Proceedings

Constraining $f(R)$ gravity theories with cosmological surveys

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Abstract: In this work we will study 3 different models for $f(R)$ modified gravity (Starobinsky, Hu-Sawicki and Exponencial) and constraint the parameters for each model using observational data from different surveys to find viable cosmological scenarios.

Keywords: $f(R)$ Modifed Gravity, Starobinsky model, Hu-Sawicki model, Exponential model, BBN

I. Introduction

Although the theory of General Relativity (GR) has been one of the most successful theories so far it didn't stop physicist from introducing modifications and proposing changes to create a more general and an ideal theory of gravity. In fact the first alternative to GR was introduced in 1922[8], shortly after it's publication in 1915. The theory studied in this paper, the $f(R)$ gravity, was introduced in 1970 by Hans Adolph Buchdahl[7].

As there are many potential forms of $f(R)$ gravity, since it merely replaces the Ricci scalar R with an arbitrary function of that same scalar $f(R)$ in the standard Einstein–Hilbert action, it is easy to see that it can become difficult, if not impossible, to find generic tests, especially considering that those deviations from GR can be made arbitrarily small.

$f(R)$ theories of gravity have been active fields of research, mainly in cosmic inflation[7], and given the arbitrary nature of the function used to generalize the Ricci scalar it is important to put these theories to the test to narrow down the several theoretical models introduced so far.

Today we know that GR might have some shortcoming, such as the mystery behind dark energy, dark matter and the speculative nature of inflation, and so it's possible that those phenomena can be explained by a theory of modified gravity.

With that in mind a lot of theories of modified gravity have been proposed throughout the years and, because of the Lovelock's theorem, we can classify the theories of gravity in four different ways, depending on how each violates each postulate of this theorem:

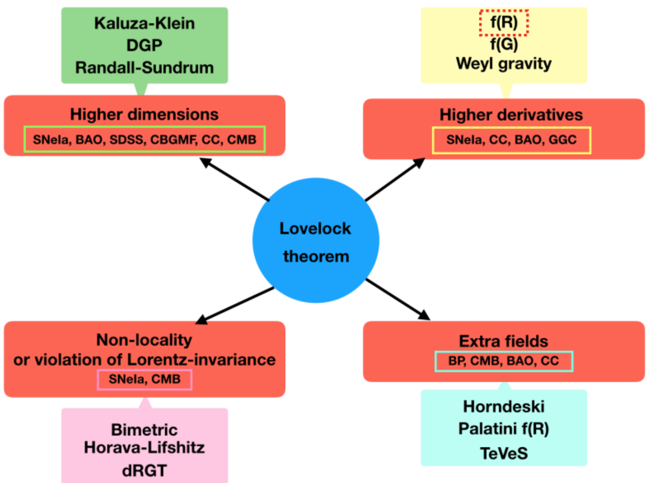

Figure 1. Organization of the different theories of gravity based on the violations of Lovelock's theorem and the observational surveys that have already been used to constraint each theory so far.

As we can see most of the observational surveys used to constraint modified theories of gravity are of cosmological nature. This is because gravitational phenomena whit in our solar system is in agreement with GR, which means that all of these theories of modified gravity must agree with GR at our scale.

There are also experiments to study gravitational fields at the a lower scale but those will not be considered in this paper.

With cosmology going everyday from a more quantitative science, and new observational data coming in the next few years, we will be able to further constraint these and many other models of modified gravity.

On Section II we will introduce $f(R)$ modified gravity and the three models we will analyze throughout the paper.

On Section III we show the observational surveys used to provide observational data.

Finally on Section IV we will show the statistical results for a set of cosmological parameters for each $f(R)$ model considering early and late time samples.

Unless specified most of the content shown in this paper is based on [1] and part of the introduction is also based on [2] and [3].

II. $f(R)$ Modified Gravity

As with every modified theory of gravity, it starts by modifying the Einstein-Hilbert, in this case by adding an arbitrary function $f(R)$, where R is the Ricci scalar:

$$S[g_{\mu\nu}, \psi] = \int \frac{f(R)}{2k} \sqrt{-g} d^4x + S_m[g_{\mu\nu}, \psi] \tag{1}$$

Where $G = c = 1$, $k = 8\pi$ and S_m is the action for matter. Computing the Einstein tensor ($G_{\nu\mu} = R_{\nu\mu} - g_{\nu\mu}R/2$) using Equation (1) we obtain the Einstein field equations:

$$G_{\mu\nu} = \frac{1}{f_R}[f_{RR}\nabla_\mu\nabla_\nu R + f_{RRR}(\nabla_\mu R)(\nabla_\nu R) - \frac{g_{\nu\mu}}{6}(Rf_R + f + 2kT) + kT_{\mu\nu}] \tag{2}$$

Where $T_{\nu\mu}$ is the energy momentum tensor that we consider to be an ideal fluid composed by baryons, dark matter and radiation.

To obtain viable cosmological models we consider a flat, homogeneous and isotropic space-time which is given by the FLRW metric:

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2] \tag{3}$$

We are now in conditions to derive the equations for a $f(R)$ modified gravity. First we can compute the second order equation for the Ricci scalar:

$$\ddot{R} = -3H\dot{R} - \frac{1}{3f_{RR}}[3f_{RRR}\dot{R}^2 + 2f - f_R R + kT] \tag{4}$$

With the previous results we can now derive both of the Friedmann equations, where $H = \dot{a}/a$ is the well known Hubble parameter:

$$H^2 = -\frac{1}{f_{RR}}[f_{RR}H\dot{R} - \frac{1}{6}(Rf_R - f)] - \frac{kT_t^t}{3f_R} \tag{5}$$

$$\dot{H} = -H^2 - \frac{1}{f_R}[f_{RR}H\dot{R} + \frac{f}{6} + \frac{kT_t^t}{3}] \tag{6}$$

We can also derive the effective energy density and pressure of the fluid in a $f(R)$ modified gravity:

$$\rho_{eff} = \frac{1}{kf_R}[\frac{1}{2}(f_R R - f) - 3f_{RR}H\dot{R} + k\rho(1 - f_R)] \tag{7}$$

$$P_{eff} = -\frac{1}{3kf_R}[\frac{1}{2}(f_R R + f) + 3f_{RR}H\dot{R} - k(\rho - 3Pf_R)] \tag{8}$$

We can now determine the equation of state (EoS) for this fluid in $f(R)$:

$$\omega_{eff} = \frac{3H^2 + 3kP - R}{3(3H^2 - k\rho)} \tag{9}$$

Now that we know our cosmological fluid behaves in a general $f(R)$ theory, let's consider the following models:
1. Starobinsky model:

$$f(R) = R + \lambda R_S[(1 + \frac{R^2}{R_S^2})^{-q} - 1] \tag{10}$$

2. Hu-Sawicki model:

$$f(R) = R - R_{HS}\frac{c_1(\frac{R}{R_{HS}})^n}{c_2(\frac{R}{R_{HS}})^n + 1} \tag{11}$$

3. Exponential model:

$$f(R) = R + \beta R_*(1 - e^{-R/R_*}) \tag{12}$$

Where all of the free parameters mentioned above will be constrained later using observational data. However it was proposed a new parameterization, called the JJE parameterization, for general EoS of $f(R)$ defined as:

$$\omega_{JJE}(z) = -1 + \frac{\omega_0}{1 + \omega_1 z^{\omega_2}}cos(\omega_3 + z) \tag{13}$$

This parameterization[1] wasn't added by chance. Integrating Equation (4), Equation (5) and Equation (6) numerically we can see that we can parameterize the EoS of each model as a more general EoS with different numerical values for $\omega_0, \omega_1, \omega_2$ and ω_3 for the 3 different models considered.

However finding the best fit values for the parameters of Equation (13) is a difficult task, especially in the CLASS code used to run the simulations. So we introduce a modified version of the EoS:

$$\omega_{f(R)} = -1 + \frac{\omega_0 cos(\alpha\nu(z))}{1 + \omega_1 z^{\omega 2}} \ , \ \nu(z) = 2\pi/(\sqrt{6}z + 1)^{1/2} \tag{14}$$

In order to constraint the free parameters of the models we used gravitational tests performed whit in our solar system as they are in agreement in GR and are the most accurate gravitational data so far. After introducing those constraints the best fit values are:

	Starobinsky		Hu-Swicki		Exponential	
Parameter	ω_{JJE}	$\omega_{f(R)}$	ω_{JJE}	$\omega_{f(R)}$	ω_{JJE}	$\omega_{f(R)}$
ω_0	0.145	0.09	0.049	0.024	0.384	0.358
ω_1	0.106	0.11	0.310	0.310	0.000014	0.03
ω_2	4.491	7	4.395	4.000	11.370	11.00
ω_3	7.374	-	7.348	-	0.684	-
α	-	1	-	1	-	1

Table 1. Best fit values for EoSs Equation (13) and Equation (14) for each of the considered models.

[1] We can see that for high values of z we can recover $\omega \to -1$ for a general $f(R)$ EoS.

With the previous table we have all the free parameters we need to define our cosmological fluid for each of the 3 different models considered.

With that in mind it's time to move on to the next section, where we will introduce an overview of the data used from different observational surveys.

III. Observational Data

Considering that this paper aims to study these models both at early and late time universe we will use data from Supernovae Type Ia (SNeIa) luminous distance, Baryon acoustic oscillations (BAO) redshift surveys, high-z measurements of H(z) from Cosmic Chronometers (CC), Cosmic Microwave Background (CMB) data from Planck 2018 and detection of Lyman-α (Lyα) radiation in both emission and absorption lines.

Instead of providing the values for each of these observational surveys we will instead provide a quick conceptual overview of each of the surveys mentioned before.

1. The SNeIa samples used are given by Pantheon[2], a program that was used to provide us with the data regarding the measurements of the type Ia Supernova, the well known "standard candles".
2. BAO[4][9] are density fluctuations regarding visible baryonic matter caused by density waves in the primordial plasma. These matter clusters provide a "standard ruler" for cosmological lengths as there is a maximum distance these waves would travel before the plasma could cool down into neutral matter which stopped the expansion and freezing them into place.
3. CC consist in passively evolving old galaxies whose redshifts are known that allow us to extract the values of $H(z)$ for several values of z. These are known as the "standard clocks".
4. The CMB is the background radiation that we measure coming from all directions in the sky. This relic radiation gives us the last scattering surface for photons and so it includes information regarding the early stages of the universe, especially during photon decoupling in the recombination epoch.
5. Ly-α[5] is the spectral analysis of hydrogen in large scale structures both in emission and absorption. This provides information regarding the structure formation of these structures, dark matter, distribution of matter and the cosmological constant.

The the complete data sets used in this work we refer the reader to the main paper where all of this information is present in chapter number 5 titled "Observational samples".

With all of the data introduced in this chapter and the models introduced previously, we are now in conditions to combine these two and obtain the numerical values of the cosmological parameters.

IV. Results

In this section we will show the statistic results of several cosmological parameters obtained using the 3 models introduced previously using the data from the previous section. For the sake of readability our tables will only show the mean and the upper and lower limit for each value. For more information regarding the complete statistical analysis (best fist, 95% C.L., etc.) we refer the reader to the main paper.

To have a better understanding of how each model behaves we first considered early and late time samples separately. Using the late time data samples (Pantheon SNeIa, CC and BAO) the statistical results for each model are present in Table 2, while the statistical results for each model using early time data samples (CMB and Ly-α) are present in Table 3.

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[2] Source code at Github

Parameter	Mean $\pm\sigma$		
	Starobinsky	Hu-Sawicki	Exponencial
$100\omega_b$	$2.224^{+0.021}_{-0.024}$	$2.227^{+0.022}_{-0.023}$	$2.226^{+0.022}_{-0.024}$
ω_{cdm}	$0.1182^{+0.0012}_{-0.0013}$	$0.118^{+0.0013}_{-0.0012}$	$0.1181^{+0.0012}_{-0.0013}$
ω_0	$0.002704^{+0.078}_{-0.082}$	$0.002035^{+0.079}_{-0.088}$	$-0.006569^{+0.08}_{-0.082}$
M	$-19.4^{+0.012}_{-0.014}$	$-19.4^{+0.013}_{-0.014}$	$-19.4^{+0.012}_{-0.013}$
H_0	$68.85^{+2.1}_{-2}$	$68.47^{+0.69}_{-0.64}$	$69.11^{+1.9}_{-2.2}$
Ω_m	$0.3004^{+0.0077}_{-0.0078}$	$0.2994^{+0.008}_{-0.64}$	$0.2995^{+0.0072}_{-0.0081}$
Ω_0	$0.7028^{+0.021}_{-0.017}$	$0.7005^{+0.0083}_{-0.008}$	$0.7053^{+0.021}_{-0.017}$

Table 2. Statistical results for a set of cosmological parameters for each model using late time samples.

Parameter	Mean $\pm\sigma$		
	Starobinsky	Hu-Sawicki	Exponencial
$100\omega_b$	$2.24^{+0.013}_{-0.0138644}$	$2.242^{+0.016}_{-0.013}$	$2.241^{+0.015}_{-0.015}$
ω_{cdm}	$0.1196^{+0.001345}_{-0.001345}$	$0.1199^{+0.0013}_{-0.0015}$	$0.1199^{+0.001455}_{-0.001455}$
ω_0	$0.02378^{+0.018}_{-0.018}$	$0.2574^{+0.289}_{-0.289}$	$0.02409^{+0.012}_{-0.012}$
H_0	$68.02^{+0.645}_{-0.645}$	$67.2^{+1.2}_{-0.68}$	$67.96^{+0.79}_{-0.53}$
$ln10^{10}A_s$	$3.043^{+0.013}_{-0.018}$	$3.044^{+0.016}_{-0.017}$	$3.041^{+0.016}_{-0.016}$
n_s	$0.9667^{+0.004205}_{-0.004205}$	$0.9663^{+0.004}_{-0.004}$	$0.9644^{+0.0033}_{-0.0048}$
τ_{reio}	$0.05409^{+0.0081}_{-0.0068}$	$0.05366^{+0.0068}_{-0.0091}$	$0.05253^{+0.0072}_{-0.0077}$
Ω_m	$0.3071^{+0.034}_{-0.034}$	$0.3153^{+0.0089}_{-0.013}$	$0.3081^{+0.043}_{-0.043}$
$\Omega0_{fld}$	$0.6928^{+0.0086}_{-0.0083}$	$0.6846^{+0.013}_{-0.0089}$	$0.6918^{+8.832}_{-8.832}$
Y_p	$0.2479^{+6.3e-5}_{-6.2e-5}$	$0.2479^{+6.6e-5}_{-5.8e-5}$	$0.2479^{+6.467e-5}_{-6.467e-5}$
$\sigma 8$	$0.8227^{+0.0098}_{-0.0076}$	$0.8193^{+0.01}_{-0.0095}$	$0.8217^{+0.00772}_{-0.00772}$

Table 3. Statistical results for a set of cosmological parameters for each model using early time samples.

Now let's see how each model behaves when considered early or late time samples, and then see how both of of these scenarios compare against known observations. To do so we made two plots for each model.

In the left plot we show the value of the Helium-4 fraction Y_P, represented by dotted red line, and the uncertainty of the top limit, in blue, compared the value of $100\omega_b$. The green C.L. will represent the values obtained by the considered model.

In the right plot we will show how the early time samples, in green, compares to the late time samples, in red, for the exact same model. On the x axis we will have the value of H_0 and one the y axis the value of Ω_m. We will also compare the previous results to the values given by Planck 2018, in purple, and the values from [6], in yellow.

For the Starobinsky model the results are in Figure 2, for Hu-Sawicki in Figure 3 and for Exponential in Figure 4.

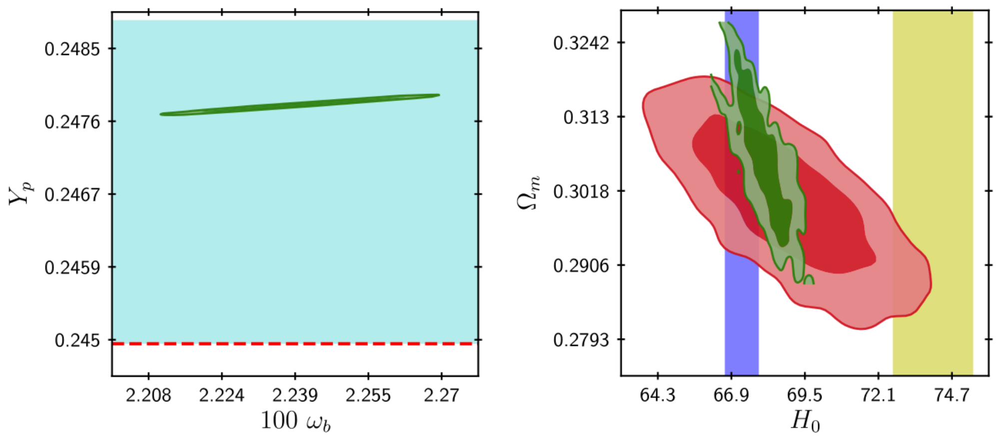

Figure 2. C.L. analysis for the Starobinsky model. To understand the colors refer to the main text

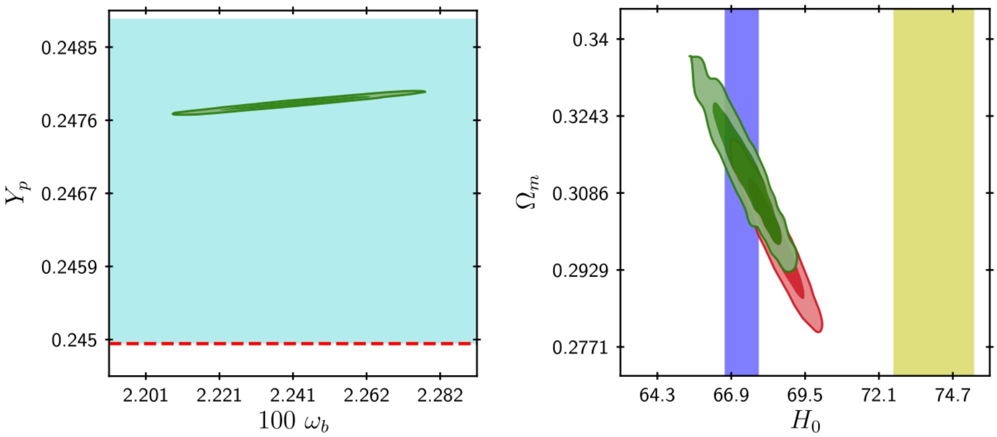

Figure 3. C.L. analysis for the Hu-Sawicki model. To understand the colors refer to the main text

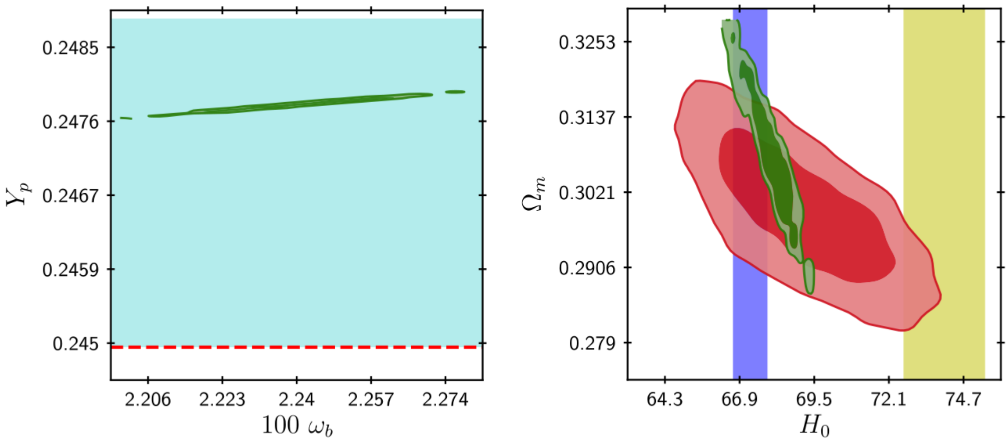

Figure 4. C.L. analysis for the Exponential model. To understand the colors refer to the main text

In Table 4 we compute the tensions between Table 2, Table 3 and the value of H_0 from [6] compute the tension between two models using Equation (15).

$$T_{H_0} = \frac{|H_0 - H_0^{survey}|}{\sqrt{\sigma_{H_0}^2 - \sigma_{survey}^2}} \tag{15}$$

Starobinsky								
[6] vs Planck+Lyα			[6] vs SNeIa + CC + BAO			SNeIa + CC + BAO vs Planck+Lyα		
H_0	H_0	T_{H_0}	H_0	H_0	T_{H_0}	H_0	H_0	T_{H_0}
74.03	67.72	4.0456	74.03	68.75	2.0828	68.75	67.72	0.4688

Hu-Sawicki								
[6] vs Planck+Lyα			[6] vs SNeIa + CC + BAO			SNeIa + CC + BAO vs Planck+Lyα		
H_0	H_0	T_{H_0}	H_0	H_0	T_{H_0}	H_0	H_0	T_{H_0}
74.03	67.6	3.46	74.03	68.41	3.5597	68.41	67.6	0.513

Exponential								
[6] vs Planck+Lyα			[6] vs SNeIa + CC + BAO			SNeIa + CC + BAO vs Planck+Lyα		
H_0	H_0	T_{H_0}	H_0	H_0	T_{H_0}	H_0	H_0	T_{H_0}
74.03	68.2	3.587	74.03	68.86	1.9744	68.86	68.2	0.282

Table 4. Tension values for $\omega_{f(R)}$ parameterisation for each model and different observational surveys

V. Discussion and Conclusions

We can see, comparing Table 2 with Table 3, that all values from the 3 different models are approximately equal to each other except for one which is the value of ω_0 obtained using early time samples for the Hu-Sawicki model, however it does come with a great uncertainty associated with it.

From Figure 2, Figure 3 and Figure 4 we can see that all 3 models lie in the C.L. region of the primordial helium abundance, and are consistent with the data from Planck 2018 but in tension with the data from [6]. Also worth mentioning is that with both early and late time samples all models appear to have a negative correlation in the Ω_m vs H_0 plot, where in the Y_P vs $100\omega_b$ plot the models show a very small, but non-zero, positive correlation.

Overall these 3 $f(R)$ models are in agreement with each other and with the data coming from Plank 2018, but in tension with the data found in [6]. The usage of early or late time samples while constraining these models had almost no effect on the Hu-Sawicki model but had some changes in the C.L. of the Starobinsky and Exponential models but always preserving the negative correlation found in all 3 of them.

As such we've shown that these models can indeed produce viable cosmological scenarios and could be an alternative to GR on the cosmological scale.

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