Cosmologia Física

Homework 6 due 28 May 2026 (iatereno@fc.ul.pt)

Exercise 1: Parameter estimation

The distance modulus, $\mu = 5\log_{10} D_L + 25$, depends on the cosmological parameters through the luminosity distance D_L. In the following, consider that D_L is given by the approximation:

$$D_L = \frac{c}{H_0}\left[z + \frac{z^2}{2}\left(1 + \Omega_\Lambda - \frac{\Omega_{\rm m}}{2}\right)\right].$$

1.1)

a) Is the observable μ more sensitive to $\Omega_{\rm m}$ or to Ω_Λ?

b) μ depends on a combination of the two parameters of the model. This implies that a measurement of μ correlates the parameters. The estimated parameter values become correlated or anti-correlated?

c) In a flat model (and considering only a fluid with two components) the two parameters are not independent: are they correlated or anti-correlated?

d) From the results of b) and c) would you say that assuming a flat model in the parameter estimation procedure is a strong prior that has a large impact on the results or a weak prior with negligible impact?

1.2) Consider a very small supernova survey that detected 3 supernovae and measured their distance modulus with uncorrelated errors. The data vector obtained is: distance modulus $\mu = 42.10 \pm 0.25$ for the object SN 2002dc at z=0.475; $\mu = 43.14 \pm 0.21$ for SN 2003bd at z=0.67; and $\mu = 44.25 \pm 0.14$ for SN 2001hb at z=1.0. (These values are taken from the SNLS survey).

Consider also the Fisher matrix for the distance modulus, which is given by

$$F_{ij} = \sum_z \left[\left(\frac{\partial \mu_z}{\partial p_i}\right)_{\rm fid} \frac{1}{\sigma_z^2} \left(\frac{\partial \mu_z}{\partial p_j}\right)_{\rm fid}\right]$$

a) Compute the Fisher matrix of this survey, assuming $\Omega_{\rm m} = 0.3$ and $\Omega_\Lambda = 0.7$ as the fiducial model. (Note that we do not need to give a fiducial value to the Hubble constant, because the Fisher matrix will be independent of H_0.)

b) Notice that the Fisher matrix just obtained is singular (which does not happen when using the full integral expression for D_L). What is the feature in the $D_L(z)$ approximate formula that is responsable for this behaviour?

c) Being singular, the associated contour ellipse stretches to infinity, and we cannot get finite constraints on the individual parameters. This means that μ only has information about one effective parameter: the combination $\Omega = \Omega_\Lambda - \Omega_{\rm m}/2$. Use the Fisher matrix method in just one dimension for the effective parameter Ω, to find the constraint on that parameter. Write the result in the form: $\Omega = \Omega_{\text{fid}} \pm \sigma_\Omega$.

1.3) Consider again the supernova survey of three supernovae presented above, but now including the stretch bias in the estimator. This implies that the μ estimator needs to be corrected with an additive parameter, i.e.,

$$\mu = 5\log_{10} D_L + 25 + \alpha\,(s-1),$$

where, for simplification, the 3 SNe all have the same stretch factor s, and so the additive term is a constant bias parameter, $b = \alpha\,(s-1)$.

The bias also correlates the measurements of the 3 supernovae, implying that the errors are now given by a non-diagonal covariance matrix:

$$C = \begin{matrix} (0.25)^2 & 0.007 & 0.003 \\ 0.007 & (0.21)^2 & 0.005 \\ 0.003 & 0.005 & (0.14)^2 \end{matrix}$$

a) The impact of the bias is larger at low or at high redshift?

b) Compute the Fisher matrix, for the biased SNe survey on the (Ω, b) parameter space, where $\Omega = \Omega_\Lambda - \Omega_\text{m}/2$, using again the second-order approximation formula for D_L.

Hint: Remember the Fisher matrix is now given by its full form to account for the cross-correlations:

$$F_{ij} = \left(\frac{\partial \mu_z}{\partial p_i}\right)^T_{\text{fid}} C^{-1}_{zz'} \left(\frac{\partial \mu_{z'}}{\partial p_j}\right)_{\text{fid}}$$

c) Compute the marginalized uncertainty of Ω, i.e. marginalize the Fisher matrix over the bias parameter b.

d) Compare this result with the constraint obtained in the unbiased case. Did the uncertainty on the cosmological parameter Ω increase or decrease by correcting the bias? Is this the expected behaviour? Why?