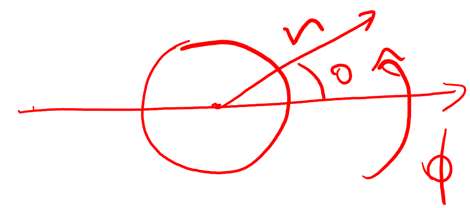
Stokes flow around a sphere

(Acheson, page 223)

Axisymmetric flow

$u_\phi = 0$



$$\boldsymbol{u} = [u_r(r, \theta), u_\theta(r, \theta), 0]$$

By using the Stokes stream function, we automatically satisfy the continuity equation (div **V** =0)

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \Psi}{\partial \theta}, \qquad u_\theta = -\frac{1}{r \sin\theta} \frac{\partial \Psi}{\partial r}$$

Then

$$\nabla \wedge \boldsymbol{u} = \left[0, 0, -\frac{1}{r \sin\theta} E^2 \Psi\right]$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\right)$$

NS in the Stokes regime

$$\vec{\nabla} P \cong \mu \nabla^2 \vec{V} \quad \Rightarrow \quad \nabla p = -\mu \nabla \wedge (\nabla \wedge \boldsymbol{u})$$

since $\quad \nabla \wedge (\nabla \wedge \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}$

We obtain

$$\frac{\partial p}{\partial r} = \frac{\mu}{r^2 \sin\theta} \frac{\partial}{\partial \theta} E^2 \Psi,$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{-\mu}{r \sin\theta} \frac{\partial}{\partial r} E^2 \Psi,$$

Eliminating the pressure cross derivatives we find

$$E^2(E^2\Psi) = 0$$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right)\right]^2 \Psi = 0$$

Boundary condition at r=a: no slip

$$\left.\frac{\partial \Psi}{\partial r}\right|_{r=a} = \left.\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right|_{r=a} = 0$$

At infinity:

$$u_r \sim U \cos\theta \quad \text{and} \quad u_\theta \sim -U \sin\theta \quad \text{as } r \to \infty$$

$$\Rightarrow \quad \Psi \sim \tfrac{1}{2} U r^2 \sin^2\theta$$

Which suggests a solution of the form

$$\Psi = f(r)\sin^2\theta$$

then

$$E^2(E^2\Psi) = 0 \quad \Rightarrow \quad \left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 f = 0$$

$$f = r^\alpha$$

The solution is a polynomial in r, with the condition (use $f=r^{\alpha}$ in the previous equation):

$$[(\alpha-2)(\alpha-3)-2][\alpha(\alpha-1)-2]=0$$

$$\Rightarrow \quad f(r)=\frac{A}{r}+Br+Cr^2+Dr^4$$

Uniform flow at infinity: $C=\frac{1}{2}U$ and $D=0$

At r=a, $f(a)=f'(a)=0$

We find

$$\Psi=\frac{1}{4}U\left(2r^2+\frac{a^3}{r}-3ar\right)\sin^2\theta$$

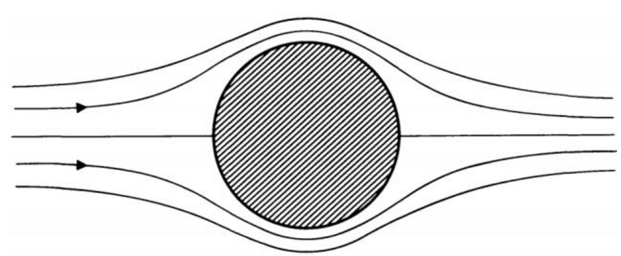


Fig. 7.2. Low Reynolds number flow past a sphere.

Drag force

To calculate the pressure, we use

$$\frac{\partial p}{\partial r} = \frac{\mu}{r^2 \sin\theta} \frac{\partial}{\partial \theta} E^2 \Psi,$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{-\mu}{r\sin\theta}\frac{\partial}{\partial r} E^2\Psi,$$

For the previous streamfunction:

$$E^2\Psi = \tfrac{3}{2} U a r^{-1} \sin^2\theta$$

Integrating

$$p = p_\infty - \tfrac{3}{2}\frac{\mu U a}{r^2}\cos\theta$$

Stress components in cylindrical coordinates

$T_{ij} = -p\delta_{ij} + 2\mu e_{ij}$

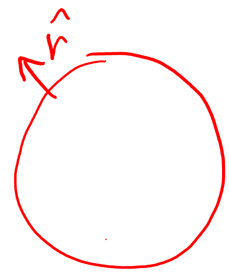
$T_i = T_{ij} n_j$

normal only in "r"

$$t_r = T_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r},$$

$$t_\theta = T_{r\theta} = \mu r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{\mu}{r}\frac{\partial u_r}{\partial \theta},$$

$$t_\phi = T_{r\phi} = 0,$$



Strain rate

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r},$$

$$e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r},$$

$$2e_{\theta\phi} = \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi},$$

$$2e_{\phi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right),$$

$$2e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}.$$

Using the streamfunction, we can calculate the velocity field and the stress components

$$t_r = -p_\infty + \tfrac{3}{2}\frac{\mu U}{a}\cos\theta, \qquad t_\theta = -\tfrac{3}{2}\frac{\mu U}{a}\sin\theta.$$

By symmetry, we expect the net force on the sphere to be on the direction of the uniform stream, and the appropriate component of the stress is

$\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$

$$t_z = t_r\cos\theta - t_\theta\sin\theta = -p_\infty\cos\theta + \tfrac{3}{2}\frac{\mu U}{a}$$

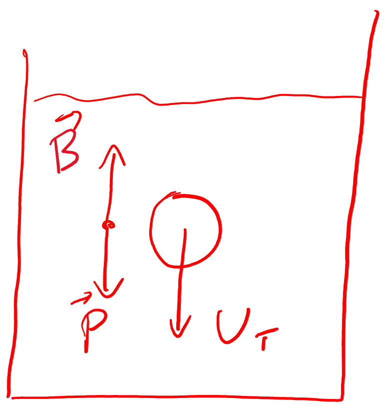
The drag on the sphere is therefore

$$D = \int_0^{2\pi} \int_0^{\pi} ta^2 \sin\theta \, d\theta \, d\phi = 6\pi\mu U a.$$

This is the Stokes law. This is valid for low Re (measurements start to deviate from Stokes law for Re = 0.5).

For a ball falling through a viscous liquid, we also have the buoyancy force

$$6\pi\mu U_T a = \tfrac{4}{3}\pi a^3 (\rho_{\mathrm{sphere}} - \rho_{\mathrm{fluid}}) g.$$



Stokes flow around a sphere (alternative derivation)

Faber

When its inertial term is neglected, the Navier–Stokes equation becomes

$$-\nabla p^* - \eta\nabla \wedge (\nabla \wedge \boldsymbol{u}) = 0, \qquad (6.63)$$

which, since

$$\nabla \wedge (\nabla \wedge \boldsymbol{u}) = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u},$$

is equivalent for an effectively incompressible fluid such that $\nabla\cdot\boldsymbol{u}$ is zero to

$$\nabla^2\boldsymbol{u} = \frac{1}{\eta}\nabla p^*. \qquad (6.64)$$

This is the basic equation of motion for creeping flow. Its solutions for $\boldsymbol{u}$ consist in general of a *particular integral*, $\boldsymbol{u}_{\text{PI}}$, and a *complementary function*, $\boldsymbol{u}_{\text{CF}}$. The latter is a solution of $\nabla^2\boldsymbol{u} = 0$, which means that it is normally a solution of $\nabla \wedge \boldsymbol{u} = 0$ and can therefore be described by a potential ϕ_{CF}. In the present problem the complementary function has to be chosen in such a way that it corresponds to uniform flow in the x_1 direction at large distances from the sphere, so in the spherical polar coordinates defined in fig. 4.6 we may expect [§4.7]

$$\phi_{\text{CF}} = UR\cos\theta + AR^{-2}\cos\theta,$$

or

$$u_{R,\mathrm{CF}} = (U - 2AR^{-3}) \cos \theta,$$
$$u_{\theta,\mathrm{CF}} = (-U - AR^{-3}) \sin \theta,$$

where the coefficient A remains to be determined.

We cannot hope to match the boundary condition that $\boldsymbol{u} = 0$ at $R = a$ for all values of θ unless $u_{R,\mathrm{PI}}$ and $u_{\theta,\mathrm{PI}}$ are likewise proportional to $\cos\theta$ and $\sin\theta$ respectively. But application of the divergence operator $(\boldsymbol{\nabla}\cdot)$ to (6.63) shows at once that p^* obeys Laplace's equation,

$$\nabla^2 p^* = 0. \tag{6.65}$$

Where the flow is axially symmetric, as it is here, p^* must therefore be expressible, like ϕ_{CF}, in solid harmonic functions. If it is defined to be zero at large values of R where $\boldsymbol{u} = \boldsymbol{U}$, then the only credible possibility is that

$$p^* = BR^{-2} \cos \theta, \tag{6.66}$$

where the coefficient B is independent of θ and R. In that case $\boldsymbol{\nabla}p^*$ is proportional to R^{-3}, and $\boldsymbol{u}_{\mathrm{PI}}$ must therefore be proportional to R^{-1}. Let us try

$$u_{R,\mathrm{PI}} = CR^{-1} \cos \theta.$$

Then in order to satisfy the condition

$$\nabla \cdot \boldsymbol{u}_{\mathrm{PI}} = \frac{1}{R^2}\frac{\partial(R^2 u_{R,\mathrm{PI}})}{\partial R} + \frac{1}{R \sin\theta}\frac{\partial(\sin\theta\, u_{\theta,\mathrm{PI}})}{\partial\theta} = 0$$

we must set

$$u_{\theta,\mathrm{PI}} = -\frac{1}{2} C R^{-1} \sin\theta .$$

These guesses have now to be checked by substitution into (6.64). Both sides of that equation are, of course, vectors, but to simplify the analysis we shall consider only their components in the longitudinal x_1 direction; it can easily be verified that when these are equal to one another the transverse components are equal to one another also. On the left-hand side we have

$$\nabla^2 u_{1,\mathrm{PI}} = \left\{\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right\}(u_R\cos\theta - u_\theta\sin\theta),$$

which simplifies to

$$\nabla^2 u_{1,\mathrm{PI}} = \frac{1}{2} C \frac{1}{R^3 \sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}(2\cos^2\theta + \sin^2\theta)\right\}$$

$$= -\frac{C}{R^3}(2\cos^2\theta - \sin^2\theta).$$

On the right-hand side we have

$$\frac{1}{\eta}\frac{\partial p^*}{\partial x_1} = \frac{1}{\eta}\left(\cos\theta\,\frac{\partial p^*}{\partial R} - \frac{1}{R}\sin\theta\,\frac{\partial p^*}{\partial \theta}\right)$$

$$= -\frac{B}{\eta R^3}\,(2\cos^2\theta - \sin^2\theta).$$

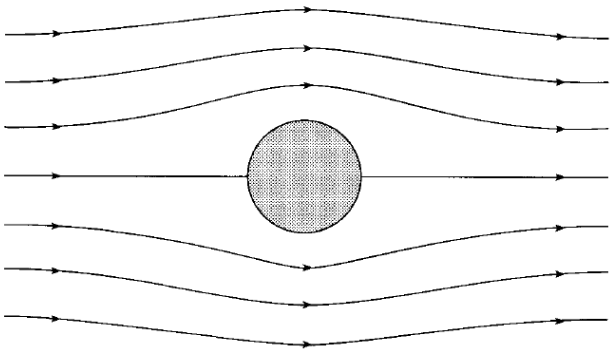


Figure 6.12 Lines of flow past a sphere according to Stokes's solution.

These expressions can indeed be made equal to one another, by choosing $C = B/\eta$. Finally, to ensure that both u_R and u_θ vanish at $R = a$ we need to let $A = -\frac{1}{4}a^3U$, $C = -\frac{3}{2}aU$.

The full solution, which is the only solution which satisfies the given boundary conditions, is therefore

$$u_R = u_{R,\mathrm{CF}} + u_{R,\mathrm{PI}} = U\cos\theta\left(1 - \frac{3a}{2R} + \frac{a^3}{2R^3}\right),$$
$$u_\theta = u_{\theta,\mathrm{CF}} + u_{\theta,\mathrm{PI}} = -U\sin\theta\left(1 - \frac{3a}{4R} - \frac{a^3}{4R^3}\right). \tag{6.67}$$

The principal respects in which in which it differs from the solution of Euler's equation worked out in §4.7, on the basis of potential theory alone, are:

(i) it satisfies the no-slip boundary condition at the sphere's surface;
(ii) it describes a velocity u_θ in the equatorial ($\theta = \pi/2$) plane which increases monotonically towards U with increasing R instead of decreasing;
(iii) the terms in a/R which it contains represent a perturbation of the flow field which is of a *long-range* nature.

Pressure

According to this solution, the excess stress which acts upon the surface of the sphere has a normal component given by

$$p_R^* = p^* - 2\eta \left(\frac{\partial u_R}{\partial R}\right)_{R=a} = -\frac{3\eta U \cos\theta}{2a},$$

[(6.11)] and a shear component acting in the direction of increasing θ given by

$$s_{\theta R} = \eta a \left\{\frac{\partial}{\partial R}\left(\frac{u_\theta}{R}\right) + \frac{1}{a^2}\frac{\partial u_R}{\partial \theta}\right\}_{R=a} = -\frac{3\eta U \sin\theta}{2a}$$

[(6.3) and (6.53)]. Taken together, these components are equivalent to a uniform force per unit area in the direction of $\boldsymbol{U}$ of magnitude $3\eta U/2a$. The total drag force in the direction of $\boldsymbol{U}$ is therefore

$$F_D = 4\pi a^2 \frac{3\eta U}{2a} = 6\pi\eta a U. \qquad (6.68)$$

This expression constitutes *Stokes's law*.

Discussion

It is only in the limit when velocity U and Reynolds Number Re $(= 2\rho aU/\eta)$ tend to zero that the assumption on which Stokes's law is based is fully consistent with the details of his solution. Since the leading term in $\boldsymbol{u}$ is $\boldsymbol{U}$, while the next terms in (6.67) are proportional to aU/R, the inertial term in the Navier–Stokes equation, $\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$, is of order $\rho U^2 a/R^2$ at large values of R according to Stokes, while the viscous term $\eta\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\boldsymbol{u})$ is of order $\eta aU/R^3$. Far from being negligible, the inertial term is clearly liable to exceed the viscous term at distances such that

$$R > \frac{\eta}{\rho U} = \frac{2a}{Re}.$$

The inconsistency may suggest to the reader that we cannot trust equations (6.67) to describe the velocity distribution in the immediate vicinity of the sphere, and that we therefore cannot trust Stokes's law, unless Re is really quite small compared with unity. It is only when Re reaches about 0·5, however, that deviations from the law become detectable experimentally.

Needless to say, if Stokes's law applies in a frame of reference such that the sphere is stationary then it applies also in the frame in which the distant fluid is stationary and the sphere is moving instead. Thus a solid sphere of radius a and density ρ_{sol}, falling down the axis of a vertical cylinder of sufficiently large radius which is filled with liquid of density ρ_{liq}, may be expected to reach a terminal velocity U such that

$$6\pi\eta aU = \frac{4}{3}\pi a^3(\rho_{sol} - \rho_{liq})g, \qquad (6.69)$$

provided that

$$Re = \frac{4a^3 \rho_{\rm liq}(\rho_{\rm sol} - \rho_{\rm liq})g}{9\eta^2} < 0{\cdot}5. \tag{6.70}$$

If the falling sphere is itself liquid, with viscosity η', circulating currents arise within it as it falls which modify the flow pattern outside the sphere. The modified form of Stokes's law which applies in these circumstances is

$$F_{\rm D} = \frac{4\pi\eta a U(\eta + \frac{3}{2}\eta')}{\eta + \eta'}. \tag{6.71}$$

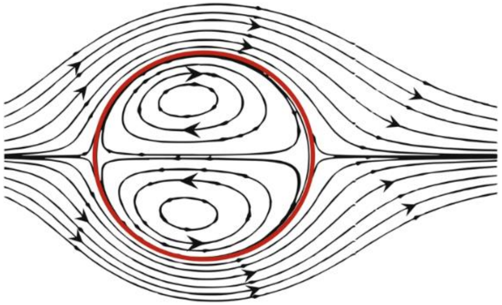
This evidently reduces to (6.68) when $\eta' \gg \eta$, e.g. under the conditions of Millikan's celebrated experiment, where the spheres were oil drops moving through air. At the opposite extreme where $\eta' \ll \eta$, however, e.g. where the spheres are very small bubbles of gas rising (rather than falling) through soda water or champagne, it reduces to $F_{\rm D} = 4\pi\eta a U$, so the terminal velocity of such bubbles should be

$$U = \frac{a^2 \rho_{\rm liq} g}{3\eta} \tag{6.72}$$

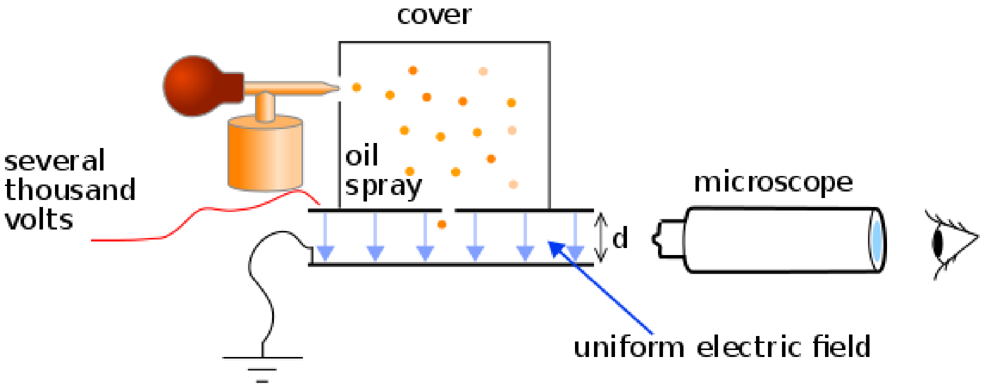
[(6.69), but with 6 replaced by 9 and with $\rho_{\rm sol}$ replaced by $\rho_{\rm gas}$; $\rho_{\rm gas}$ is negligible compared with $\rho_{\rm liq}$]. In fact, (6.72) does not describe the terminal velocity of rising soda water bubbles at all accurately. That is partly because the Reynolds Number normally exceeds 0·5 but also, it seems, because impurities adsorbed on the gas–liquid interface endow this interface with some measure of rigidity. It can be shown, incidentally, that the stresses which act on a gas bubble which is rising steadily with $Re \ll 1$ do not tend to distort it; it should – and does – remain spherical.

Challenge: exercise 18 of list 4

$\vec{V}_{\rm A} = \vec{V}_{\rm B}$ and $\tau_{s,\rm A} = \tau_{s,\rm B}$



$$\eta' \gg \eta$$



$$\eta' \ll \eta$$