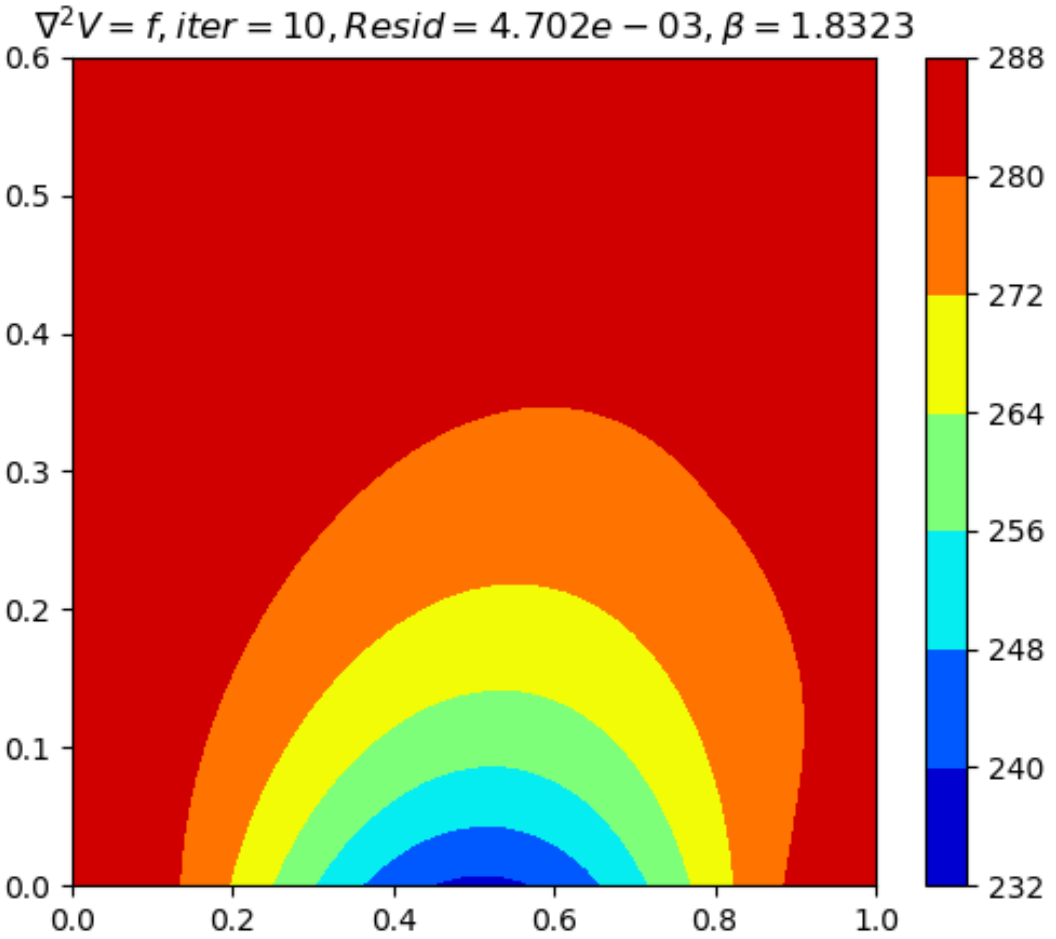
Aula 20

Transferência de calor em regime estacionário



Lei de Fourier da condução

Num meio conductor, o fluxo de calor (Wm^{-2}) é proporcional ao gradiente de temperatura, transportando calor em direção às regiões mais frias:

$$\vec{q} = -\chi \nabla T = -\chi \left(\frac{\partial T}{\partial x} \vec{\imath} + \frac{\partial T}{\partial y} \vec{\jmath} + \frac{\partial T}{\partial z} \vec{k} \right)$$

χ ($Wm^{-1}K^{-1}$) é a condutividade térmica do meio, depende da sua composição e da temperatura.

A taxa de variação da temperatura num ponto depende da distribuição do fluxo de calor:

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot (-\chi \nabla T)$$

ρ é a densidade, c_p é o calor específico do meio.

Na presença de fontes internas de calor…

Incluindo fontes internas de calor (reações químicas, decaimento radioativo, ou outras) dadas por $\dot{q}_V$ (Wm^{-3}), obtém-se a forma mais geral da lei de Fourier:

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot (-\chi \nabla T) + \dot{q}_V$$

Em equilíbrio $T = const$, tem-se:

$$-\nabla \cdot (-\chi \nabla T) + \dot{q}_V = 0$$

No caso $\chi = const$ (material homogéneo com pouco gradiente de temperatura), fica a equação de Poisson:

$$\nabla^2 T = -\frac{1}{\chi}\, \dot{q}_V$$

Na ausência de fontes internas, temos a equação de Laplace:

$$\nabla^2 T = 0$$

poisson.py v2 com condições fronteira opcionais (von Neumann)

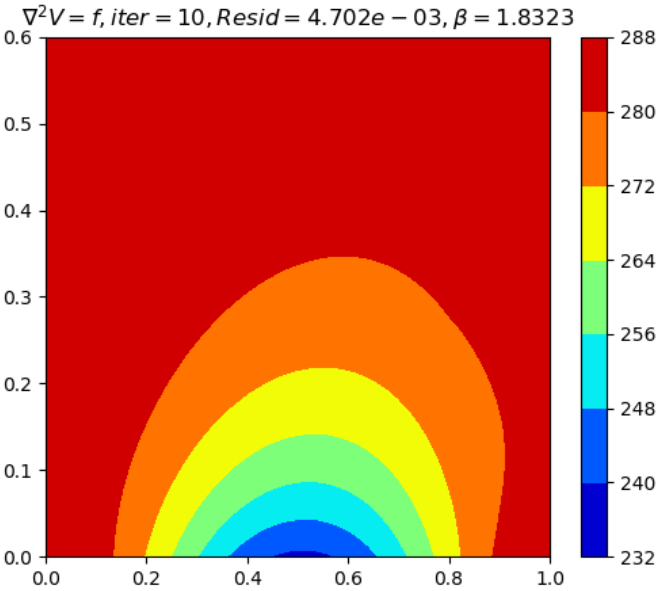
```
def poisson(f,V0,X,Y,maxiter,maxres,\  #posicionais (obrigatórias)
            beta0=0.,BxL=[],BxH=[],ByL=[],ByH=[],movie='',passo=1):    #kwargs
    [M,N]=f.shape; #determina a dimensão das matrizes
    …
    V=V0 #inicializa a matriz soluçáo
    iter=0; resid=2*maxres
    while resid>maxres and iter<maxiter: #iterações
        iter=iter+1;resid=0; vmax=0
        for i in range(1,M-1):
            for j in range(1,N-1):
                R=V[i,j-1]+V[i,j+1]+V[i-1,j]+V[i+1,j]-4*V[i,j]-delta**2*f[i,j]
                V[i,j]=V[i,j]+beta*R/4;
                resid=max(resid,abs(R))
        if len(BxL)!=0: #caso contrário: Dirichlet, mantem V
            V[0,:]=V[1,:]+BxL*dx #BxL=(dT/dn) em x=0 (n normal à parede)
        if len(BxH)!=0:
            V[M-1,:]=V[M-2,:]+BxH*dx #BxH=(dT/dn) em x=Lx
        if len(ByL)!=0:
            V[:,0]=V[:,1]+ByL*dy
        if len(ByH)!=0:
            V[:,N-1]=V[:,N-2]+ByH*dy
        vmax=np.max(np.abs(V)); resid=resid/vmax #residuo relative
    return V,iter,resid,beta
```

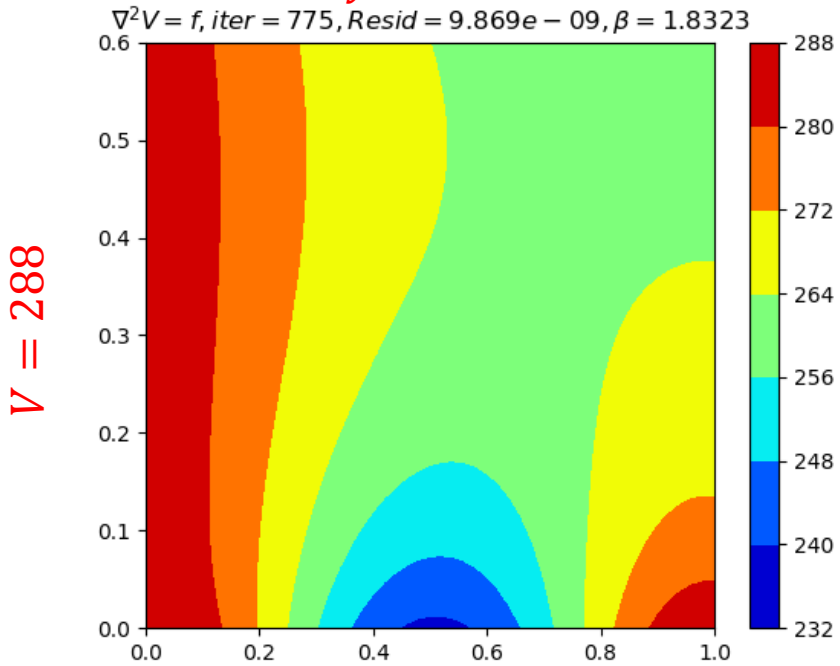
main

```
import numpy as np;import matplotlib.pyplot as plt
Lx=1.;Ly=1.2;M=51;N=31;delta=Lx/(M-1)
X=np.zeros((M,N));Y=np.zeros((M,N));ro=np.zeros((M,N));f=np.zeros((M,N))
for i in range(M):
    for j in range(N):
        X[i,j]=i*delta; Y[i,j]=j*delta
maxiter=3000;maxres=1.e-8 # erro relativo
V0=288.*np.ones((M,N)) #temperatura inicial inclui cond front Dirichlet
for ix in range(M):
    V0[ix,0]=288-50*np.sin(ix*np.pi/M)**2
BxH=np.zeros((N)); ByH=-np.ones((M))*10 #cond front von Neumann
[V,niter,res,beta]=poisson(f,V0,X,Y,maxiter,maxres,passo=10,\
     movie='T3',BxH=BxH,ByH=ByH)
plt.figure(2)
map=plt.contourf(X,Y,V,cmap='jet') # gráfico de isolinhas
plt.colorbar(map,label='T')
plt.xlabel('m');plt.ylabel('m');plt.axis('equal');
plt.title(r'$\nabla^{2} V=f,iter=%6i,Resid=%5.1e,\beta=%5.2f  $' %  \
                  (niter,res,beta))
```

```
for ix in range(M):
    V0[ix,0]=288-50*np.sin(ix*np.pi/M)**2
BxH=np.zeros((N));ByH=-np.ones((M))*10
```

$$\frac{\partial V}{\partial y} = 10 \text{ K/m}$$



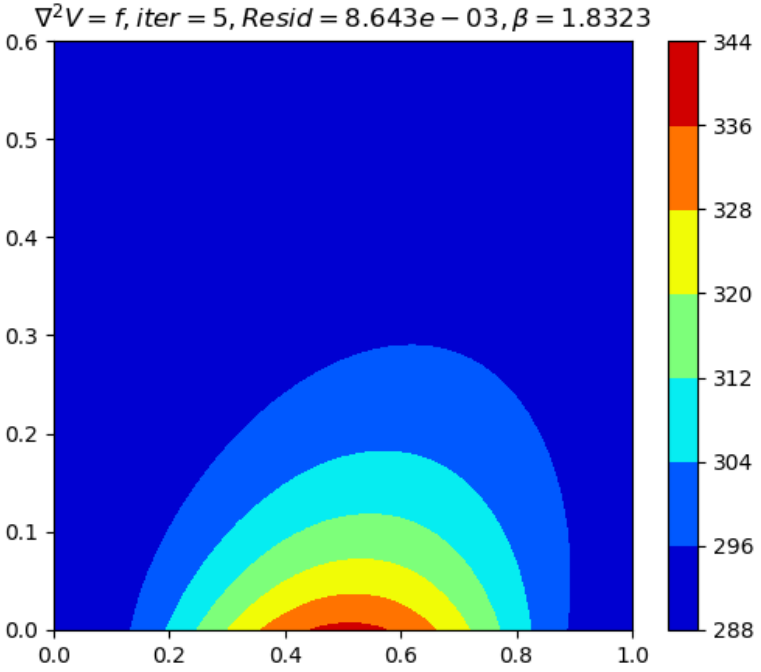


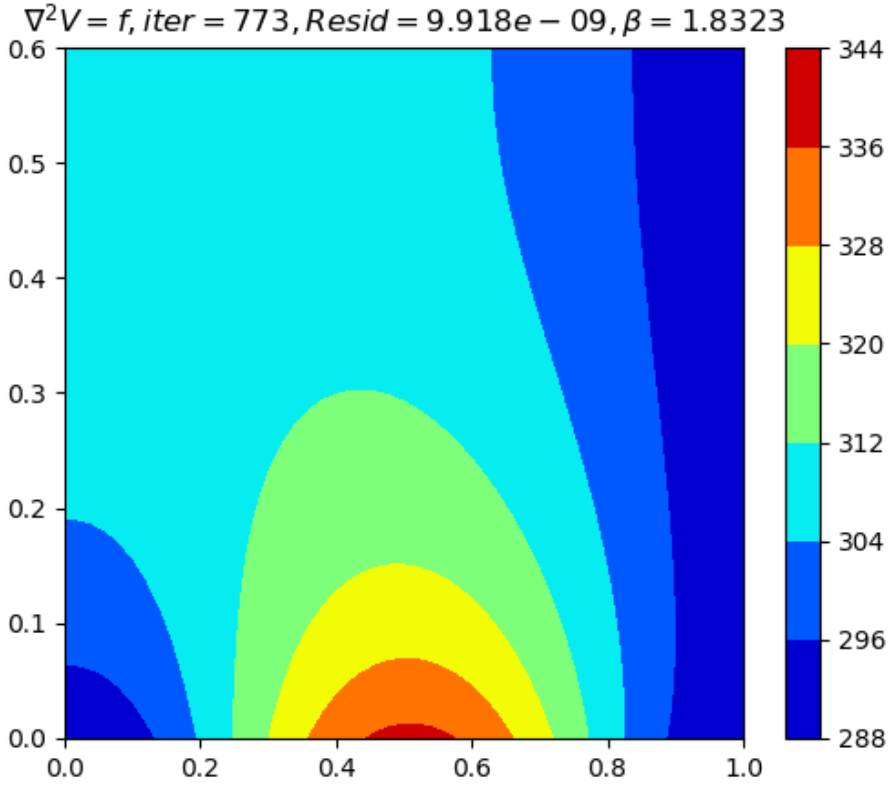
$V = 288$

$\frac{\partial V}{\partial x} = 0$

$$V = 288 - 50 * \sin\left(\frac{x\pi}{M}\right)^2$$

```
BxL=np.zeros((N));ByH=np.zeros((M))
[V,niter,res,beta]=poisson(f,V0,X,Y,maxiter,
maxres,beta0,BxL=BxL,ByH=ByH)
```

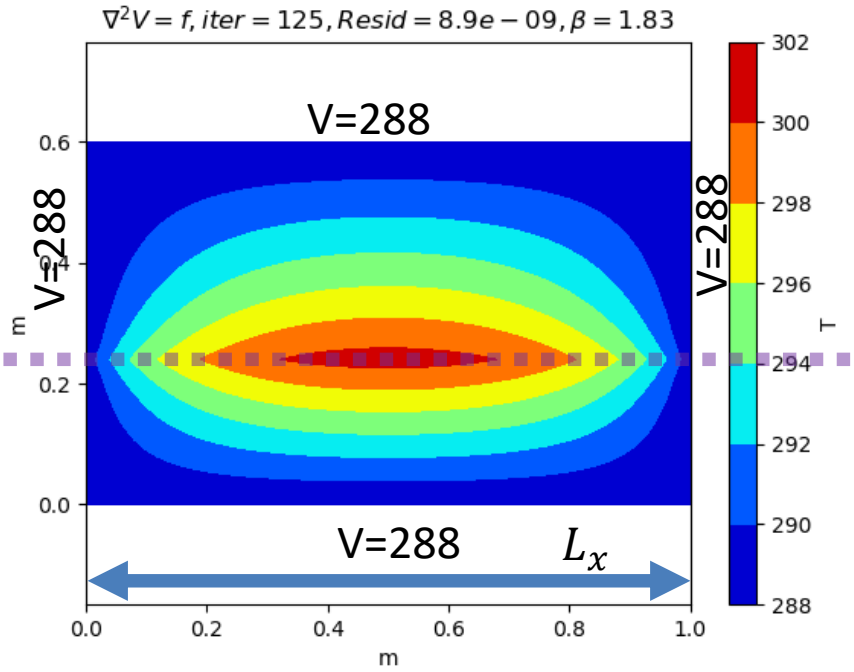




Introduzindo fontes de calor (internas)

$$\nabla^2 T = -\frac{1}{\chi}\,\dot{q}_V$$

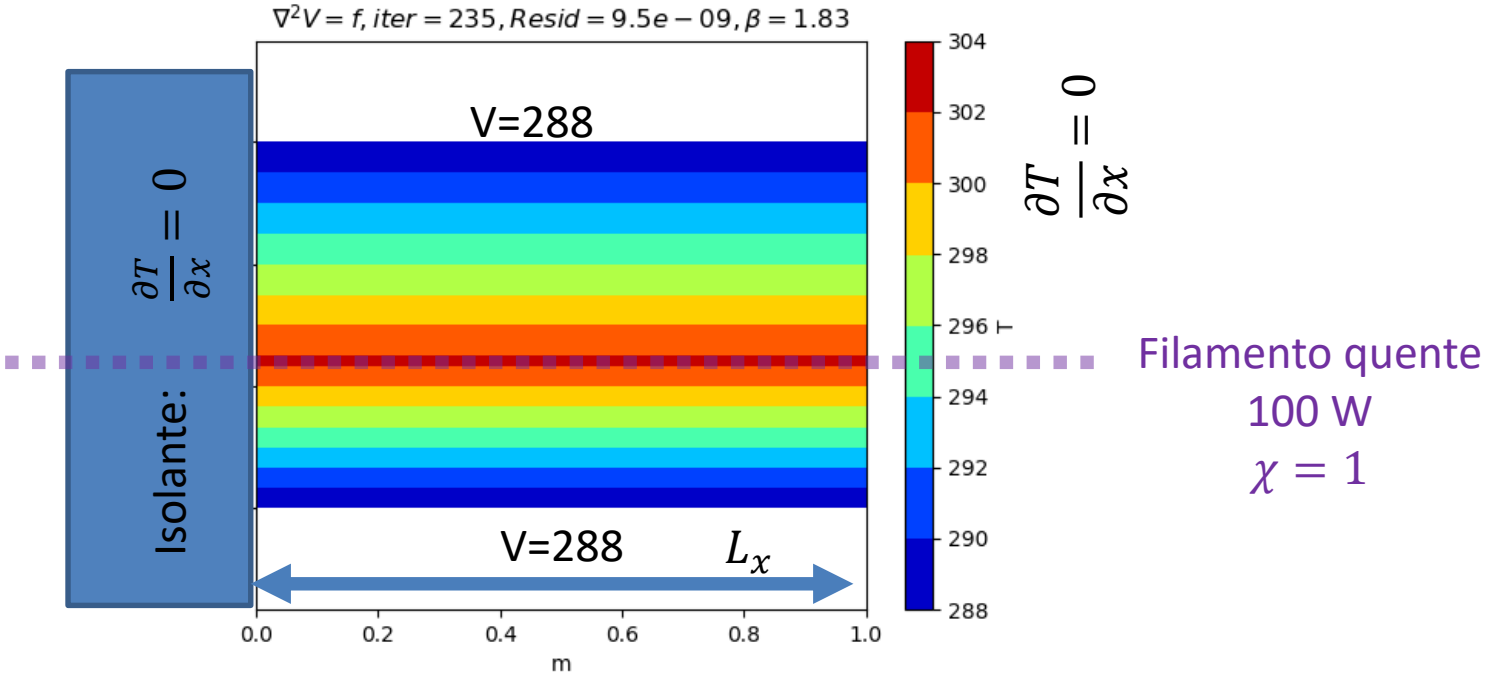
```
f=np.zeros((M,N))
for k in range(0,M):
    f[k,12]=-100/(Lx*delta)
[V,niter,res,beta]=poisson(f,V0,X,Y,maxiter,maxres)
```



Filamento quente
100 W
$\chi = 1$

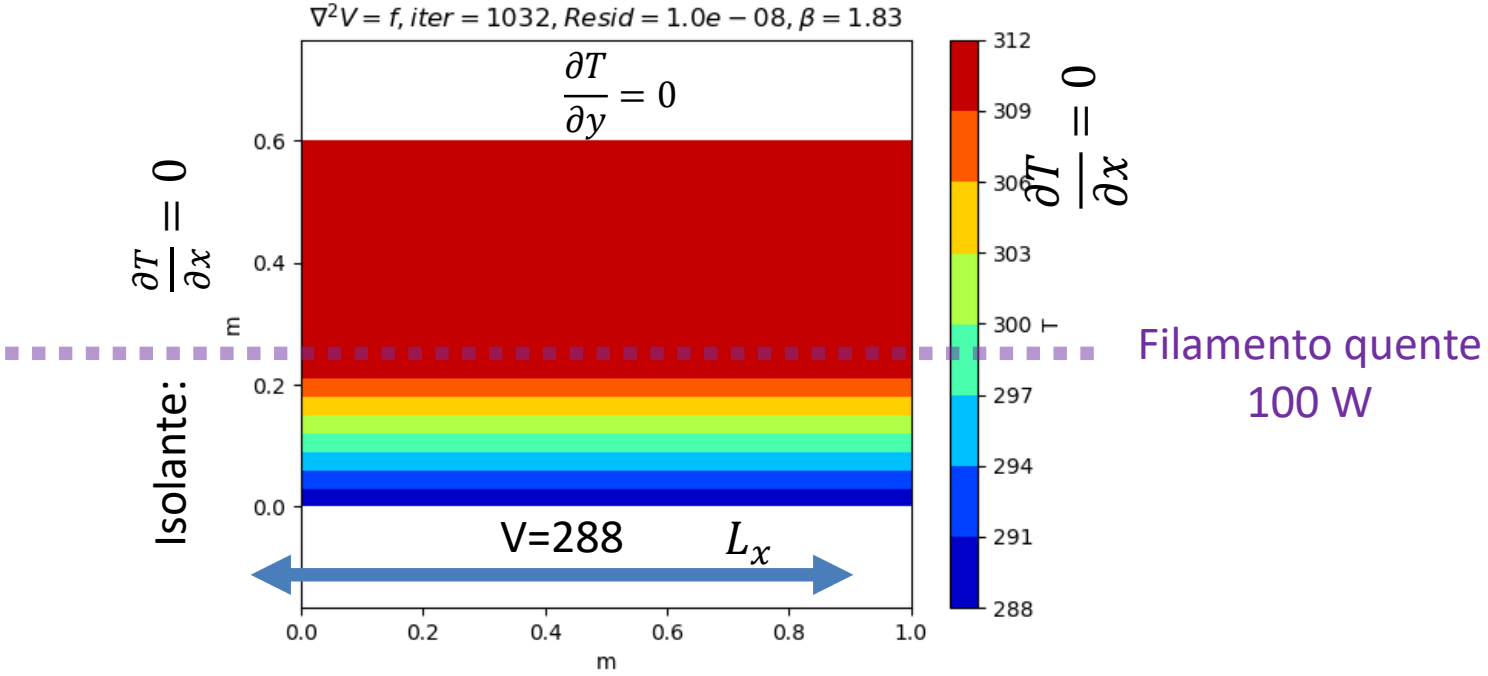
Mesmo, forçamento, outras condições fronteira

```
f=np.zeros((M,N))
for k in range(0,M):
    f[k,12]=-100/(Lx*delta)
[V,niter,res,beta]=poisson(f,V0,X,Y,maxiter,maxres,BxL=np.
zeros((N)),BxH=np.zeros((N)))
```

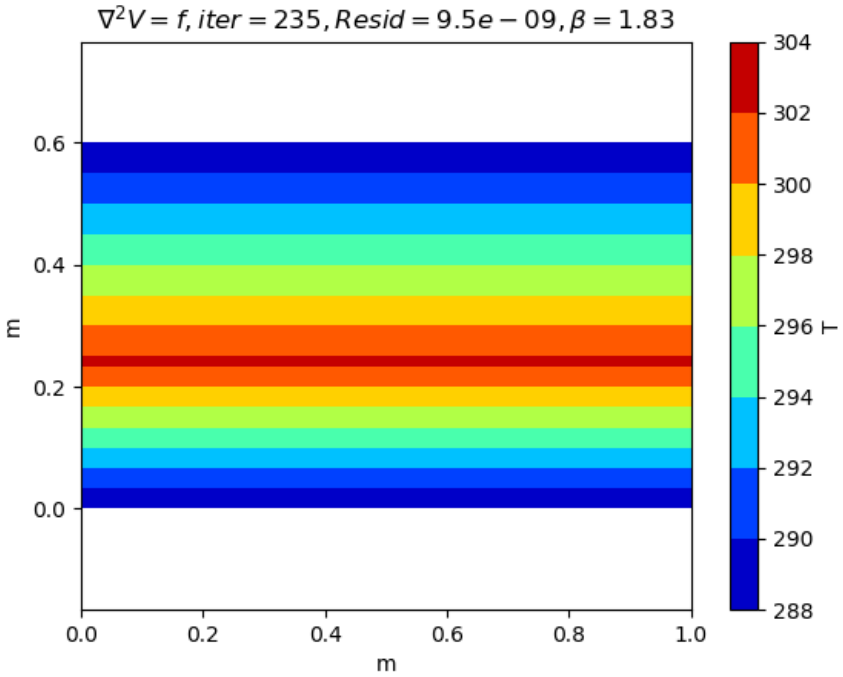


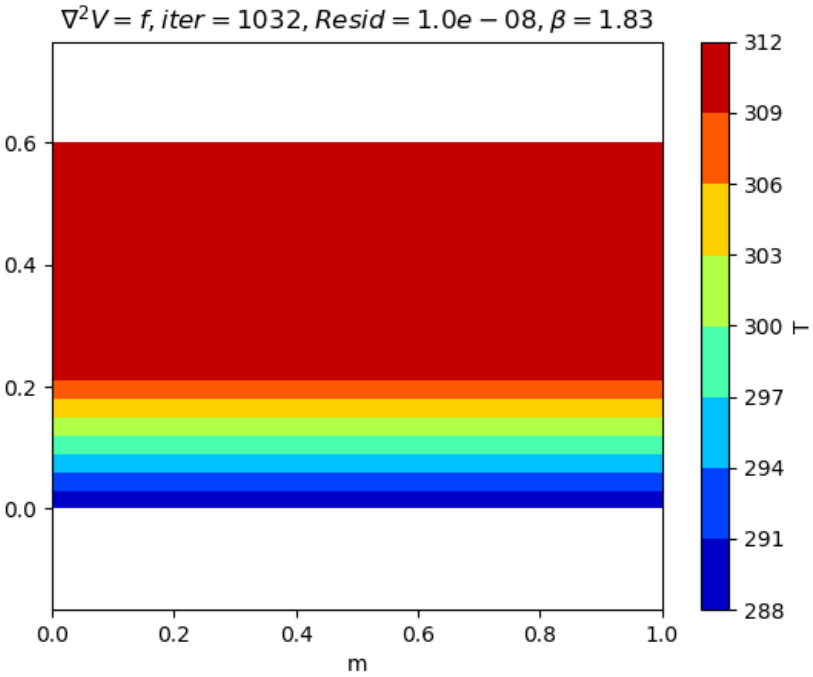
Mesmo, forçamento, outras condições fronteira

```
f=np.zeros((M,N))
for k in range(0,M):
    f[k,12]=-100/(Lx*delta)
[V,niter,res,beta]=poisson(f,V0,X,Y,maxiter,maxres,BxL=np.
zeros((N)),BxH=np.zeros((N)),ByH=np.zeros((M)))
```



Porque fica mais quente quando isolado y=Ly?

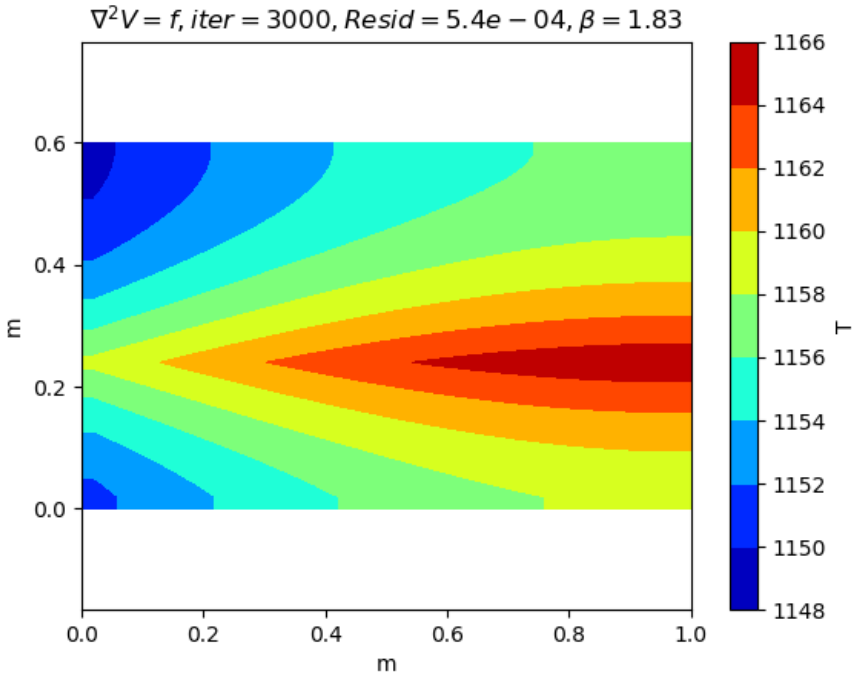


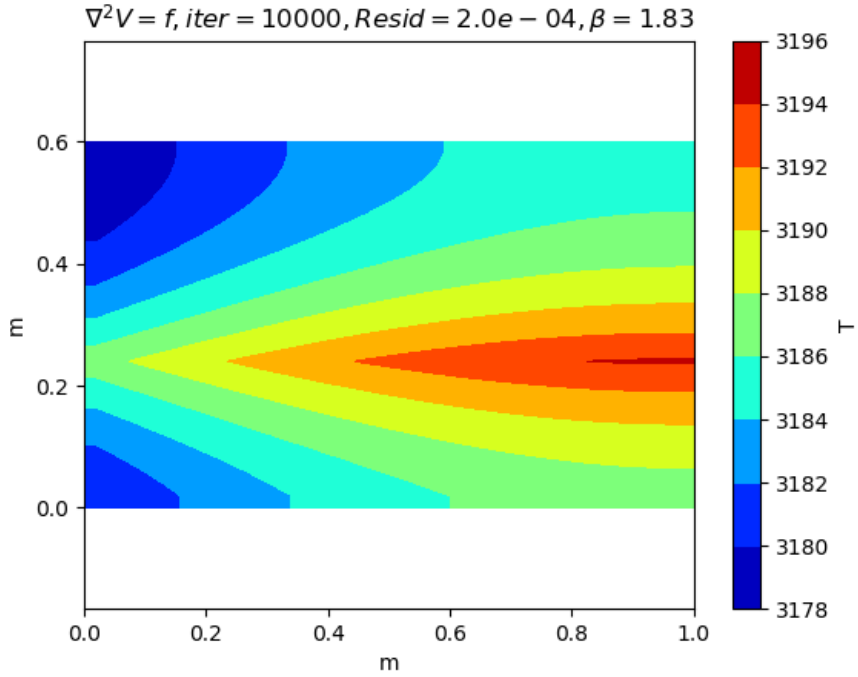


Porque todo o calor fornecido (100W) tem que ser conduzido através da fronteira y=0 (é preciso o dobro do gradiente $\vec{q} = -\kappa \nabla T$)

```
[V,niter,res,beta]=poisson(f,V0,X,Y,maxiter,
maxres,BxL=np.zeros((N)),BxH=np.zeros((N)),B
yH=np.zeros((M)),ByL=np.zeros((M)))
```

Não converge: a temperatura subiria sempre porque todas as paredes são não condutoras. A assimetria resulta da ordem de varrimento…





Condução em corpos heterogéneos (κ variável)

$$-\nabla \cdot (-\chi \nabla T) = -\dot{q}_{\mathrm{V}}$$

Caso 2D: $\dot{q}_S$ em Wm^{-2}

$$-\frac{\partial}{\partial x}\left(-\chi \frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\chi \frac{\partial T}{\partial y}\right) = -\dot{q}_S$$

Numa malha discreta (idêntico para y), em diferenças centradas:

$$-\frac{\partial}{\partial x}\left(-\chi \frac{\partial T}{\partial x}\right) \approx 1/\Delta x \left(\chi_{i+\frac{1}{2}} \frac{T_{i+1} - T_i}{\Delta x} - \chi_{i-\frac{1}{2}} \frac{T_i - T_{i-1}}{\Delta x}\right)$$

Com

$$\chi_{i+1/2} = \frac{\chi_i + \chi_{i+1}}{2}, \chi_{j-1/2} = \frac{\chi_{i-1} + \chi_i}{2}$$

$$\frac{1}{\Delta x}\left(\chi_{i+\frac{1}{2},j}\frac{T_{i+1,j}-T_{i,j}}{\Delta x}-\chi_{i-\frac{1}{2},j}\frac{T_{i,j}-T_{i-1,j}}{\Delta x}\right)$$
$$+\frac{1}{\Delta y}\left(\chi_{i,j+\frac{1}{2}}\frac{T_{i,j+1}-T_{i,j}}{\Delta y}-\chi_{i,j-\frac{1}{2}}\frac{T_{i,j}-T_{i,j-1}}{\Delta y}\right)+f_{i,j}=R_{i,j}$$

$$\frac{1}{\Delta x^2}\left(\chi_{i+\frac{1}{2},j}T_{i+1,j}-\chi_{i+\frac{1}{2},j}T_{i,j}-\chi_{i-\frac{1}{2},j}T_{i,j}+\chi_{i-\frac{1}{2},j}T_{i-1,j}\right)$$
$$+\frac{1}{\Delta y^2}\left(\chi_{i,j+\frac{1}{2}}T_{i,j+1}-\chi_{i,j+\frac{1}{2}}T_{i,j}-\chi_{i,j-\frac{1}{2}}T_{i,j}+\chi_{i,j-\frac{1}{2}}T_{i,j-1}\right)$$
$$+f_{i,j}=R_{i,j}$$

Sobre-relaxação simultânea

$$\frac{\chi_{i-\frac{1}{2},j}}{\Delta x^2} T_{i-1,j} + \frac{\chi_{i+\frac{1}{2},j}}{\Delta x^2} T_{i+1,j}\, x + \frac{\chi_{i,j-\frac{1}{2}}}{\Delta y^2} T_{i,j-1} + \frac{\chi_{i,j+\frac{1}{2}}}{\Delta y^2} T_{i,j+1}$$

$$-\left(\frac{\chi_{i-\frac{1}{2},j}}{\Delta x^2} + \frac{\chi_{i+\frac{1}{2},j}}{\Delta x^2} + \frac{\chi_{i,j-\frac{1}{2}}}{\Delta y^2} + \frac{\chi_{i,j+\frac{1}{2}}}{\Delta y^2}\right) T_{i,j} + f_{i,j} = R_{i,j}$$

$$T_{i,j}^{novo} = T_{i,j} + \beta \left(\frac{\chi_{i-\frac{1}{2},j}}{\Delta x^2} + \frac{\chi_{i+\frac{1}{2},j}}{\Delta x^2} + \frac{\chi_{i,j-\frac{1}{2}}}{\Delta y^2} + \frac{\chi_{i,j+\frac{1}{2}}}{\Delta y^2}\right)^{-1} R_{j,k}$$

χ variável: $-\nabla \cdot (-\chi \nabla T) = -\dot{q}_V$

```
def poissonT(f,V0,chi,X,Y,maxiter,maxres,\
             beta0=0.,BxL=[],BxH=[],ByL=[],ByH=[]):
    …
    V=V0
    chix=np.zeros((M,N))
    chiy=np.zeros((M,N))
    for i in range(1,M):
        for j in range(N):
            chix[i,j]=0.5*(chi[i,j]+chi[i-1,j])
    for i in range(M):
        for j in range(1,N):
            chiy[i,j]=0.5*(chi[i,j]+chi[i,j-1])
    chix=chix/dx**2
    chiy=chiy/dy**2
```

relaxação $T_{i,j}^{novo} = T_{i,j} + \beta \left(\frac{\chi_{i-\frac{1}{2},j}}{\Delta x^2} + \frac{\chi_{i+\frac{1}{2},j}}{\Delta x^2} + \frac{\chi_{i,j-\frac{1}{2}}}{\Delta y^2} + \frac{\chi_{i,j+\frac{1}{2}}}{\Delta y^2} \right)^{-1} R_{j,k}$

```
itera=0
resid=2*maxres #garante a primeira iteração
pngs=[]
while resid>maxres and itera<maxiter: #iterações
    itera=itera+1
    resid=0; vmax=0
    for i in range(1,M-1):
        for j in range(1,N-1):
            chi4=(chix[i,j]+chix[i+1,j]\
                  +chiy[i,j]+chiy[i,j+1])
            R=chiy[i,j]*V[i,j-1]+ chiy[i,j+1]*V[i,j+1]+\
              chix[i,j]*V[i-1,j]+chix[i+1,j]*V[i+1,j]\
              -chi4*V[i,j]-f[i,j]
            V[i,j]=V[i,j]+beta*R/chi4;
            resid=max(resid,abs(R))
```

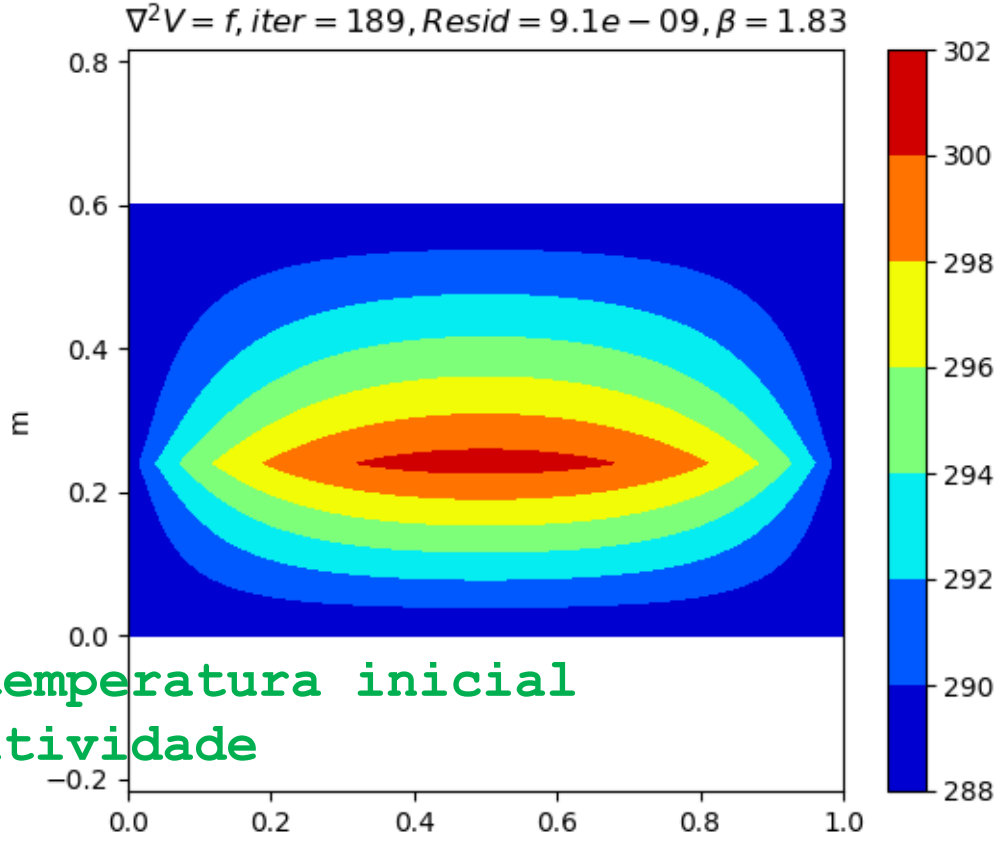
Condições fronteira (de von Neumann)

(notar que as derivadas são perpendiculares à fronteira de dentro para fora)

```
#Neumann X (caso contrário, mantém V fixo na fronteira)
        if len(BxL)!=0:
            V[0,:]=V[1,:]+BxL*dx
        if len(BxH)!=0:
            V[M-1,:]=V[M-2,:]+BxH*dx
#Neumann y (caso contrário, mantém V fixo na fronteira)
        if len(ByL)!=0:
            V[:,0]=V[:,1]+ByL*dy
        if len(ByH)!=0:
            V[:,N-1]=V[:,N-2]+ByH*dy

        vmax=np.max(np.abs(V))
        resid=resid/vmax #residuo relativo
```

teste




```
V0=288.*np.ones((M,N)) #temperatura inicial
chi=np.ones((M,N)) #condutividade
f=np.zeros((M,N))
for k in range(0,M): #filamento retilíneo
    f[k,12]=-100/(Lx*delta)
BxH=np.zeros((N))
ByH=-np.ones((M))*10
ByL=np.zeros((M))
[V,niter,res,beta]=poissonT(f,V0,chi,X,Y,maxiter,maxres)
```

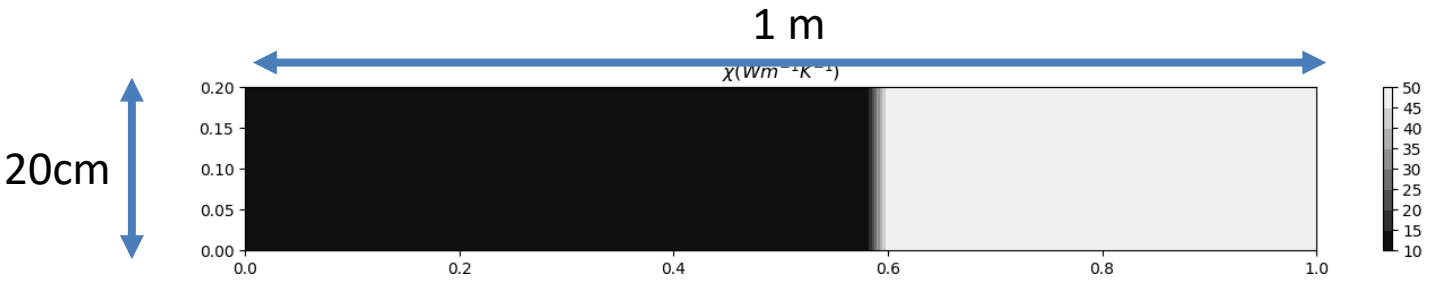
Notas

No caso em que $\chi = const$ a nova função dá os resultados obtidos anteriormente.

No entanto agora $f = -\dot{q}_S$, e no caso anterior era $f = -\frac{\dot{q}_S}{\chi}$.

Condução de calor através de uma parede heterogénea

```
V0=273.*np.ones((M,N)) #potencial inicial e cf em y=Ly
V0[:,0]=300 #condição fronteira em y=0
chi=np.ones((M,N))*50 #condutividade
chi[0:30,:]=10        #condutividade
f=np.zeros((M,N)) #não há fontes de calor internas
[T,niter,res,beta]=poissonT(f,V0,chi,X,Y,maxiter,maxres,\
      BxL=np.zeros((N)),BxH=np.zeros((N)))
```



Fluxos de calor em cada fronteira