UNIVERSO PRIMITIVO: INFLAÇÂO E ESTRUTURA DE LARGA ESCALA
Mestrado em Física Astronomia
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Exercise Sheet 4

1. Consider a particle physics model with 4 families of massless neutrinos (all with one helicity state).
   1.1. Compute the ratio between the neutrino temperature and photon temperature, T_ν/T_γ, just after electron-positron annihilation ($T < 0.5$ MeV).
   1.2. Compare/comment your findings with the same ratio for three neutrino families (derived in classroom).
   1.3. Compute the effective degrees of freedom in entropy and energy at that time.
   1.4. What would be the present temperature of the neutrinos (express your answer in Kelvin)?

2. Recall the Riccati equation derived in classroom for weakly interactive massive particles (WIMPs) written as ($Y \equiv N_X$):
$$\frac{dY}{dx} = -\frac{\lambda}{x^2}[Y^2 - Y_{eq}^2]$$
where $x = M_X/T$, M_X is the mass of the WIMP particles, T is the photon temperature and λ can be treated as a constant. Let $\Delta \equiv Y - Y_{eq}$ be the variable that measures the deviation of Y from its equilibrium value.
   2.1. Prove that:
$$\frac{d\Delta}{dx} = -\frac{dY_{eq}}{dx} - \frac{\lambda}{x^2}[\Delta^2 + 2Y_{eq}\Delta]$$
   2.2. Simplify this equation using the approximation $Y \simeq Y_{eq} \Rightarrow \Delta \simeq 0$ and $d\Delta/dx \simeq 0$, valid for the temperature range $1 < x < x_f$, where $x_f = M_X/T_f$ is the freeze-out temperature. [Hint: note that under these approximations, the first term inside the square brackets is smaller than the second term]
   2.3. Derive an expression for Δ assuming $Y_{eq} \approx e^{-x}$. How does it depend on x and λ?
   2.4. Re-derive Δ, now using $Y_{eq} = N_X^{eq} = n_X^{eq}/s$ [Hint: assume that the WIMP particles are already non-relativistic and write their equilibrium density, n_X^{eq}, and the specific entropy of the fluid, s, as a function of x].

3. Considering that the equilibrium number density of protons, neutrons and a nuclear species with Z protons and $A - Z$ neutrons (where A is the nuclear atomic mass and Z the charge of the nucleus) can be written as:
$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right)$$
where $i = \{p, n, A\}$, show that the number density of the nucleus A is given by:
$$n_A = \frac{g_A}{2^A} A^{3/2} \left(\frac{m_B T}{2\pi}\right)^{3(1-A)/2} n_p^Z n_n^{A-Z} \exp\left(\frac{B_A}{T}\right)$$
where $B_A = Zm_p + (A - Z)m_n - m_A$ is the biding energy of the nucleon A. [Hint: Note that the chemical potential of the nucleus, μ_A, is related with the chemical potentials of the protons, μ_p, and neutrons, μ_n, by $\mu_A = Z\mu_p + (A - Z)\mu_n$. Use also the approximations $m_A = Am_B$, with $m_B = m_p \approx m_n$].