstein tensor has to be related to it.

In particular, the Einstein tensor is defined as: $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$

where R is the **Ricci scalar**, and the **Ricci tensor** is a contraction of the Riemann tensor:

$$R_{ij} = R^{a}_{iaj}$$

and is computed as,

$$R_{ab} = \Gamma^{a}_{bc,a} - \Gamma^{a}_{ba,c} + \Gamma^{a}_{da}\Gamma^{d}_{bc} - \Gamma^{a}_{dc}\Gamma^{d}_{ba}$$

The connection Γ^a_{bc} is the quantity that enables to “connect” the local geometry around one point of the curved space (or space-time) with the local geometry around another point of the same space (or space-time). In other words, it describes how the basis vector change from point to point due to the curvature.

The connection is thus a needed quantity when computing derivatives in a curved space: the covariant derivative (;) of a function, includes the “normal” derivative (,) of the function and the derivative of the basis:

derivative of a vector: $\lambda^a{}_{;b} = \lambda^a{}_{,b} + \Gamma^a{}_{bc}\lambda^c$ or $\lambda_{a;b} = \lambda_{a,b} - \Gamma^c{}_{ba}\lambda_c$

derivative of a tensor: $T^{ab}{}_{;c} = T^{ab}{}_{,c} + \Gamma^a{}_{cd} T^{db} + \Gamma^b{}_{cd} T^{ad}$

derivative of a scalar: $\phi_{;a} = \phi_{,a}$ naturally, the connection is not needed when differentiating a scalar quantity

In GR the connection is completely determined by the metric, as,

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd}\left[g_{ad,b} + g_{bd,a} - g_{ab,d}\right]$$

but in general the connection and the metric could be two independent quantities related to curvature. Note: this is the case in the Palatini approach to gravity.

Hence, the Einstein tensor is computed from the connection and the metric. We saw that the homogeneous Universe is described by the Robertson-Walker metric:

$$g_{ab} = \begin{bmatrix} -1 & & & \\ & a^2 & & 0 \\ & 0 & a^2 f^2 & \\ & & & a^2 f^2 \sin^2\theta \end{bmatrix}$$

(here f is f_K)

$$g^{ab} \text{ is the inverse } = \begin{bmatrix} -1 & & & \\ & 1/a^2 & & 0 \\ & 0 & 1/a^2 f^2 & \\ & & & 1/a^2 f^2 \sin^2\theta \end{bmatrix}$$

$$g^a{}_b = \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & 0 & 1 & \\ & & & 1 \end{bmatrix} = \mathbb{1}$$

Computing the various terms of the connection, metric and tensors
(for all combinations of indexes, running from 0 [time] to 1,2,3 [space]):

RW $\Rightarrow$ $\Gamma^0_{00} = 0$; $\Gamma^0_{0i} = \Gamma^0_{i0} = 0$; $\Gamma^0_{ij} = \delta_{ij}\,\dot{a}a$; $\Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j$; $\Gamma^i_{jk} = 0$

; $\Gamma^i_{00} = 0$

2 or 3 $\underline{0}$ given 0

1 $\underline{0}$ and 2 space

no $\underline{0}$ also gives 0

$(\delta^i_i = 3)$

$$\Rightarrow \quad R_{00} = -3\left(\frac{\ddot{a}}{a}\right)$$

$$R_{ij} = \delta_{ij}\left(2\dot{a}^2 + \ddot{a}a\right)$$

Ricci tensor is diagonal, like the metric and T^a_b

$\Rightarrow$ Ricci scalar $\quad R = g^{ab}R_{ab} \Leftrightarrow R = R_{00} - \frac{1}{a^2}\delta^{ij}R_{ij} \Leftrightarrow R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]$

the resulting Einstein tensor is:

$\Rightarrow$ Einsteintensor für RW:

$$
\begin{cases}
G_{00} = \dfrac{f'^{2} + 2ff'' - 1}{a^{2}f^{2}} - \dfrac{3}{c^{2}}\left(\dfrac{\dot{a}}{a}\right)^{2} \\[2ex]
G_{11} = \dfrac{1 - f'^{2}}{f^{2}} + \dfrac{1}{c^{2}}\left(\dot{a}^{2} + 2a\ddot{a}\right) \\[2ex]
G_{22} = \dfrac{f^{2}}{c^{2}}\left(\dot{a}^{2} + 2a\ddot{a}\right) - ff'' \\[2ex]
G_{33} = G_{22}\sin^{2}\theta
\end{cases}
\qquad f = f_{k}(\chi)
$$

It is diagonal, like the RW metric, having only 4 non-zero elements.

Energy-momentum tensor

The energy-momentum tensor, also called stress-energy tensor, represents the total energy present in the Universe in various forms.
It contains the energy-momentum contributions from all sources of gravity.

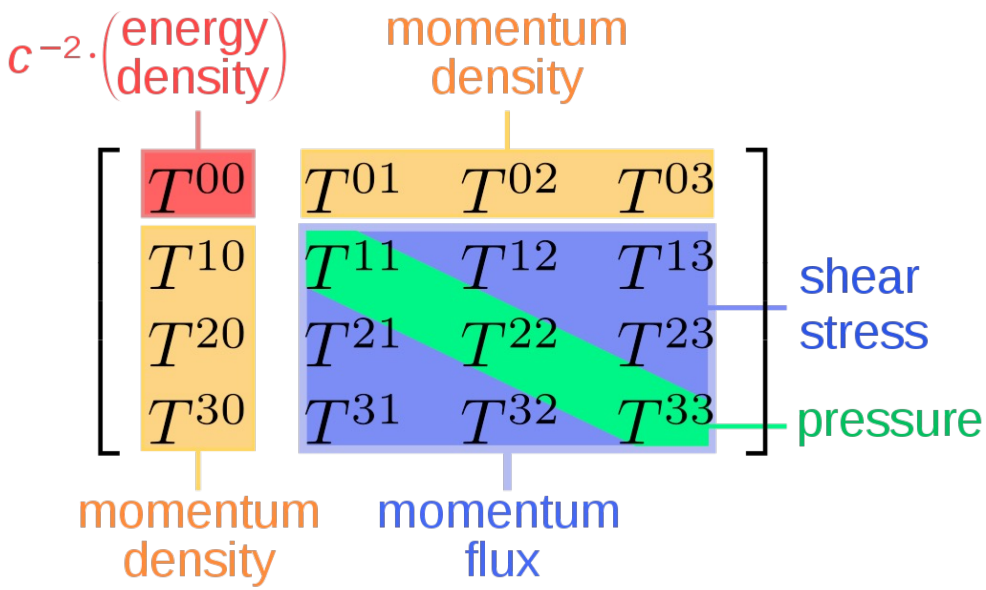
In the homogeneous universe, each source of gravity (e.g. matter, radiation, dark energy, curvature) is treated as a fluid, and together they form the **cosmological fluid.** The general form of the 4D energy-momentum tensor of the cosmological fluid is:

$$T_{ab} = (\rho + p)\, u_a u_b + p\, g_{ab} + q_a u_b + q_b u_a + \pi_{ab}$$

This expression shows what are the physical quantities that contribute to the energy-momentum. They are:

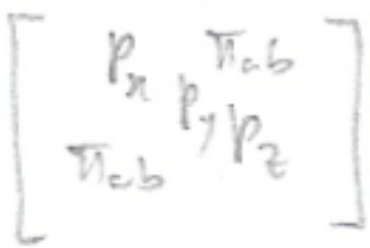
- ρ - energy **density** (T_{00})
- q_a - **momentum** density and flux (T_{0i})
- Π - **anisotropic stress** (T_{ij})
- p - **pressure** (T_{ii})

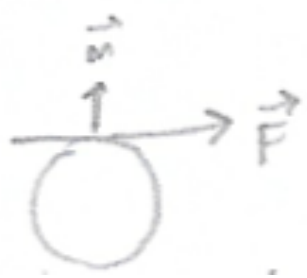
and u^a is the 4-vector velocity of the fluid reference frame → $u^a = (-1,0)$ for comoving



Note that all these quantities have dimensions of energy/volume.

The spatial part is the stress-tensor, containing isotropic and anisotropic pressure:





isotropic force :
produces expansion/contraction

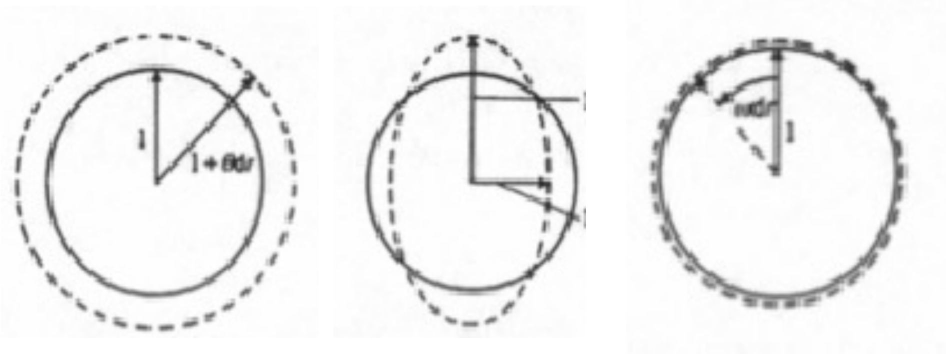
anisotropic force :
produces shear (ellipticity)

The stress tensor can be decomposed in three contributions:

$$\begin{bmatrix} p+\sigma_1 & 0 \\ 0 & p+\sigma_1 \end{bmatrix} + \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & -\sigma_1 \end{bmatrix} + \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix}$$

(simplified for 2D spatial dimensions)

expansion shear rotation



However, in the homogeneous Universe, the metric is RW, and T_{ab} is forced to have the same symmetries → off-diagonal terms are necessarily zero, and the energy-momentum tensor is that of a **perfect fluid**:

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab}$$

$$T_{00} = (\rho + p)(-1)^2 + p(-1) = \rho + p - p = \rho \quad \longrightarrow T^0{}_0 = -\rho$$

$$T_{ii} = 0 + p = p \quad \longrightarrow T^i{}_i = p$$

$$T_{ab} = \begin{bmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{bmatrix}$$

Density and pressure are thus the only relevant quantities. How can they be computed (for each constituent of the cosmological fluid)?

The elements of the energy-momentum tensor may be obtained from the **least action principle**, by varying the action with respect to the metric.

The **action** is

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}$$

It is defined in the 4D volume, and the integration measure must include the determinant g of the metric.

In cartesian coordinates, t,x,y,z :

$$g_{ab} = \begin{bmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{bmatrix} \Rightarrow \quad g = -a^6, \qquad \sqrt{-g} = a^3$$

In GR, the **Lagrangian** density of the **homogeneous and empty Universe** is given by the Ricci scalar:

$$\mathcal{L} = \frac{R}{16\pi G}$$

So we need to compute the derivative of the action with respect to the metric $\frac{\delta S}{\delta g^{ab}}$

$$\delta S = \frac{1}{16\pi G} \delta(\sqrt{-g}R)\, \delta g^{ab} = 0$$

We need to compute,

$$\delta(\sqrt{-g}R) = \delta\left(\sqrt{-g}\, g^{ab} R_{ab}\right) = \delta(\sqrt{-g})\, g^{ab} R_{ab} + \sqrt{-g}\, g^{ab}\delta R_{ab} + \sqrt{-g}\, R_{ab}\, \delta g^{ab}$$

The derivative of the determinant is

$$\delta g = g\, g_{ab}\, \delta g^{ab}$$

$$\Rightarrow \delta\sqrt{-g} = \frac{1}{2}(-g)^{-1/2}\delta g = -\frac{1}{2}\frac{1}{\sqrt{-g}}\, g\, g_{ab}\, \delta g^{ab} = -\frac{1}{2}\sqrt{-g}\, g_{ab}\, \delta g^{ab}$$

Inserting above, we write,

$$\delta\left(\sqrt{-g}\, g^{ab} R_{ab}\right) = -\frac{1}{2}\sqrt{-g}\, g_{ab} R\, \delta g^{ab} + \sqrt{-g}\, g^{ab}\, \delta R_{ab} + \sqrt{-g}\, \delta g^{ab} R_{ab}$$

$$= \underbrace{\sqrt{-g}\left(R_{ab} - \frac{1}{2} g_{ab} R\right)}_{\text{This is } T_{ab} \text{ (from Einstein eq.)}} \delta g^{ab} + \sqrt{-g}\, g^{ab} \underbrace{\delta R_{ab}}_{\text{the derivation of the Ricci tensor is zero}}$$

$$\delta R_{ab} = 0$$

and so:

$$\delta S = 0$$

$$\delta S = \frac{1}{16\pi G}\sqrt{-g}\left(R_{ab} - \frac{1}{2} g_{ab} R\right) = 0$$

$$\Rightarrow \quad R_{ab} - \frac{1}{2} g_{ab} R = 0 \quad \Rightarrow \quad T_{ab} = 0$$

This is the result for the energy-momentum tensor of the empty Universe (which is zero, as expected)

Now, if we consider a **matter-energy component**, described by a Lagrangian L, the action becomes

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} + L \right)$$

and the variational principle leads to:

$$\delta S = 0 \iff \delta \left(\frac{\sqrt{-g} R \, \delta g^{ab}}{16\pi G} + \sqrt{-g} L \right) = 0 \iff \sqrt{-g} \frac{8\pi G \, T_{ab} \, \delta g^{ab}}{16\pi G} + \delta(\sqrt{-g} L) = 0$$

$$\iff \boxed{T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L)}{\delta g^{ab}}}$$

This shows that if we consider a (energy/particle) field in the Universe (for example a dark matter particle or a dark energy field) described by a potential V and a field ϕ, we can compute its energy-momentum tensor elements as function of the field's V and ϕ → the theoretical approach

If a Lagrangian cannot be computed for the new field, we can alternatively give directly a prescription for the density and pressure → the phenomenological approach

Einstein equations

Having the Einstein and the energy-momentum tensors, we can write the Einstein equations

$$G_{00} = \frac{f'^2 + 2ff'' - 1}{a^2 f^2} - \frac{3}{c^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{c^4}\rho$$

$$G_{11} = \frac{f'^2 - 1}{a^2 f^2} - \frac{1}{c^2}\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{2\ddot{a}}{a}\right] = \frac{8\pi G}{c^4} p$$

$$G_{22} \text{ or } G_{33} \rightarrow \frac{1}{a^2}\frac{f''}{f} - \frac{1}{c^2}\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{2\ddot{a}}{a}\right] = \frac{8\pi G}{c^4} p$$

Note that for example for K > 0

$$f'_K(\chi) = \cos(K^{1/2}\chi)$$

$$f''_K(\chi) = -K^{1/2}\sin(K^{1/2}\chi)$$

and so $(f'^2-1)/f^2 = -K$ and $f''/f = -K$

Inserting this in G_{00} we get $$G_{00} = \frac{3K}{a^2} + \frac{3}{c^2}\left(\frac{\dot{a}}{c}\right)^2$$

and the 00 element of the Einstein equations is:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}}$$

Friedmann equation

Adding the elements 00 and ii we get a second equation:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)}$$

Second Friedmann equation

The second Friedmann equation turns out to be the **Raychaudhuri equation** for the case of a perfect fluid in the RW space-time.

To see this, consider the general **decomposition** of a (3D timelike) vector field:

$$u_{i;j} = \frac{1}{2}(u_{i;j} + u_{j;i}) - \frac{1}{3}h_{ij}u_{k;k} + \frac{1}{2}(u_{i;j} - u_{j;i}) + \frac{1}{3}h_{ij}u_{k;k}$$

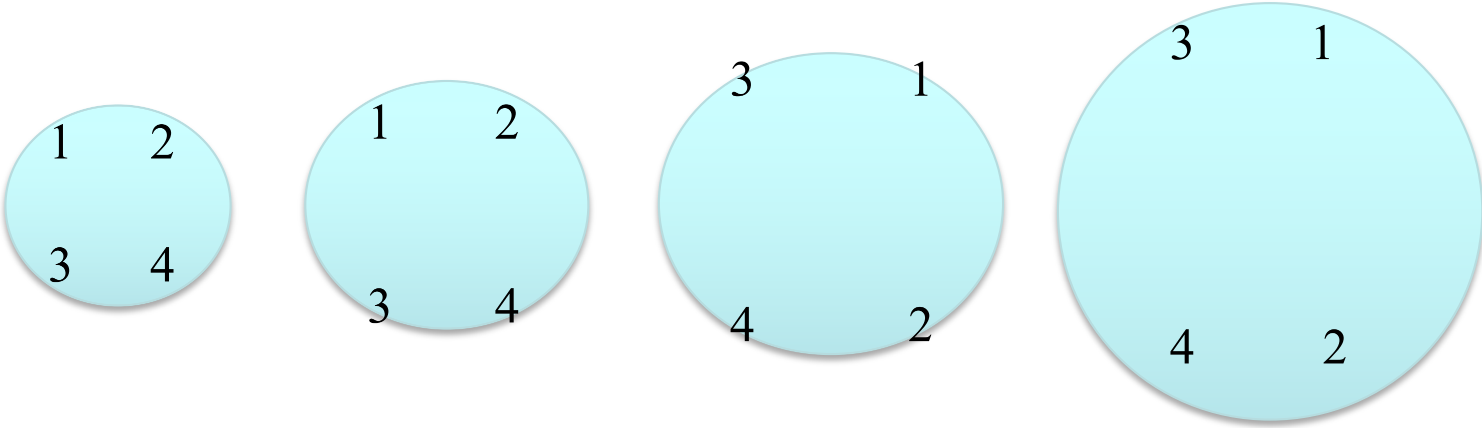
symmetric and traceless: shear σ

anti-symmetic: rotation ω

trace (diagonal): expansion θ (or convergence)

h_{ij} is the metric projected in the 3D space defined by the vector field

An important application is the case of the vector field of a set of comoving galaxies in the cosmic flow.



t_1

t_2 : expansion

t_3 : expansion and rotation

t_4 : expansion and shear

We are interested in **the time evolution of this vector field**.

In particular, working from the expression for the decomposition, the evolution of the expansion/convergence θ = div(u) is found to be given by:

$$\dot{\theta} + \frac{1}{3}\theta^2 + \sigma^2 - \omega^2 = + R_{ab} v^a v^b$$

This equation for the evolution of the trace of the separation vector of a congruence of timelike geodesics is the Raychaudhuri equation

From the Einstein equation we get:

$$R_{ab} u^a u^b = -4\pi G(\rho + 3p)$$

and using comoving coordinates:

$$\theta = v_{i;i} = \mathrm{div}(\dot{a}\chi) = \mathrm{div}\left(\frac{\dot{a}}{a}r\right) = \frac{\dot{a}}{a}\mathrm{div}(r) = \frac{\dot{a}}{a}3 \qquad (r = a\chi)$$

note the factor 3, that comes from the divergence in 3D, like in the ij terms of the Einstein equations and so

$$\dot{\Theta} = \frac{d}{dt}\left(3\frac{\dot{a}}{a}\right) = 3\,\frac{\ddot{a}a - \dot{a}\dot{a}}{a^2} = 3\frac{\ddot{a}}{a} - 3\left(\frac{\dot{a}}{a}\right)^2$$

Inserting in the Raychaudhuri equation we get:

$$\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p) - (\sigma^2 - \omega^2)$$

We see that **density** and **pressure** are sources of attractive gravity → **contributing to attraction, or decelerated expansion**

Shear is also a source of attractive gravity

Rotation is a source of repulsive gravity → **contributing to repulsion, or accelerated expansion** (like a centrifugal force)

So a rotating cosmological fluid could be an alternative to dark energy. But from where would it get its rotation → also from some mysterious extra energy? from internal coherent rotation of all dark matter particles at a fundamental level?

However, for a perfect fluid in a homogeneous and spherical symmetric universe, shear and rotation are zero, and we see that the Raychaudhuri equation is indeed identical to the second Friedmann equation.

The two Friedmann equations are **constraint equations**, connecting the field (i.e. the metric) quantities, a and K, to the source quantities ρ and p.

(Note that a, ρ and p evolve in time, while K is constant)

The evolution of the sources are determined by **energy conservation equations**, which in principle are independent from the Einstein equations.

Energy conservation equations are equivalent to conservation of the energy-momentum tensor:

$$T^{ab}{}_{;b} = 0$$

From here, in principle we can get 4 energy conservation equations.
For the RW metric and perfect fluid, there is only one:

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0}$$

Continuity equation

In the RW case, the continuity equation is already contained in the Einstein equations and can be found by combining the time-derivative of Friedmann equation with the second Friedmann equation:

Derivative → $\left(\frac{\ddot{a}}{a}\right) 2\frac{\dot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^3 2 = \frac{8\pi G}{3}\dot{\rho} + 2K\dot{a}\frac{1}{a^3}$

Insert $\left[\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho+3p)\right. \Rightarrow \overline{-\frac{8}{3}\pi G(\rho+3p)\frac{\dot{a}}{a}} - \left(\frac{\dot{a}}{a}\right)^3 2 = \overline{\frac{8\pi G}{3}\dot{\rho}} + 2K\dot{a}\frac{1}{a^3}$

Now, what is $\left(\frac{\dot{a}}{a}\right)^3 2 + 2K\dot{a}\frac{1}{a^3}$.

Insert again eq. Friedmann

→ $\left(\frac{\dot{a}}{a}\right)^2 2\frac{\dot{a}}{a} + 2K\dot{a}\frac{1}{a^3}$

$\left(\frac{8\pi G}{3}\rho\right)\frac{2\dot{a}}{a} + \cancel{\frac{K}{a^2}2\frac{\dot{a}}{a}} - \cancel{2K\dot{a}\frac{1}{a^3}}$

So,

$\Rightarrow$ Derivative of Fried. eq. is,

$$-\frac{8\pi}{3}G(\rho+3p)\frac{\dot{a}}{a}-\frac{8\pi G}{3}\dot{\rho}=\frac{8\pi G}{3}\rho\,\frac{2\dot{a}}{a}$$

[d Fried + Fried + Raych = Cons.]

$$-\rho\frac{\dot{a}}{a}-3p\frac{\dot{a}}{a}-\dot{\rho}-\rho\,2\frac{\dot{a}}{a}=0$$

$$\boxed{\dot{\rho}+3\frac{\dot{a}}{a}(\rho+p)=0}$$

Conservation eq.
or Continuity eq.

The cosmological fluid components

We saw that the cosmological fluid is described by density and pressure.
In addition, there are in general several different **species** in the cosmological fluid, and so we need to consider the densities and pressures of all of the species.

In general, for each species, the properties are not independent. For a perfect fluid, density and pressure are related through an **equation-of-state** $w(t)$ (like in thermodynamics, when p,V,T, etc may be related under certain conditions).

$$w(t) = p(t) / [\rho(t)\, c^2]$$

Note: in the inhomogeneous universe, perturbations in density may also be related to perturbations in pressure. That relation determines the speed of sound in the fluid.

Note: there may also be constraints, relating the density and pressure of different species.

We have one equation of energy conservation, involving density and pressure. For some species, the pressure is known or may be determined independently. For those cases, the continuity equation may then be solved to find $\rho(t)$.

Matter

Matter (of any type: baryonic or dark matter) is defined by

p=0 (also called "dust")

In this case, the continuity equation is easily solved:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \implies \frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \implies \frac{1}{\rho}d\rho = -3\frac{1}{a}da$$

$$d \ln \rho = -3 \, d \ln a \implies \rho \propto a^{-3}$$

This means the density dilutes linearly with the expansion of the volume a^3.

Note that the factor 3 comes from having 3 spatial dimensions. In the Einstein equations this appears from terms like $\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$, and $\delta^i_i = 3$

In the Newtonian derivation, it appeared more directly from the volume in the first law of thermodynamics: $d/dt\,(a^3) = 3a^2\dot{a}$

Note that we found the functional form of $\rho(t)$ but not its amplitude.
In reality the solution is

$$\ln \rho + C_1 = -3 \ln a + C_2,$$ where C_1 and C_2 are integration constants.

We can choose the constants in any form we wish. A usual way is to choose the "initial" conditions at t_0 = today. The solution is then:

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3}$$

With $a_0 = 1$, we are left with 1 free parameter: **the matter density today $\rho_{m,0}$**

Radiation

Radiation is the flux of relativistic particles present in the Universe, mostly photons from the CMB, but also neutrinos from the cosmic neutrino background. They have radiation pressure that will be a source of gravity, in addition to their energy density.

Let us compute this pressure.

Fluid made of radiation. Particles move with c $\Rightarrow$ density fluctuations (sound or pressure waves) should propagate with $c_s = c$. So p should be $p = \rho$?

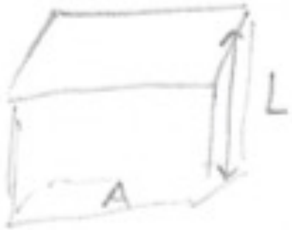
Let us try to compute it:

Consider 1 photon arrives at an area A

$p = \frac{F}{A}$

A photon distribution arrives. At which rate?

Put a mirror do get a rate:



$V = AL$

photons bounce

$$\rho_{1\gamma} = \frac{E}{V} = \frac{E}{AL}$$

$$P_{1\gamma} = \frac{F}{A} \quad \text{or} \quad = \frac{\text{moment transfer}}{\text{time} \times A}$$

A photon arrives every $t = \frac{2L}{c}$ (bouncing)

and transfers a moment $2p$

So $$\text{Pressure}_\gamma = \frac{2p}{2\frac{L}{c}A} = \frac{pc}{V} = \frac{E}{V} \Rightarrow \underline{\text{Press}_\gamma = \rho_\gamma}$$

So, is w = 1 ?

But note that this result is only valid for 1D – when all photons come from the same direction.

Each photon arrives from a direction θ

$\Rightarrow$ moment transfered is just $2p\cos\theta$ and not $2p$

and $t = \dfrac{2L}{c\cos\theta}$ (travels $\dfrac{2L}{\cos\theta}$ inside the box)

$$\Rightarrow \text{Pressure}_\gamma = \frac{2p\cos\theta}{\frac{2L}{c\cos\theta}A} = \frac{pc}{LA}\cos^2\theta = \boxed{P_\gamma \cos^2\theta}$$

The photons come isotropically from all directions $\Rightarrow$ on average they come from the mean direction

Mean of $\cos^2\theta$ in 3D is:

$$\langle\cos^2\theta\rangle = \frac{\int_0^{2\pi}\int_0^{\pi}\cos^2\theta \, d\theta \, \operatorname{sen}\theta \, d\varphi}{\int_0^{2\pi}\int_0^{\pi} d\theta \, \operatorname{sen}\theta \, d\varphi} = 2\pi \frac{\int_0^{\pi}\cos^2\theta \operatorname{sen}\theta \, d\theta}{2\pi\int_0^{\pi}\operatorname{sen}\theta \, d\theta}$$

$$\begin{pmatrix}\cos\theta = x \\ dx = -\operatorname{sen}\theta \, d\theta\end{pmatrix} \rightarrow \langle\cos^2\theta\rangle = \frac{-\int_1^{-1} x^2 dx}{-\int_1^{-1} 1 \, dx} = \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} dx} = \frac{\frac{x^3}{3}\Big|_{-1}^{1}}{x\Big|_{-1}^{1}} = \frac{\frac{1}{3}\times 2}{2} = \frac{1}{3}$$

So on average each photon only has $P_\gamma = \frac{1}{3}\rho_\gamma$

And for the full fluid of N photons $\rightarrow$ $P = \frac{1}{3}\rho$ $\quad w = 1/3$, while in 1D $w = 1$

The result is then w = 1/3

We know the pressure of the radiation species, now we can find the density evolution:

$p = 1/3\,\rho$

$$\dot\rho + 3\frac{\dot a}{a}\frac{4}{3}\rho \Rightarrow \dot\rho + 4\frac{\dot a}{a}\rho = 0 \Rightarrow \frac{\dot\rho}{\rho} = -4\frac{\dot a}{a} \Rightarrow d\ln\rho = -4\, d\ln a$$

$$\rho = \rho_0\left(\frac{a}{a_0}\right)^{-4}$$

The radiation energy density dilutes faster than the matter density, and this is because it is affected by both expansion and redshift.

Again, at $a_0 = 1$ we have found another cosmological parameter: **the radiation density today $\rho_{r,0}$**.

Note that from the Stefan-Boltzmann law: $\rho_r \sim T^4$

(this is the temperature of the radiation fluid, which is the temperature of the Universe)

given the density evolution → this implies that $T \sim 1/a$

So, T is also a unique indicator of the instants in the Universe evolution, just like the redshift : they provide **model-independent** indicators of the events of the Universe.

From the comparison of the two cases (matter and radiation), we see that for a species with **constant (non-evolving) equation-of-state**, the solution for the density evolution is:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(w+1) \Rightarrow \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \Rightarrow \rho \propto a^{-3(1+w)}$$

On the other hand, from the second Friedmann equation we see that

$$\ddot{a} > 0 \text{ if } \rho + 3p < 0 \Rightarrow 3p < -\rho \Rightarrow p < -\frac{1}{3}\rho \Rightarrow w < -1/3$$

So, the larger is w of a species, the faster is the dilution of its energy density, and the stronger is its contribution to gravity → a faster **deceleration** of the Universe (or a contraction, if the Universe had not started from an expanding beginning)

Conversely, the smaller is w, the slower is the dilution of its energy density (could even remain constant, or increase), and the weaker is its contribution to gravity (can even be repulsive if $w < -1/3$) → a slower deceleration of the Universe, or even an **acceleration**.

Curvature

Inserting the density evolution of matter and radiation in Friedmann equarion, this becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[\frac{\rho_{m,0}}{a^3} + \frac{\rho_{r,0}}{a^4}\right] - \frac{\mathrm{K}}{a^2}$$

We see that the curvature term has the same structure of the others. So, even though the curvature is a parameter of the metric, it also has a gravitational effect like an effective density. By analogy, a curvature density may be defined, and its evolution is then,

$\rho_K (a) = \rho_{K,0}\, a^{-2}$,

introducing the parameter **curvature density today** $\rho_{K,0} = -3K/8\Pi G$

Note that to keep the structure of the equation, the curvature density is defined as the negative of the curvature, and so the negative curvature is the one that contributes to a positive density.

Now we can reason the other way around and find the curvature pressure (or equation-of-state) associated to a density a^{-2} evolution. It is $-3\,(1+w) = -2$, i.e.,

$w = -1/3$, in the limit between attractive and repulsive regimes

Cosmological constant

The Einstein equation has the freedom to contain an extra degree of freedom that we did not yet consider, known as the cosmological constant Λ:

$$G_{ab} + \Lambda g_{ab} = 8\Pi G\, T_{ab}$$

This way, the Friedmann equation has a new term:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}$$

This allows us to define an effective energy density with evolution: $\rho_\Lambda(a) = \rho_{\Lambda,0}\, a^0$, that keeps a constant amplitude during all the evolution of the Universe, given by the **cosmological constant density parameter** $\Lambda / 8\Pi G$

The equation-of-state associated to a constant is -3 (1+w) = 0, i.e.,

w = - 1, well into the repulsive regime

This represents an extreme case of negative pressure (an outward tension), where p = - ρ. It is an unknown fluid, with exotic properties.

Dark energy: phenomenological approach

Any cosmological species with $w < -1/3$, capable of producing acceleration may be considered dark energy, and this includes the cosmological constant.

Since the first observations that the Universe is accelerating, there has been great activity in building dark energy models.

An interesting approach is the **phenomenological approach**, where the functional form of the evolution of the density, or the pressure or the equation-of-state is imposed and parameterized.

This method produces models with more free parameters than the standard ΛCDM, which need then to be fitted by observations.

There is no theory of the dark energy species to provide the functional forms.

CPL

A popular method is to **parameterize the equation-of-state** based on a Taylor expansion around a pivotal redshift. This is known as the **Chevalier-Polarski-Linder** dark energy:

$$w(a) = w(a=1) + \frac{dw}{da}\Big|_{a=1}(1-a) + \mathcal{O}(1-a)^2$$

This species introduces three new cosmological parameters, the **equation-of-state today** w_0, the derivative of the **equation-of-state today** w_a, and the **dark energy density today** $\rho_{DE,0}$

$$w_0 = w(a=1)$$

$$w_a = \frac{dw}{da}(a=1)$$

If w_a is set to zero and only w_0 is used (with $w_0 \neq -1$, otherwise it would just be Λ) then this is called the **wCDM** model.

The density evolution for these models is computed as usual from the continuity equation,

$$d\ln\rho = -3\,(1+w(a))\,d\ln a$$

$$\Rightarrow \rho(a) = \rho_0\, a^{-3}\, e^{-3\int_0^a \frac{w(a)}{a}\,da}$$

For CPL, with $w(a)=w_0 + w_a (1-a)$, the density evolution is,

$$\rho(a) = \rho_0 \, a^{-3(1+w_0+w_a)} \, e^{3w_a a}$$

Since current data favors the ΛCDM model, most dark energy parameterizations are built to have a **ΛCDM limit**, and the data best-fit to the DE parameters are usually values close to this limit.

In the case of CPL, this means **w_0 slightly larger than -1 and w_a close to zero**.

In this “cosmology”, the CPL dark energy density stays constant in the early universe (as a cosmological constant), and as the scale factor approaches 1, the exponential term starts to dominate and the DE density increases, being able to produce a faster acceleration than ΛCDM.

Phenomenological models can thus be tuned to include the desired behaviors (in this case, a faster acceleration).

UDM

Another example is the **Unified dark matter - dark energy model**

This dark energy species behaves both as matter and as dark energy, in different periods of the Universe.

To produce this behavior, the **parameterization is made on the density** and not on the equation of state. This way, we can directly choose the desired feature. One example is:

$$\rho = \begin{cases} \rho_t \left(\frac{a_t}{a}\right)^3 & a < a_t \\ \rho_\Lambda + (\rho_t - \rho_\Lambda)\left(\frac{a_t}{a}\right)^{3(1+\alpha)} & a > a_t \end{cases}$$

Before a transition scale factor, $a = a_t$ the species behaves as dark matter, with the density decreasing with a^3.

After the transition, a constant term ρ_Λ arises that will eventually dominate and in the late universe the density will tend to that constant value.

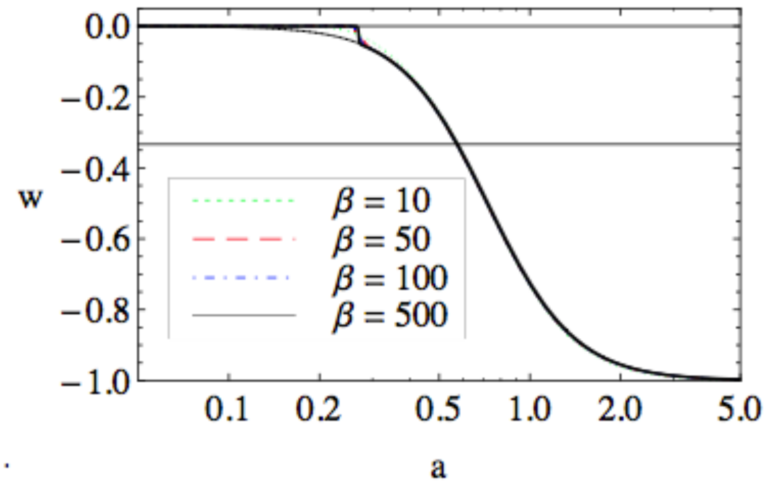
Note: one advantage of giving a **prescription for the density** instead of the equation-of-state, is that the continuity equation is a differential equation for the density, but does not involve derivatives of p or w. This way, **having ρ, we can differentiate it and directly get p (and w)** without introducing an additional parameter for p (or w).

The other way around, **having p (or w), we need to integrate to get ρ**, which introduces an additional parameter.

This species introduces two new cosmological parameters, the **dark energy density today** $\rho_{UDM,0}$, the **dark energy density at transition** $\rho_{UDM,t}$ (or alternatively the transition scale factor).

To ensure a fast but smooth transition between the two regimes, a Heaviside-type function can be used, which introduces a third parameter β.

The behavior of the equation-of-state, computed from the continuity equation is the expected one, starting at w=0 (matter) and reaching a=1 with w < -1/3 (dark energy)



Dark energy: theoretical approach

Most dark energy models are not built phenomenologically, but are built as a **physical model**, defining its Lagrangian and deriving its energy-momentum tensor.

Quintessence

Quintessence was one of the first physical DE models proposed, and it is based on a **scalar field** ϕ.

$$\mathcal{L}_\phi = +\frac{1}{2}\underbrace{g^{ab}\,\partial_a\phi\,\partial_b\phi}_{\text{kinetic}} - \underbrace{V(\phi)}_{\text{potential}}$$

Note: (For homogeneous $\phi = \phi(t)$ this is

$$\mathcal{L}_\phi = +\frac{1}{2}\,\partial_0\phi\,\partial^0\phi - V(\phi)$$

or $\boxed{\mathcal{L}_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)}$)

As we saw, the energy-momentum tensor is computed from

$$T_{ab} = -\delta(\sqrt{-g}\,L_\phi)\frac{2}{\sqrt{-g}}\frac{1}{\delta g^{ab}}$$

The goal is to obtain ρ and p as function of ϕ and V. This approach does not introduce additional free parameters for the density and pressure, but density and pressure parameters will be related to the model's underlying parameters: e.g., **amplitude of the scalar field, amplitude of the potential, slope of the potential**, etc. So, in this case, the observations will constrain the parameters of the physical model.

Now, computing T_{ab} yields,

$$\delta(\sqrt{-g}\,L_\phi) = \delta(\sqrt{-g})\,L + \sqrt{-g}\,\delta L = -\frac{1}{2}\sqrt{-g}\,g_{ab}\,\delta g^{ab}\,L + \sqrt{-g}\left(+\frac{1}{2}\right)\delta g^{ab}\,\underbrace{\partial_a\phi\,\partial_b\phi + 0}_{\text{No dependence of } \phi \text{ and } V \text{ on } g_{ab}}$$

$$= -\frac{1}{2}\sqrt{-g}\,\delta g^{ab}\left(g_{ab}L - \partial_a\phi\,\partial_b\phi\right)$$

So we get $\Rightarrow$ $\boxed{T_{ab} = -\partial_a\phi\,\partial_b\phi + L\,g_{ab}}$

We may compute $T_{00} = -\partial_0\phi\,\partial_0\phi + \left(\frac{1}{2}\dot\phi^2 - V\right) g_{00} = -g_{00}\,\partial^0\phi\,\partial_0\phi + \frac{1}{2}\dot\phi^2 \cdot g_{00} - V g_{00}$ or

or $T_{00} = \dot\phi^2 - \frac{1}{2}\dot\phi^2 + V$ or $T_{00} = \frac{1}{2}\dot\phi^2 + V$

Note: $T_{00} = g_{00}\,T^0{}_0 = -T^0{}_0$, como $T^0{}_0 = -\rho$

$\Rightarrow T_{00} = +\rho$

$$\Rightarrow \rho = \frac{1}{2}\dot\phi^2 + V$$

$$\Rightarrow T_{11} = \overset{\partial_1\phi}{\overset{\uparrow}{0}} + L\,g_{11} = \left(\frac{1}{2}\dot\phi^2 - V\right) g_{11}$$

$$T_{11} = g_{11}\,T^1{}_1 = g_{11}\,p = p$$

$$\Rightarrow p = \frac{1}{2}\dot\phi^2 - V$$

Like for an inflationary field, the case of **slow-rolling** , i.e., $\dot\phi^2 << V$

leads to ρ ~V and p ~ - V → w ~ -1 → a dark energy behavior.

We have thus found ρ and p as function of ϕ and V and checked that a scalar field may have dark energy properties. Inserting the ρ and p scalar field expressions in the continuity equation, we get an equation for the time evolution of the scalar field:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V$$

$$p = \frac{1}{2}\dot{\phi}^2 - V$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V\right) + 3H\left(\frac{1}{2}\dot{\phi}^2 + V + \frac{1}{2}\dot{\phi}^2 - V\right) = 0$$

$$\Leftrightarrow \dot{\phi}\ddot{\phi} + \dot{\phi}V' + 3H\dot{\phi}^2 = 0 \qquad \text{where} \qquad V' = \frac{dV}{d\phi}$$

The continuity equation for the evolution of the scalar field ϕ (a) is then

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0$$

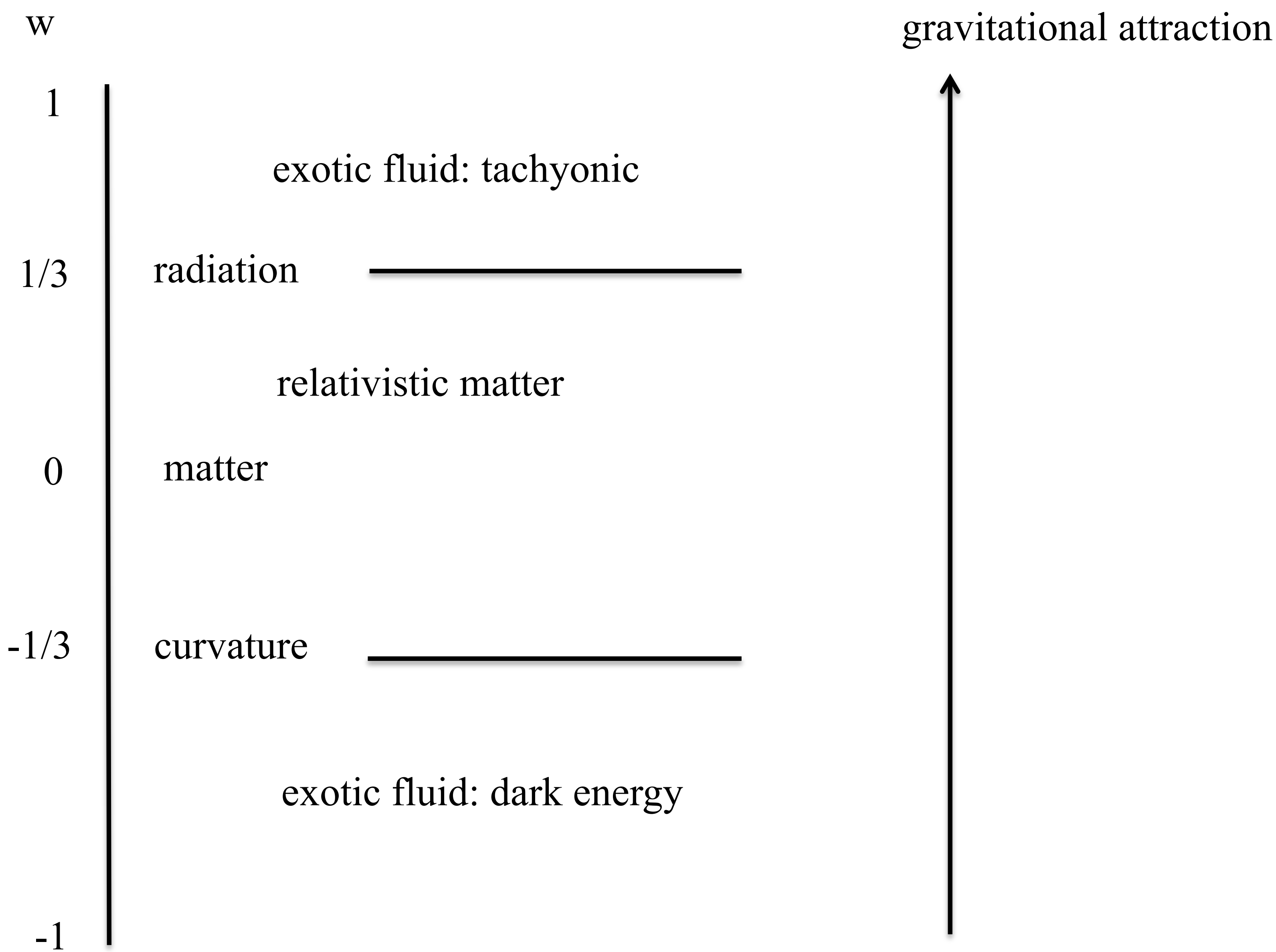
(also known as the **Klein-Gordon equation**, which is the equation of motion of a quantum field).

Note that we need to choose the **potential** V (ϕ) in order to fully describe the dark energy model. This choice defines a particular quintessence model and may introduce additional free (cosmological) parameters.

Alternatively, the energy conservation (Klein-Gordon) may be found from the Euler-Lagrange equation (instead of using the continuity equation):

$$\frac{d}{dt}\left(\frac{\partial \sqrt{-g}L}{\partial(\partial^{\nu}\phi)}\right) - \frac{\partial(\sqrt{-g}L)}{\partial\phi} = 0$$

Summary


w
gravitational attraction
1
exotic fluid: tachyonic
1/3
radiation
relativistic matter
0
matter
-1/3
curvature
exotic fluid: dark energy
-1

The choice of which species to include in the cosmological fluid (e.g., only matter, matter + radiation + dark energy) + their T_{ab} properties (e.g., only density, anisotropic stress, type of w(t)) + the functional form of fundamental quantities of the Universe (e.g. $\rho(t)$) derived from the equations of the theory → defines the **model.**

Examples: CDM, ΛCDM, Milne, Einstein-de Sitter, etc.

However, the models have free parameters and are only completely defined once the values of the **parameters** are known. The parameters values are constrained with observations

Examples: Concordance model (ΛCDM with $\rho_{m,0} = 0.3$, $\rho_{\Lambda CDM,0} = 0.7$, $h = 0.70$)
Planck cosmology (ΛCDM with $\rho_{m,0} = 0.32$, $\rho_{\Lambda CDM,0} = 0.68$, $h = 0.67$)

Example: with different density parameter values, the **"matter + radiation" model** may have completely different properties:

$\rho_{r,0} << \rho_{m,0}$

Epoch of radiation and epoch of matter, CMB is an important feature, baryonic matter clusters slowly, DM needed

$\rho_{r,0} <<<< \rho_{m,0}$

No epoch of radiation, matter dominates at all times, CMB is not an important feature, baryonic matter may cluster fast, DM may not be needed

The main purpose of the cosmological surveys is to constrain the parameters of the cosmological functions from astrophysical observations.

Only with precise and accurate estimates of the cosmological parameters can the cosmological model be fully established.

Dynamics of the expansion

It is usual to normalize the density parameters ρ_0 by the **critical density.**

The critical density ρ_c is the ρ_0 density today of a flat universe with a cosmological fluid containing only matter. So, from Friedmann equation:

$$\left(\underbrace{\frac{\dot{a}}{a}}_{\to H_0}\right)^2 = \frac{8\pi G}{3}\rho_c - \overbrace{\frac{k}{a^2}}^{\to 0} \quad \rightarrow \quad \rho_c = 3H_0^2 / 8\Pi G$$

Since in this case there is only one density parameter (one species), its value is completely determined by the Hubble constant, and there is no need to determine it independently from observations → **the Friedmann equation provides a constraint** → if there are N species there are only N-1 free density parameters.

It follows that

$\rho_c = 1.88 \times 10^{-26}\ h^2$ Kg m^{-3} (where H_0 was left undetermined)

Note this is a very small value.

For example, if the volume between the Earth and the Moon,
$V = 4/3\ \Pi\ (384400\ \text{km})^3$, would be filled with matter with this mean density, this would correspond to a mass of 2.2 Kg (assuming h=0.7).

Now, normalizing the density parameters by the critical density, we define the dimensionless **Ω density parameters:**

$$\Omega_m = \rho_{m,0} / \rho_c\ ,\ \ \Omega_r = \rho_{r,0} / \rho_c\ ,\ \ \Omega_K = \rho_{K,0} / \rho_c = - K / H_0^2\ ,\ \Omega_\Lambda = \rho_\Lambda / \rho_c = \Lambda / 3H_0^2$$

Note that with this definition, the Ω parameters are only defined today. There is no analogous definition of a Ω_m (a) function.

Note that because of the dependence of the critical density on the Hubble parameter, the values of Ω implicitly depend on the value of h. It is also usual to define h-independent parameters, called the **physical densities**:

$$\omega_m = \Omega_m h^2, \quad \omega_r = \Omega_r h^2, \text{ etc}$$

We can now write Friedmann's equation for the case of a cosmological fluid with matter, radiation, curvature and cosmological constant:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_{m,0}}{a^3} + \frac{8\pi G}{3}\frac{\rho_{r,0}}{a^4} - \frac{K}{a^2} + \frac{\Lambda}{3}$$

Inserting the critical density, we find:

$$H^2(a) = \frac{3H_0^2}{8\pi G}\left[\frac{8\pi G}{3}\frac{\Omega_m}{a^3} + \frac{8\pi G}{3}\frac{\Omega_r}{a^4} - \frac{K}{a^2}\frac{1}{\rho_c} + \frac{\Lambda}{3}\frac{1}{\rho_c}\right]$$

$$\boxed{H^2(z) = H_0^2\left[\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right]}$$

Usually this part, factoring out H_0 is labeled E(z)

Note that at z=0, the Friedmann equation reduces explicitly to the density constraint condition:

$$\Omega_r + \Omega_m + \Omega_K + \Omega_\Lambda = 1$$

$$H^2(a) = H_0^2 \left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_K a^{-2} + \Omega_\Lambda \right]$$

Given a **model** and the **parameters values** (**which at this level are the densities Ω and H_0**) we need to integrate the Friedmann equation to get the solution for a(t).

Note that the Friedmann equation is already a solution for H(a).

Solutions a(t) are easily found by solving integrals numerically.

Let us see some cases.

Note: all physical models (the ones that are not only mathematical solutions) need to include radiation, a fundamental species in the Universe. However, the measurement of the CMB radiation shows that Ω_r is very small. In terms of impact to the background dynamics it is only relevant in the early universe. We will not consider it in most of the following models.

Cosmological models with only one species

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \frac{\Omega}{a^m} \qquad (m = 3(1+w))}$$

$$\Leftrightarrow \quad \dot{a}^2 a^{m-2} = H_0^2 \Omega \qquad \Leftrightarrow \quad \dot{a}\, a^{\frac{m}{2}-1} = H_0 \sqrt{\Omega}$$

$$\Leftrightarrow \quad \int_0^t a^{\frac{m}{2}-1}\, da = H_0 \sqrt{\Omega}\, dt \qquad \Leftrightarrow \quad (a^{m/2} \propto t)$$

$$\Leftrightarrow \quad \frac{2}{m} a^{m/2} = H_0 \sqrt{\Omega}\, t$$

Note the integration is made from t=0 (where a=0) and so it does not introduce another free parameter

$$\Leftrightarrow \quad a^{m/2} = H_0 \sqrt{\Omega} \frac{m}{2} t$$

$$\Leftrightarrow \quad \left[a = \left(H_0 \sqrt{\Omega} \frac{m}{2} t \right)^{2/m} \right]$$

Note that with only one species, its energy density is necessarily $\Omega = 1$

So, for a one-species dominated fluid we find:

$$\boxed{\Omega(a) \propto a^{-3(1+w)} \quad \text{and} \quad a \propto t^{\frac{2}{3(1+w)}}}$$

Examples are:

Einstein-de Sitter universe (sCDM): only matter, $\Omega_m = 1$, $w = 0$

From the previous result: $a(t) = \left(\frac{3H_0}{2} t\right)^{2/3}$ expansion rate: ~2/3

We can also compute the evolution of $H(t) = \dot{a}(t) / a(t) \sim 1/t$ → in the EdS universe the **Hubble radius** grows faster than the scale factor → $r_H \sim t \sim a^{3/2}$

The expansion rate solution a(t) can be inverted to compute the **age of the universe**, which is just the value of t today, when $a(t_0) = 1$. For the EdS universe:

$$t_0 = 2/(3 H_0)$$

Single-species universes are fully determined by the Hubble constant (they have only one free parameter).

If H_0 is large → the universe is younger (for a given model)

The inverse of the Hubble constant defines the **Hubble time, t_H** $= 1/H_0$

$1\,pc = 3.0857 \times 10^{16}\,m$

$H_0 = 100\,h\,km/s/Mpc$

$1\,yr = 31\,556\,926\,s$

Its value is: $t_H = 3.08577 \times 10^{17}\,h^{-1}\,s$

$\boxed{t_H = 9.778\,h^{-1}\,Gyr} \Rightarrow$ 13.97 Gyr (h=0.7)

From Friedmann's equation, we see that for any model, the age of universe i.e. the solution for t(a), is an integral times $1/H_0$.

So, any age can be given in terms of a Hubble time (that absorbs the uncertainty on the H_0 value).

Radiation-dominated universe: only radiation, $\Omega_r = 1$, $w = 1/3$

$$a(t)^2 = 2 H_0 \Omega_r^{1/2} t$$

$$a \propto t^{1/2}$$

Note the expansion is slower than in EdS because, due to pressure, “gravity is stronger”

The age of the universe in this case is $t_0 = 1/2\ t_H$

→ the radiation-dominated slow expansion leads to a universe that is younger than the one with a matter-dominated faster expansion

Milne universe: only curvature, $\Omega_K = 1$, $w = -1/3$

$$a(t) = H_0\, t \qquad\qquad \text{Fast expansion}$$

The age of the Milne universe is exactly the Hubble time $t_0 = t_H$

Note that we are consistently finding that **models with faster expansion rates lead to older universes.**

Does this seem counter-intuitive?

de Sitter universe: only cosmological constant, $\Omega_\Lambda = 1$, $w = -1$

In this case, the formula of the general solution is undetermined, and we need to go back to the Friedmann equation to find the solution,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} \qquad \left(\frac{\Lambda}{3} = \Omega_\Lambda H_0^2\right)$$

This tells us that if there is only a non-evolving species, then the Hubble function remains constant: $H(a) = H_0$, and the Friedmann equation is: $\mathring{a}(t) = a(t)\, H_0$

The solution is an exponential expansion → $a(t) = C \exp(H_0\, t)$

Given the condition $a(t_0) = 1$, the constant is $C = 1 / \exp(H_0 t_0)$ → $\mathbf{a(t) = \exp[H_0(t-t_0)]}$

Inverting this solution, we can find t(a): $H_0\, t = \ln[a \exp(H_0\, t_0)]$, i.e.

$$\mathbf{t(a) = t_0 + t_H \ln(a)}$$

If we go from a=1 to a=0, ln(a) is negative, and the time decreases from t_0 to $t(a=0) = -\infty$ → **the age of the universe is infinite.**

Cosmological models with two species

Matter and radiation: $\Omega_m + \Omega_r = 1$ $\left(a_{eq} = \frac{\Omega_r}{\Omega_m} \right) = (1 - \Omega_m) / \Omega_m$

Now there is one free density parameter → **different cosmologies are possible from one model**

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3}\right) = H_0^2\, \Omega_m a^{-3} \left(1 + \frac{a_{eq}}{a}\right)$$

$$\frac{da}{dt} = H_0 \frac{\sqrt{\Omega_m}}{\sqrt{a}} \sqrt{1 + \frac{a_{eq}}{a}} = H_0 \frac{\sqrt{\Omega_m}}{\sqrt{a}} \sqrt{1 + \frac{1}{y}}$$

$y = a / a_{eq}$

It is possible to write an integral expression for t(a) and solve it analytically:

$$t(a) = \frac{t_H}{\sqrt{\Omega_m}}\, a_{eq}^{3/2} \int_0^y \frac{y}{\sqrt{1+y}}\, dy$$

$$t(a) = \frac{t_H}{\sqrt{\Omega_m}} a_{eq}^{3/2} \left(2y\sqrt{y+1}\Big|_0^y - \int_0^y 2\sqrt{y+1}\,dy \right)$$

$$= \frac{t_H}{\sqrt{\Omega_m}} a_{eq}^{3/2} \left(2y\sqrt{y+1}\Big|_0^y - \frac{4}{3}(y+1)^{3/2}\Big|_0^y \right)$$

$$t(a) = \frac{t_H}{\sqrt{\Omega_m}} a_{eq}^{3/2} \quad (y+1)^{1/2}\left(2y - \frac{4}{3}(y+1)\right)\Big|_0^y =$$

$$= (y+1)^{1/2}\left(2y - \frac{4}{3}y - 4/3\right)\Big|_0^y = 2(y+1)^{1/2}\left(y - \frac{2}{3}y - \frac{2}{3}\right)\Big|_0^y =$$

$$= 2(y+1)^{1/2}\left(\frac{1}{3}y - \frac{2}{3}\right)\Big|_0^y = \frac{2}{3}(y+1)^{1/2}(y-2)\Big|_0^y =$$

$$= \frac{2}{3}(y+1)^{1/2}(y-2) - \frac{2}{3}(-2) = \frac{2}{3}\left[(y+1)^{1/2}(y-2)+2\right]$$

$$\boxed{t(a) = \frac{t_H}{\sqrt{\Omega_m}} a_{eq}^{3/2} \frac{2}{3}\left[\sqrt{\frac{a}{a_{eq}}+1}\left(\frac{a}{a_{eq}}-2\right)+2\right]}$$

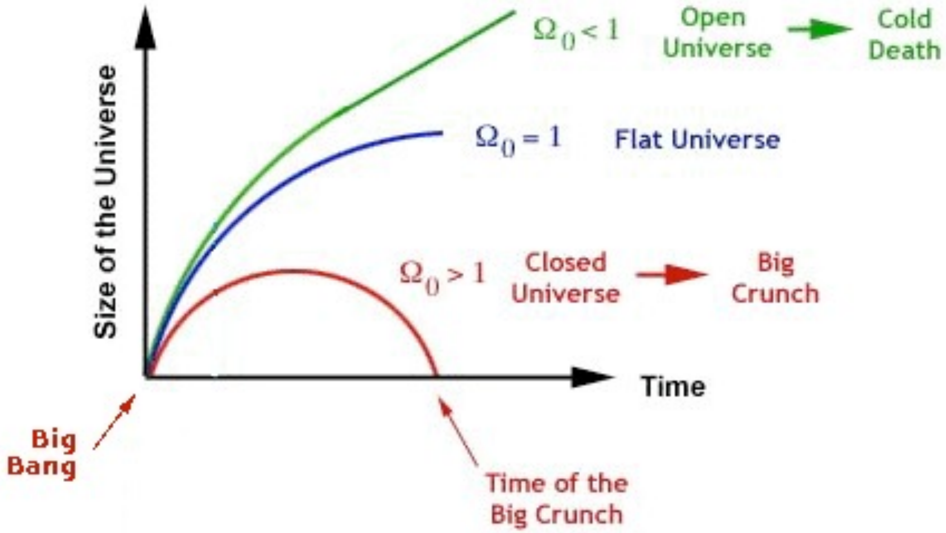
Note the free parameters are t_H (i.e. H_0) and a_{eq} (i.e. Ω)

Matter and curvature: $\Omega_m + \Omega_K = 1$

$$\frac{da}{dt} = H_0\sqrt{|\Omega_K|}\left(\frac{a_K}{a}+1\right)^{1/2} \qquad a_K = \frac{\Omega_m}{\Omega_K}$$

This model can have various cosmologies, grouped in three types:

- $\Omega_K > 0$: **Open CDM (oCDM)**, a(t) expands fast, and the universe is older
- $\Omega_K = 0$: **Standard CDM (sCDM)**, a(t) expands slower
- $\Omega_K < 0$: **Friedmann-Einstein**, a(t) expands slower and contracts



These are the three well-known classical GR cosmologies

Cosmological models with three species

ΛCDM: Matter, curvature and cosmological constant: $\Omega_m + \Omega_K + \Omega_\Lambda = 1$

Note: remember ΛCDM also includes radiation, that we neglect here.

We are left then with two free density parameters, and we can place all the possible ΛCDM models in **the (Ω_m , Ω_Λ) plane**.

Let us find the possible qualitative behaviours of the various models:

The goal is to analyse the general evolution behaviour, not the actual a(t) rate.

The general possibilities are expanding or contracting.

From observations, we know the universe is expanding now. So either it has always expanded and will continue to do so in the future, or it had already a contracting phase (or it will have in the future). In this case, H(t) will be (or was) <0

So, we may look for **H(a) = 0 as an indicator of a transition from expansion to collapse (or collapse to expansion).**

This means that, using the Friedmann equation, it is useful to consider the **third-order polynomial f(a)**:

$$\left[\Omega_m + \Omega_\Lambda a^3 + (1 - \Omega_m - \Omega_\Lambda)\, a\right] = f(a)$$

Its roots f(a)=0 (for a>0) will correspond to the instants of **transition**

if root a < 1 → transition in the past
if root a > 1 → transition in the future

The flat line ($\Omega_\Lambda = 1 - \Omega_m$)

Consider the particular case of $\Omega_K = 0$.

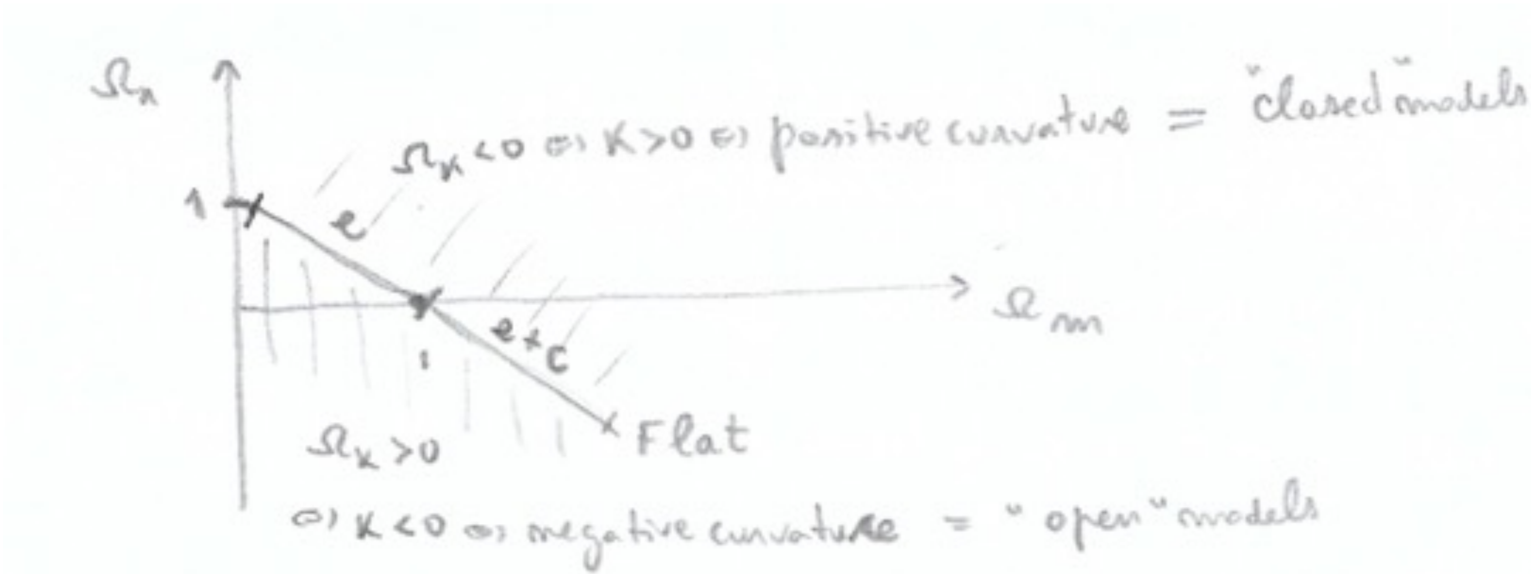
Then all ΛCDM models lie in the flat line $\Omega_\Lambda = 1 - \Omega_m$

and the transition polynomial simplifies to $f(a) = \Omega_m + (1-\Omega_m)\,a^3$

with roots $a = \left(\dfrac{\Omega_m}{\Omega_m - 1}\right)^{1/3}$

This means that, for models with $\Omega_m > 1$, there is a transition and the larger is Ω_m the earlier the transition occurs.

For models with $\Omega_m < 1$ there is no transition



Flat models lie on this line, and they can be of two types: always expanding (e), or expanding + contracting (e+c).

The line also separates positive curvature and negative curvature models.

The no-Λ line ($\Omega_\Lambda = 0$)

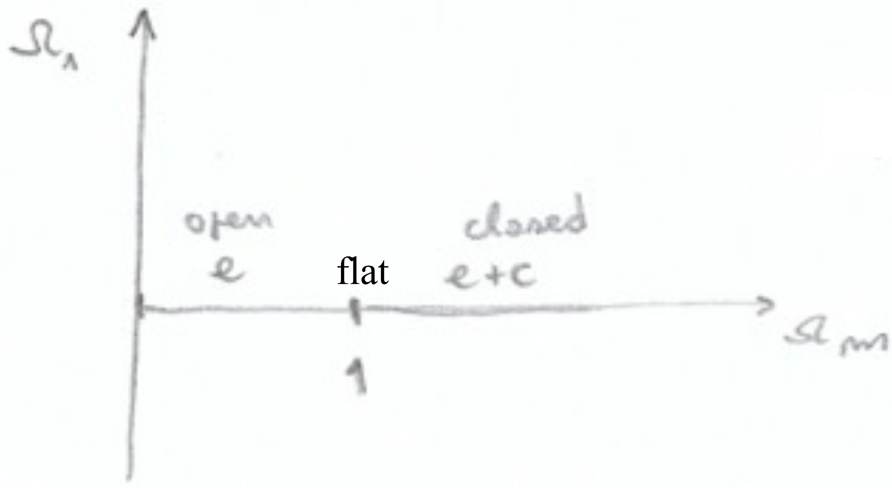
In this particular case of $\Omega_K = 1 - \Omega_m$ we have

$$f(a) = \Omega_m + (1 - \Omega_m)\, a$$

$$f(a) = 0 \Rightarrow \left[a = \frac{\Omega_m}{\Omega_m - 1} \right]$$

$\Omega_m < 1 \Rightarrow$ No roots $a > 0$

$\Omega_m > 1 \Rightarrow$ Roots $\quad \uparrow \Omega_m \Rightarrow$ sooner recollapse



We recover the 3 classical models.

Note however that in general open curvature does not imply (e),

and closed curvature does not necessarily lead to (e+c)

The collapse in the future region (a > 1)

Turning now to the general case, $f(a) = \Omega_m (1-a) + a + \Omega_\Lambda (a^3 - a)$

let us consider examples of collapse in the future:

$a = 2$

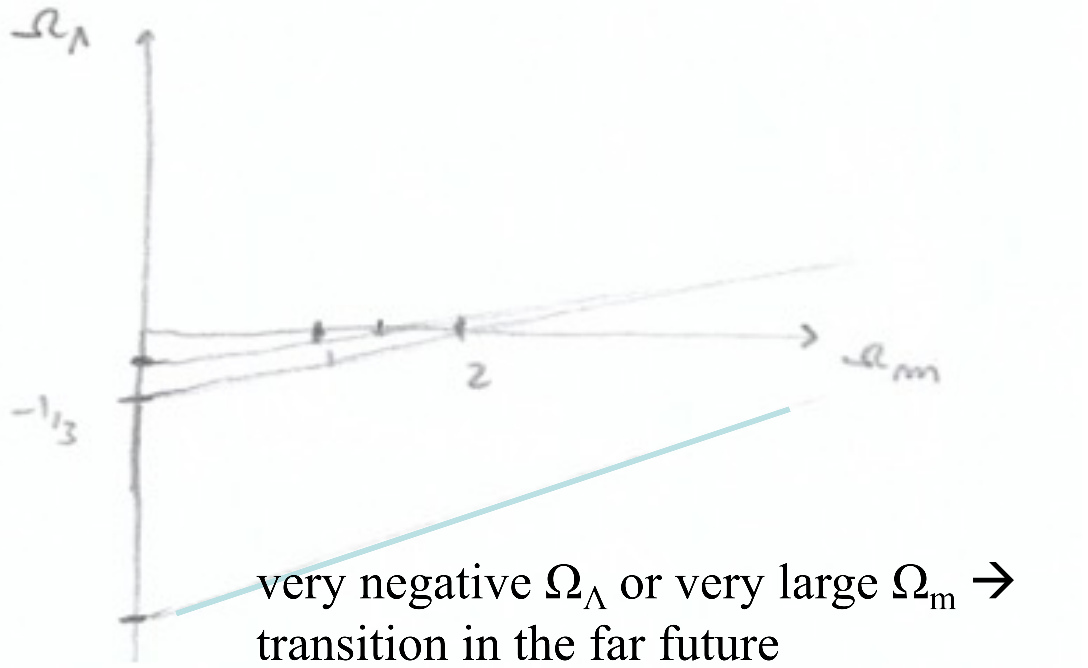
$$\Omega_m(-1) + 2 = -6\,\Omega_\Lambda$$

$$\Rightarrow \quad \Omega_\Lambda = \frac{1}{6}\Omega_m - \frac{1}{3}$$

$a = 4$

$$-3\,\Omega_m + 4 = -60\,\Omega_\Lambda$$

$$\Omega_\Lambda = \frac{1}{20}\Omega_m - \frac{1}{15}$$



Models with this property (e+c with transition in the future) lie on these straight lines (one for each value of transition).

The collapse in the past region (a < 1)

Let us consider examples of collapse in the past.

Note: since the universe is expanding today, these cases imply c+e (i.e., bouncing models with no big bang), instead of e+c → **GR allows models without Big Bang**

$$\Omega_m \underbrace{(1-a)}_{>0} + a = -\Omega_\Lambda \underbrace{(a^3-a)}_{<0}$$

⇒ The collapse area will have zeropoint at $\Omega_\Lambda > 0$ and slope > 0

$a = 0.1$

$$\Omega_m\, 0.9 + 0.1 = \Omega_\Lambda\, 0.1$$

$$\Rightarrow \Omega_\Lambda = 9\,\Omega_m + 1$$

$a \ll 1$

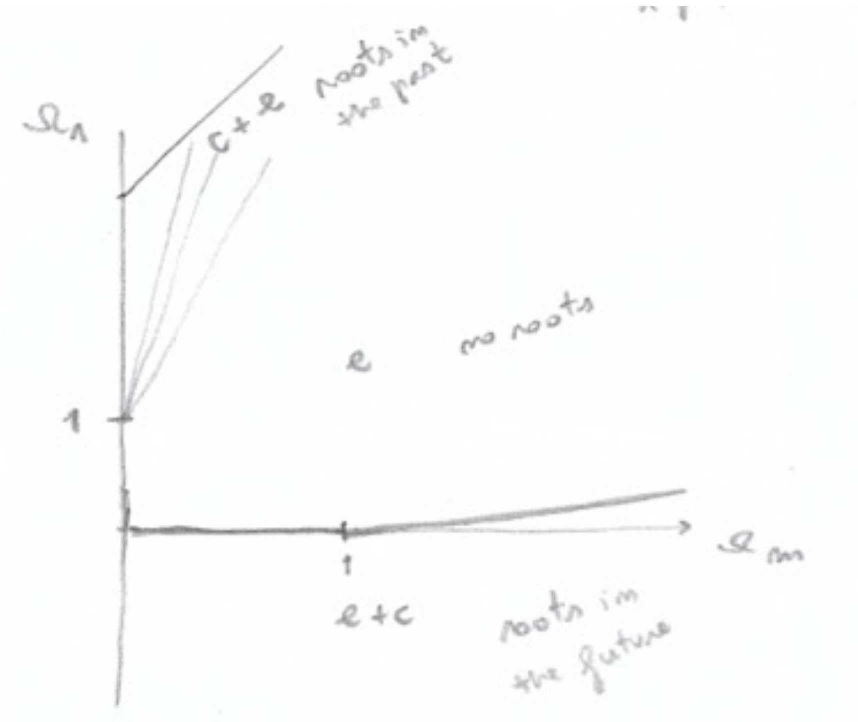
$$\Omega_m + a = \Omega_\Lambda\, a$$

If $a \ll 1$, line starts at $\Omega_\Lambda = 1$

$a = 0.9$

$$\Omega_m\, 0.1 + 0.9 = -\Omega_\Lambda\, (-0.17)$$

$$\Omega_\Lambda = 0.6\,\Omega_m + 5$$



Models with this property (c+e with transition in the past) lie on these straight lines (one for each values of transition).

Note: a measurement of the transition redshift would constrain the model → finding the line where the "real" cosmology is → **values along the same line are degenerate with respect to this observable** (the **transition redshift**)

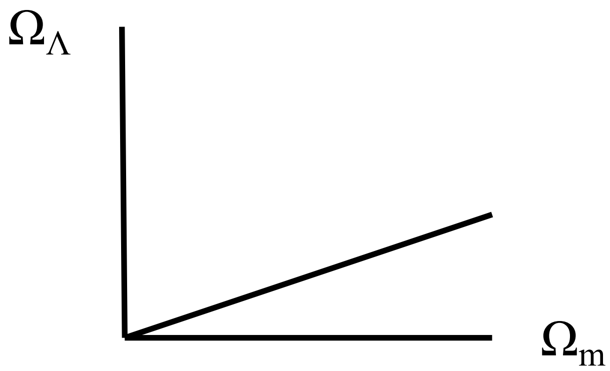
The no-acceleration line ($\Omega_m - 2\Omega_\Lambda = 0$)

Introducing the three species in the second Friedmann equation, we can find a constraint for the models that do not have acceleration today:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$\Omega_m a^{-3} + \Omega_K [1+ 3(-1/3)] a^{-2} + \Omega_\Lambda (1-3) = 0 \quad (\text{for } a = 1)$$

This is then $\Omega_m - 2\Omega_\Lambda = 0$



Note: a measurement of the acceleration of the universe would constrain the model → finding the line where the "real" cosmology is → **values along the same line are degenerate with respect to this observable** (the **acceleration**)

Note: the acceleration line intersects the curvature line. Two independent measurements (of the acceleration and the curvature) would allow us to find the intersection point of the two lines → **breaking the degeneracy of the cosmological parameters.**

The loitering line

Universes with a c+e transition but with no acceleration at the transition redshift, cannot leave the transition point → they remain trapped at that point with zero H(a) and zero acceleration.

They are called loitering cosmologies and lie on a line separating the no-big bang universes from the big bang universes.

Let us find out what are the scale factors at which the acceleration of a universe can go to zero. Again, from the second Friedmann equation, these are the scale factors that verify:

$\Omega_m a^{-3} - 2\Omega_\Lambda = 0 \rightarrow a^3 = \frac{\Omega_m}{2\Omega_\Lambda}$ (the scale factor is different for each universe)

Now, we are looking for cases where this happens at a transition, i.e., which verify f(a)=0

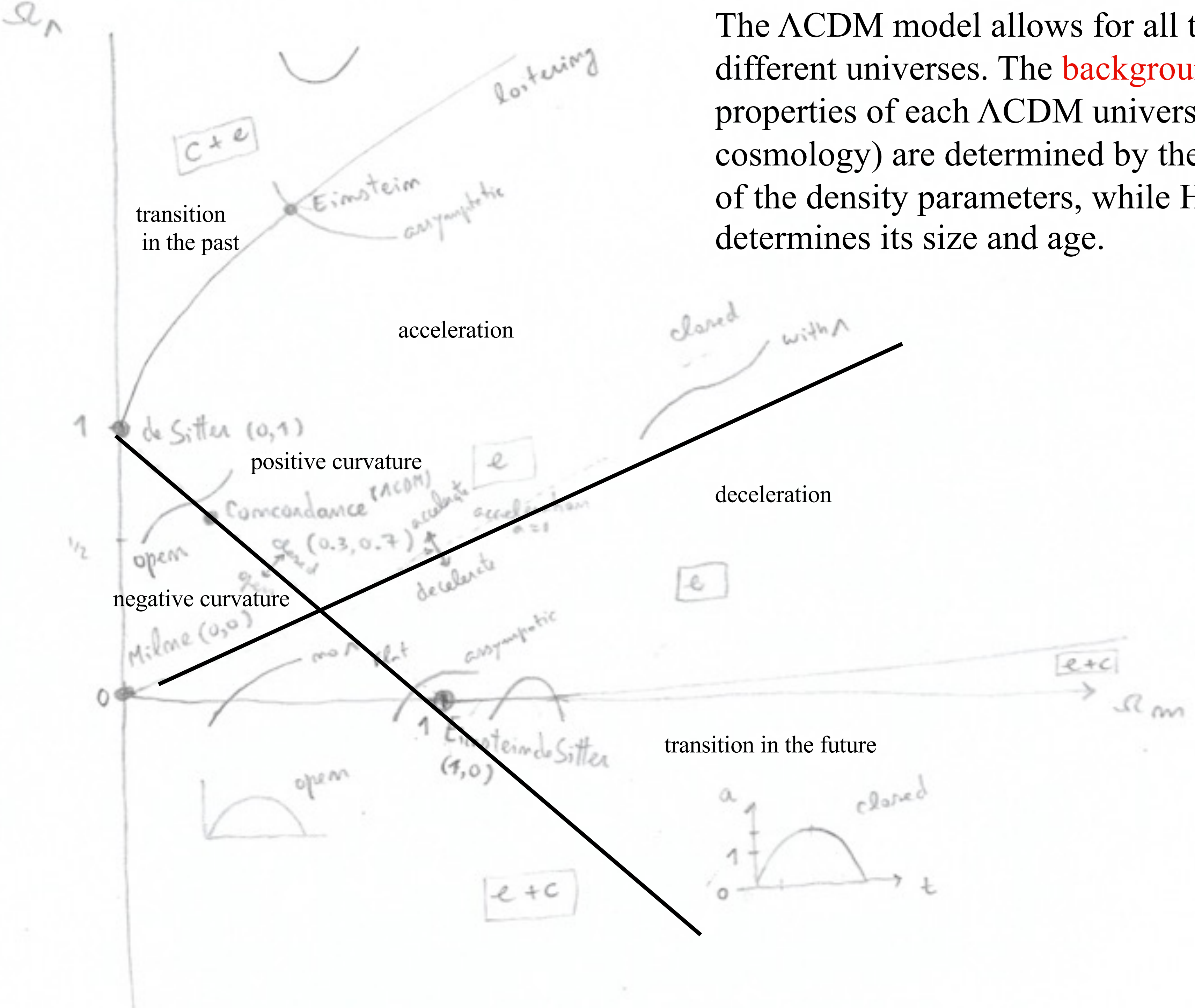
Inserting in f(a):

$$\Rightarrow \quad \Omega_m (1-a) + a = -\Omega_\Lambda (a^3 - a)$$

$$\Rightarrow \quad \Omega_m + \left(\frac{\Omega_m}{2\Omega_\Lambda}\right)^{1/3} (1-\Omega_m-\Omega_\Lambda) - \frac{\Omega_m}{2}$$

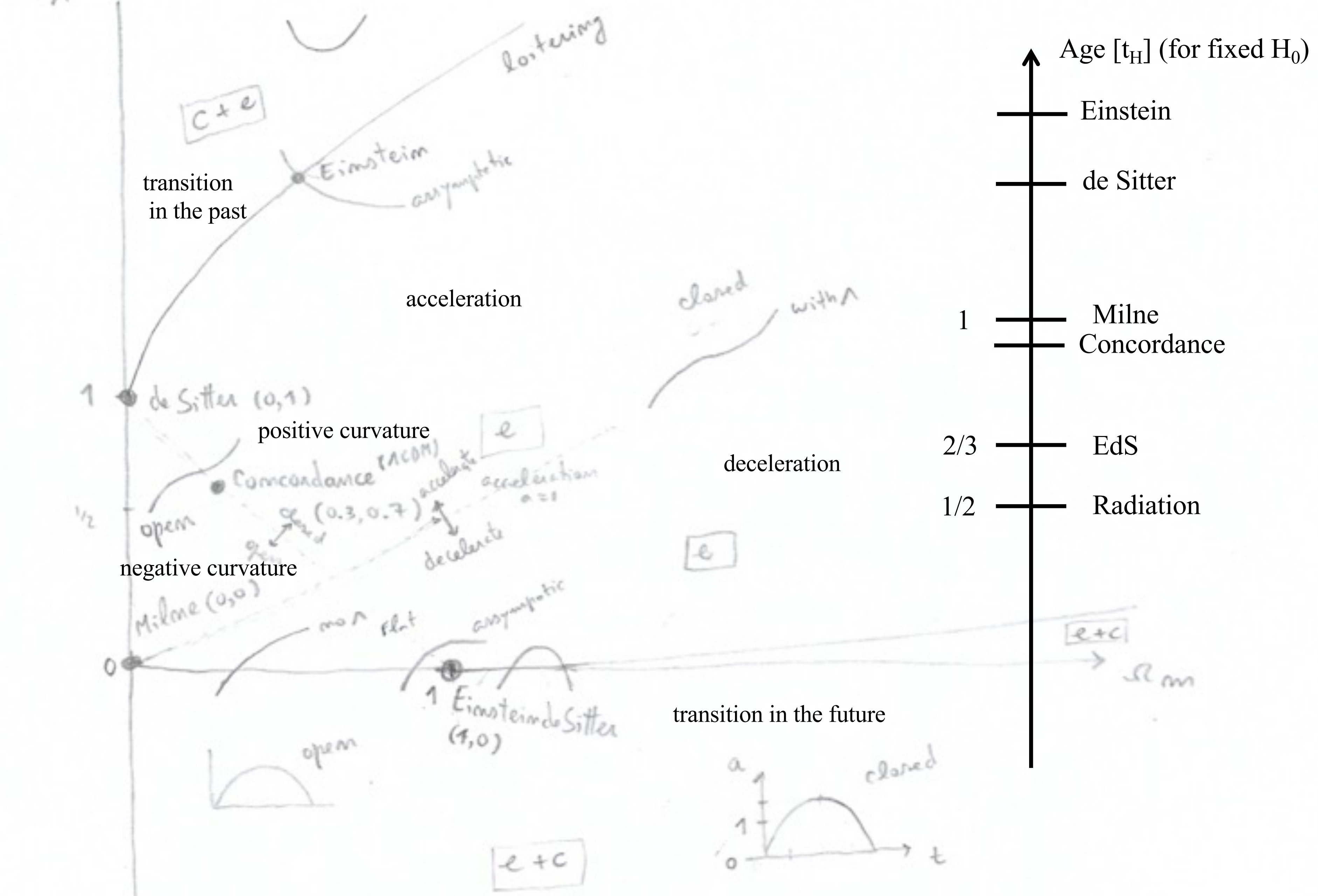
$$\Rightarrow \quad \Omega_m + 2(1-\Omega_m-\Omega_\Lambda)\left(\frac{\Omega_m}{2\Omega_\Lambda}\right)^{1/3} = 0 \longrightarrow \text{It is the loitering equation}$$

This is a curve in the (Ω_m , Ω_Λ) plane. The well-known static **Einstein universe** is on this curve.



The ΛCDM model allows for all these very different universes. The background properties of each ΛCDM universe (or cosmology) are determined by the values of the density parameters, while H_0 determines its size and age.

Some well-known cosmologies are:



Note: you can compute background properties (age and distances) of these models, using the on-line cosmology calculator: *http://www.astro.ucla.edu/wright/CosmoCalc.html*