Proceedings

Inflation: From the Horizon Problem to Quantum Fluctuations

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Abstract: This work is a pedagogical review on how inflation solves the horizon problem, how a inflationary scalar field provides the seeds for the large-scale structures that we observe and how we can determine what inflation 'looked like'.

Keywords: Inflation; De Sitter Space; Quantum Fluctuations; Quantum Mechanics; Quantum Field Theory

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1. Introduction

The theory of cosmic inflation managed to solve from a conceptual point of view some of the problems that the hot Big Bang model left unresolved. The inability to explain the fact that apparent disjoint patches of space have approximately the same density and temperature, the so-called horizon problem, is one of them[1][2]. Another problem is the flatness problem[1][2], which will not be addressed in this work because i consider that the first has a greater relevance to explain the mechanism of inflation. Hence in this work, firstly, I present the basic notions of inflation starting from the horizon problem and presenting some (equivalent) physical conditions for it to occur. Then, I expose the equations of the scalar inflationary field's dynamics as well as the conditions in which slow-rolling inflation occurs. After that, I present the relationship between the curvature fluctuations and those of the inflationary field, which will be useful later on tho explain the quantum-to-classical transition. Later on, I deduce the Mukanov-Sasaki equation with some detail and proceed to treat the quantum fluctuations associated with the inflationary field in de Sitter space. Then, I approach the comoving curvature perturbation in a simple way to show you how we move from these quantum fluctuations to the perturbations of a stochastic classical field. Finally, I show how we can use the comoving curvature perturbation power spectra to confront the theory with observations. Throughout the work I used natural units so $c = \hbar = 1$.

2. Inflation

Inflation is a theory which states that in a short period of time, the universe expanded in an accelerated manner that led it to a state of isotropy and homogeneity as we can see today. It's true that inflation solves some of the hot Big Bang problems but it should be emphasized that these problems are not intrinsic to the hot Big Bang model, because if we assume that the initial total density parameter Ω is very close to 1 and that the universe began homogeneously over superhorizon distances then it will continue to evolve homogeneously in agreement with observations. Then, these problems are predictive problems of the model because none of the above considerations can be predicted (explained) by it. This is where inflation comes in and plays its part because it's able to predict these events instead of assuming them. As Baumann says in [2] "A theory that explains these initial conditions dynamically seems very attractive".

2.1. The horizon problem

Assuming homogeneity and isotropy on large scales and after some calculations, one can write the Friedmann-Robertson-Walker (FRW) metric as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\mathrm{d}\chi^2 + S_k^2(\chi)\mathrm{d}\Omega^2\right] \tag{1}$$

where $a(t)$ is the scale factor, $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ is the solid angle, with ϕ as the azimuthal angle and θ as the polar angle, $\mathrm{d}\chi \equiv \mathrm{d}r/\sqrt{1-kr^2}$ is a redefinition of the radial coordinate, with r as the radial coordenate in spherical coordenates and k as the curvature parameter. Lastly the $S_k(\chi)$ is defined by

$$S_k(\chi) \equiv \begin{cases} \sinh\chi & k=-1 \\ \chi & k=0 \\ \sin\chi & k=+1 \end{cases}$$

The causal structure of the universe is determined by how much space light can travel in a certain time interval. Since spacetime is isotropic, the coordinate system that makes the most sense in being chosen is such that the light travels only in the radial direction χ. In that case $\mathrm{d}\Omega^2 = 0$ and hence evolution is determined by a two-dimensional line element with the FRW metric (1) becoming

$$\mathrm{d}s^2 = a^2(\tau)[\mathrm{d}\tau^2 - \mathrm{d}\chi^2] \tag{2}$$

where $\mathrm{d}\tau \equiv \mathrm{d}t/a(t)$ is the conformal time. We know that photons travel along null geodesics ($ds^2 = 0$) so their trajectory is given by

$$\Delta\chi(\tau) = \pm\Delta\tau \tag{3}$$

where the minus sign corresponds to incoming photons and the plus sign to outgoing photons. As we can see the main advantage of working with the conformal time is that the paths of the photons become straight lines oriented at 45º, instead of curved lines if we had written the FRW metric in a non conformal way. Now it's time (pun intended) to define two cosmological horizons: one of them as the limit from which it is not possible to observe any past event and the other from which it is not possible to observe any future event, respectively the (comoving) particle horizon and the (comoving) event horizon. The particle horizon is the one on which we will focus due to its importance to explain how the horizon problem is resolved by inflation. Now more physically speaking, we can define the particle horizon as the maximum comoving distance that light can propagate between an initial time τ_i and a final time τ and is given by

$$\chi_{ph}(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{\mathrm{d}t}{a(t)} \tag{4}$$

The particle horizon size can be seen as the intersection between the light cone of the point (observer) p and the spacelike line $\tau = \tau_i$ (see **Figure 1** for more detail). It's in this region where it's possible to have a causal influence between the observer p and another observer, because it's there that p is able to receive signals from the other observer. If the other observer is outside p's light cone of the past, then they are said to be in disjoint regions of spacetime and not able to communicate with each other and hence does not exist a causal influence between them. One usually defines $a_i(t_i \equiv 0) \equiv 0$, corresponding to the Big Bang singularity, and if we consider that and develop (4) to reach the particle horizon in terms of the (comoving) Hubble radius $(aH)^{-1}$, where $H \equiv \dot{a}/a$ is the Hubble parameter, we get

$$\chi_{\mathrm{ph}}(\tau) = \int_0^t \frac{\mathrm{d}t}{a} = \int_0^a \frac{\mathrm{d}a}{a\dot{a}} = \int_0^{\ln a} (aH)^{-1}\,\mathrm{d}\ln a \tag{5}$$

As we can see by (5), we proved that the causal structure of the spacetime can be related to the evolution of the Hubble radius. If the universe is dominated by a fluid such that its equation of state $w \equiv P/\rho$ is constant, then starting from Friedmann's equation one obtains

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)} \tag{6}$$

Since any normal matter source does not violate the strong energy condition (SEC) this implies that $1+3w > 0$ and since the universe always have been expanding from the very beginning $\dot{a} > 0$, we get the following time variation for the comoving Hubble radius

$$\frac{\mathrm{d}(aH)^{-1}}{\mathrm{d}t} = H_0^{-1}\frac{1}{2}(1+3w)a^{-\frac{1}{2}(1-3w)}\dot{a} > 0 \tag{7}$$

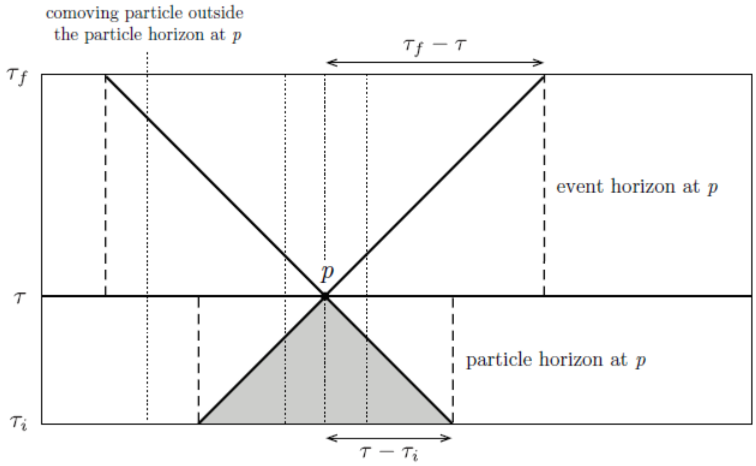

Figure 1. Spacetime diagram where we can see the concepts of particle and event horizons as well as light cones and causality. The dotted lines represent the worldlines of comoving objects. Causally disconnected regions of spacetime are separated by spacelike intervals, $ds^2 < 0$. The set of all null geodesics passing through a given point p (or event) in spacetime is called the light cone. The interior of the light cone, consisting of all null ($ds^2 = 0$ - photons) and timelike geodesics ($ds^2 > 0$ - massive particles), define the region of spacetime causally related to point p. The particle horizon is the maximal conformal distance to which we can send a signal while the event horizon is the maximal conformal distance to which we can receive a signal.

Therefore the comoving Hubble radius increases as the universe expands. If the comoving Hubble radius increases over time, then by (5) the comoving particle horizon, *the fraction of the universe in causal contact* [2], also increases over time, which implies that comoving scales entering the horizon today have been far outside the horizon at Cosmic Microwave Background (CMB) decoupling. But the near-homogeneity of the CMB tells us that the universe was very homogeneous at the time of last-scattering on scales encompassing many regions that a priori are causally independent, and this problem is the previously mentioned horizon problem.

2.2. The horizon problem solved

In the previous subsection we saw that the horizon problem arises from the fact that in conventional cosmology the comoving Hubble radius $(aH)^{-1}$ is always increasing. This suggests that the horizon problem, as well as the other Big Bang Puzzles we did not mention, is solved by a beautifully simple idea: we just need to invert the behavior of the comoving Hubble radius, by making it decrease sufficiently in the very early universe instead of strictly increasing, ie

$$\frac{d}{dt}(aH)^{-1} < 0 \tag{8}$$

which implies that the source which dominates the Universe in the very early times violate the SEC, ie $1 + 3w < 0$. Substituting (6) in (5) one can get initial conformal instant τ_i

$$\tau_i \equiv \frac{2H_0^{-1}}{(1+3w)} a_i^{\frac{1}{2}(1+3w)} \overset{a_i \to 0, w > -\frac{1}{3}}{\longrightarrow} 0 \tag{9}$$

But with SEC being violated we now get

$$\tau_i = \frac{2H_0^{-1}}{(1+3w)} a_i^{\frac{1}{2}(1+3w)} \overset{a_i \to 0, w < -\frac{1}{3}}{\longrightarrow} -\infty \tag{10}$$

which means that the Big Bang singularity is pushed to negative conformal time and therefore exists much more conformal time between the CMB decoupling and the Big Bang singularity. Thus, points that today are not in causal contact were in the very distant past. With inflation $\tau = 0$ is not the initial singularity, but instead becomes a transition point between inflationary cosmology and the standard hot Big Bang cosmology and hence exists conformal time before and after that.

2.3. Conditions for inflation

Although the condition (8) is very important as it is the solution of the horizon problem as we have just seen, the truth is that there are numerous equivalent conditions for inflation to occur. I will now indicate three of them:

- Accelerated expansion - From the Hubble radius derivative

$$\frac{\mathrm{d}(aH)^{-1}}{\mathrm{d}t} = \frac{\mathrm{d}(\dot{a})^{-1}}{\mathrm{d}t} = -\frac{\ddot{a}}{\dot{a}^2} \tag{11}$$

we see that for the Hubble radius to decrease with time, knowing that $\dot{a} > 0$ due to expansion, the condition (for inflation) is

$$\ddot{a} > 0 \tag{12}$$

As expected, inflation is marked by an accelerated expansion.

- Slowly-varying Hubble parameter - Alternatively, we can write

$$\frac{\mathrm{d}(aH)^{-1}}{\mathrm{d}t} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\left(\frac{\dot{a}H}{a^2H^2} + \frac{a\dot{H}}{a^2H^2}\right) = -\left(\frac{1}{a}\frac{\dot{a}}{a}\frac{1}{H} + \frac{1}{a}\frac{\dot{H}}{H^2}\right) = -\frac{1}{a}(1-\varepsilon) \tag{13}$$

where $\varepsilon = -\dot{H}/H^2$ is the slowly-varying Hubble parameter. Again, if we want the Hubble sphere to shrink, which is to say the Hubble radius decreases, we need to impose

$$\varepsilon = -\frac{\dot{H}}{H^2} = -\frac{1}{H}\frac{dH}{\mathrm{d}t}\frac{1}{H} = -\frac{dH}{H}\frac{1}{Hdt} = -\frac{\mathrm{d}\ln H}{dN} < 1 \tag{14}$$

Here, we have defined $dN = Hdt = \mathrm{d}\ln a$, which measures the number of e-folds N of inflationary expansion. N is used to quantify how long the inflationary period must be in order to solve the Hot Big-Bang problems (usually $N \sim 40 - 70$). Hence it means that the fractional change of the Hubble parameter per e-fold needs to be small. Another useful parameter is

$$\eta = \frac{\mathrm{d}\ln\varepsilon}{\mathrm{d}N} = \frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\frac{1}{H\varepsilon} = \frac{\dot{\varepsilon}}{H\varepsilon} \tag{15}$$

As ε has to be small during inflation, so η, which measures how fast ε changes during inflation, also has to be small and therefore one obtains the following condition

$$|\eta| < 1 \wedge \varepsilon < 1 \tag{16}$$

- Quasi-de Sitter expansion - If inflation is perfect (ie lasts forever) then ε parameter is null and therefore

$$-\frac{\dot{H}}{H^2} = 0 \Leftrightarrow \dot{H} = 0 \Rightarrow H = const \tag{17}$$

Knowing that $H = \dot{a}/a$ we obtain

$$\dot{a} = aH \Rightarrow a = e^{Ht} \Rightarrow a^2 = e^{2Ht} \tag{18}$$

This way, spacetime becomes de Sitter:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - e^{2Ht}\mathrm{d}\boldsymbol{x}^2 \tag{19}$$

As inflation must eventually end, spacetime cannot correspond exactly to de Sitter space, otherwise the universe would always expand at an accelerated rate. However, for small ε values (19) is still a good approximation for the inflationary scenario. This is the main reason why many inflation models have the de Sitter space as their framework and is also why it is said that the inflationary period is a quasi-de Sitter period. We will see later on how quantum fluctuations emerge in this spacetime.

2.4. The Physics of Inflation

In the previous subsection, we deduced some conditions that allow inflation to occur. Now let's see very quickly what are the physics of inflation which are linked with that conditions.

2.4.1. Scalar Field Dynamics

The simplest inflation models consider one inflationary scalar field ϕ called *inflaton* which is basically the inflation source. Associated with each field value there is a potential energy density, $V(\phi)$, and if the field varies with time, it also carries kinetic energy density. Like in [1][2] we will just see this field as tool which gives a parameterization to the evolution of the inflationary energy density regardless of its physical meaning. The idea here is to assume that during inflation the stress-energy tensor associated with this scalar field dominates the universe and therefore determines the evolution of the FWR background. The stress-energy tensor of a scalar field is given by (see [3] for more detail)

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right) \tag{20}$$

Assuming the FRW background and restricting to the case of a homogeneous field, $\phi(\tau, \boldsymbol{x}) = \bar{\phi}(t)$ where the bar above ϕ means the inflaton is unperturbed, the scalar stress-energy tensor takes the form of a perfect fluid ($T^0{}_0 = \rho_{\bar{\phi}}$ and $T^i_j = -P_{\bar{\phi}}\delta^i_j$) and hence we get

$$\rho_{\bar{\phi}} = \frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi}) \tag{21}$$

$$P_{\bar{\phi}} = \frac{1}{2}\dot{\bar{\phi}}^2 - V(\bar{\phi}) \tag{22}$$

and the equation of state is

$$w_{\bar{\phi}} \equiv \frac{p_{\bar{\phi}}}{\rho_{\bar{\phi}}} = \frac{\frac{1}{2}\dot{\bar{\phi}}^2 - V}{\frac{1}{2}\dot{\bar{\phi}}^2 + V} \tag{23}$$

This shows that a scalar field can produce an accelerated expansion ($1 + 3w < 0$) if and only if

$$V(\bar{\phi}) > \frac{1}{2}\dot{\bar{\phi}}^2 \tag{24}$$

Substituting (21) in Friedmann's equation for a Universe dominated by the inflationary energy density, $H^2 = \rho_{\bar{\phi}}/3M_{\rm pl}^2$ we get

$$H^2 = \frac{1}{3M_{\rm pl}^2}\left(\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})\right) \tag{25}$$

The other equation that gives us the dynamics of the (homogeneous) scalar field and the FRW geometry is the Klein-Gordon equation

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + V_{,\bar{\phi}} = 0 \tag{26}$$

where $V_{,\bar{\phi}} = \mathrm{d}V/\mathrm{d}\bar{\phi}$ (see [1] for a detailed deduction).

2.4.2. Slow-Roll Inflation

After some algebraic manipulations, one can get the following expression for the slowly-varying Hubble parameter

$$\varepsilon = \frac{\frac{1}{2}\dot{\bar{\phi}}^2}{M_{\rm pl}^2 H^2} \tag{27}$$

Recalling (25) we get

$$\varepsilon = 3\frac{\frac{1}{2}\dot{\bar{\phi}}^2}{\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})} \tag{28}$$

As we can see, the smaller the contribution of the kinetic term to the total energy density such that $\varepsilon < 1$, the more perfect is the inflation, culminating in perfect inflation when is null. This situation is called slow-roll inflation because the kinetic energy density is considered small. Taking the time-derivative of (27) and comparing to (15) we obtain

$$\eta = 2(\varepsilon - \delta) \tag{29}$$

where we have defined the dimensionless acceleration per Hubble time δ as

$$\delta \equiv -\frac{\ddot{\bar{\phi}}}{H\dot{\bar{\phi}}} \tag{30}$$

If $\{\varepsilon, |\delta|\} \ll 1$ then $\{\varepsilon, |\eta|\} \ll 1$, which means that in addition to occurring, inflation persists. We have seen before a case of slow-roll inflation when we took $\varepsilon = 0$, which is the case where inflationary period lasts forever. With these conditions, we now simplify the Friedmann equation (25) and the Klein-Gordon equation (26) as well the expressions for both ε and η. These approximations are known as slow-roll approximations. From (28) we see that the condition $\varepsilon \ll 1$ implies $\frac{1}{2}\dot{\bar{\phi}}^2 \ll V(\bar{\phi})$ which makes the Friedmann equation (25) much simpler

$$H^2 \approx \frac{V}{3M_{\rm pl}^2} \tag{31}$$

From now on we will use just V instead of $V(\bar{\phi})$. Now from (30), we see that the condition $|\delta| \ll 1$ leads the Klein-Gordon equation (26) to become

$$3H\dot{\bar{\phi}} \approx -V_{,\bar{\phi}} \tag{32}$$

Substituting (31) and (32) into (27) we get

$$\varepsilon = \frac{\frac{1}{2}\dot{\bar{\phi}}^2}{M_{\rm pl}^2 H^2} \approx \frac{M_{\rm pl}^2}{2}\left(\frac{V_{,\bar{\phi}}}{V}\right)^2 \equiv \varepsilon_{\rm v} \tag{33}$$

where $\varepsilon_{\rm v}$ is one of the potential slow-roll parameters. The other is defined by

$$|\eta_v| \equiv M_{\rm pl}^2 \frac{\left|V_{,\bar{\phi}\bar{\phi}}\right|}{V} \tag{34}$$

We say the slow-roll inflation occurs when the condition $\{\epsilon_{\rm v}, |\eta_{\rm v}|\} \ll 1$ is satisfied which is a stronger condition than $\{\epsilon, |\eta|\} \ll 1$

3. Quantum Fluctuations during Inflation

The great consequence (and achievement) of inflation is to able to convert microscopic quantum fluctuations into macroscopic seeds for the cosmological structure formation. Hence our goal now is to explain how this mechanism works.

3.1. Inflaton Perturbations

As we have seen before, the total energy density ρ of the early Universe, which is dominated by the inflationary energy density ρ_ϕ, depends on the inflaton field ϕ, whose evolution controls the end of the inflationary period. Currently, evidence suggests that the inflationary period has lasted about $\Delta t \sim 10^{-36}$ s. Due to this order of magnitude, the inflaton has a quantum behaviour. Because of the Heisenberg's uncertainty principle fluctuations lead to a local time delay in the time at which inflation ends, ie different parts of the universe will end inflation at slightly different

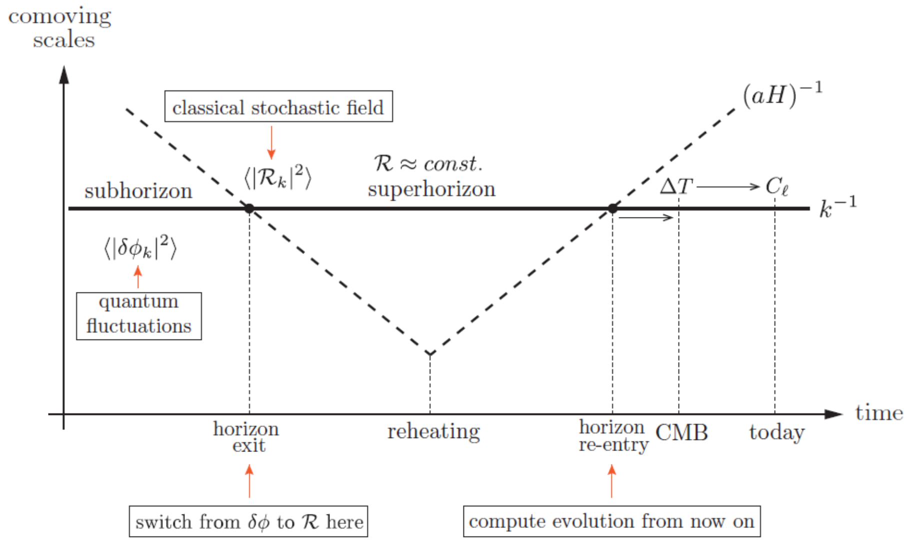

Figure 2. Curvature perturbations during and after inflation: The comoving horizon $(aH)^{-1}$ shrinks during inflation and grows in the subsequent FRW evolution. This implies that comoving scales k^{-1} exit the horizon at early times and re-enter the horizon at late times. While the curvature perturbations $\mathcal{R}$ are outside of the horizon they don't evolve, so our computation for the correlation function $\langle|\mathcal{R}_k|^2\rangle$ at horizon exit during inflation can be related directly to observables at late times. [1]

times. Different parts of the universe therefore undergo slightly different evolutions. This induces relative density fluctuations $\delta\rho(t, \boldsymbol{x})$ and lastly in the CMB temperature $\delta T(\boldsymbol{x})$. So, during inflation we define perturbations around the homogeneous solution for the inflaton

$$\phi(t, \boldsymbol{x}) \equiv \bar{\phi}(t) + \delta\phi(t, \boldsymbol{x}) \tag{35}$$

where $\delta\phi(t, \boldsymbol{x})$ is the inflaton fluctuation.

3.2. Metric Perturbations

Just like we have just did above for inflaton, we also make perturbations around the homogeneous background solution for the metric

$$g_{\mu\nu}(t, \boldsymbol{x}) \equiv \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \boldsymbol{x}) \tag{36}$$

From now on, we will work on a flat conformal FRW background

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\right] \tag{37}$$

Perturbing (37) one writes the perturbed metric as

$$\mathrm{d}s^2 = a^2(\tau)\left[(1+2A)\mathrm{d}\tau^2 - 2B_i\,\mathrm{d}x^i\,\mathrm{d}\tau - (\delta_{ij} + h_{ij})\,\mathrm{d}x^i\,\mathrm{d}x^j\right] \tag{38}$$

where A, B_i and h_{ij} are functions of space and time. It is convenient to say now that until the end of this work we will use the Latin indices to indicate spatial indices.

3.3. Gauge Choice

Fake perturbations can appear by an incovenient choice of coordinates which is called the gauge problem. A solution to the gauge problem is to fix the gauge and controlling the perturbations. One convenient gauge for computing inflationary perturbations is the spatially-flat gauge.

$$C = E = 0 \tag{39}$$

where C an E are scalars that came from SVT decomposition (see Chapter 4 of [1]). We will see later on that in this gauge we can get rid of the metric perturbations and focus only on the inflaton fluctuation $\delta\phi$. Cosmological perturbations is a complex subject and therefore it would be impossible to write a 15 pages essay describing all the details. For a nice review on cosmological perturbations see [9] or Ricardo Cipriano's essay.

3.4. Linking Scales: The Comoving Curvature Perturbation

We turn our attention to a special quantity which remains constant on superhorizon scales ($k \ll aH$), unless non-adiabatic pressure is significant - the comoving curvature perturbation. This quantity is so important because it is the bridge between the fluctuations we observe in the late time universe, such as the distributions of galaxies, (see Chapter 5 of [1] to see more examples) and the microscopic quantum fluctuations created by inflation. One can construct a gauge-invariant expression for it and then fix the spatially flat gauge (39). This will lead to

$$\mathcal{R} = C - \frac{1}{3}\nabla^2 E + \mathcal{H}(B + v) \overset{\text{spatially flat}}{\longrightarrow} \mathcal{H}(B + v) \tag{40}$$

where B is a scalar, v the 3-velocity modulus and $\mathcal{H}$ is the Hubble parameter in conformal time. Then, one can obtain the following relation

$$B + v = -\frac{\delta\phi}{\bar{\phi}'} \tag{41}$$

where $'$ denotes the conformal time derivative ∂_τ. Substituting (41) into (40) we finally get the link between macro and micro

$$\mathcal{R} = -\frac{\mathcal{H}}{\bar{\phi}'}\delta\phi \tag{42}$$

With the relation between $\mathcal{R}$ and $\delta\phi$ setted, one infer that the variance of comoving curvature perturbations for any modes $\left\langle |\mathcal{R}_k|^2 \right\rangle$ assume the following expression

$$\left\langle |\mathcal{R}_k|^2 \right\rangle = \left(\frac{\mathcal{H}}{\bar{\phi}'}\right)^2 \left\langle |\delta\phi_k|^2 \right\rangle \tag{43}$$

where $\delta\phi$ are inflaton field fluctuations in spatially flat gauge, which we will approach. Fluctuations are created on all length scales, i.e. with a spectrum of wavenumbers k. Cosmologically relevant fluctuations start their lives inside the horizon (Hubble radius)

subhorizon: $k \gg aH$

However, while the comoving wavenumber is constant the comoving Hubble radius shrinks during inflation (recall this is how we 'defined' infation), so eventually all fluctuations exit the horizon

superhorizon: $k \ll aH$

Cosmological inhomogeneity is characterized by the intrinsic curvature of spatial hypersurfaces defined with respect to the matter $\mathcal{R}$. After inflation, the comoving horizon grows, $k < aH$ so eventually all fluctuations will re-enter the horizon (see Figure 2).

3.5. Mukhanov-Sasaki Equation

On small scales (subhorizon) the dynamics of inflaton fluctuations can be described by a collection of harmonic oscillators. We will now pay special attention to the deduction of the Mukhanov-Sasaki equation as it is crucial for the description of the fluctuations dynamics. Here I present some steps that were not shown in [1]. The scalar inflaton field action is

$$S = \int \mathrm{d}\tau \mathrm{d}^3 x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \tag{44}$$

where $g \equiv \det(g_{\mu\nu})$. Using the spatially flat gauge, we can only consider the fluctuations of the inflaton and ignore the fluctuations associated with the spacetime because the metric perturbations δg_{00} and $\delta g_{i0} = \delta g_{0i}$ are suppressed relative to the inflaton fluctuations by factors of the slow-roll parameter ε and the metric perturbations $\delta g_{ij} = \delta g_{ji}$ are null because

$$C = E = 0 \Longrightarrow \delta g_{ij} = h_{ij} = 2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle} E + 2\partial_{(i}\hat{E}_{j)} + 2\hat{E}_{ij} = 0 \tag{45}$$

Then our background spacetime is just (37) instead of (38). We write the perturbed inflaton as

$$\phi(\tau, \boldsymbol{x}) = \bar{\phi}(\tau) + \frac{f(\tau, \boldsymbol{x})}{a(\tau)} \tag{46}$$

We now want to get the equations of motion from (44). To obtain them, what is typically done is to separate the action S into two: one with linear fluctuations, ie with terms of one factor of $f(\tau, \boldsymbol{x})$ which we denote by $S^{(1)}$ and other with quadratic fluctuations, ie in terms of two factors of $f(\tau, \boldsymbol{x})$ which we denote by $S^{(2)}$. Applying the covariant derivates present in (44) in the perturbed inflaton (46) and knowing that the metric tensor is diagonal for (37) (which implies $\mu = \nu$) we get

$$\begin{aligned}
\partial_\mu \phi \partial_\mu \phi &= \partial_\mu \left[\bar{\phi}(\tau) + \frac{f(\tau, \boldsymbol{x})}{a(\tau)} \right] \partial_\mu \left[\bar{\phi}(\tau) + \frac{f(\tau, \boldsymbol{x})}{a(\tau)} \right] \\
&= \partial_\tau \left[\bar{\phi}(\tau) + \frac{f(\tau, \boldsymbol{x})}{a(\tau)} \right] \partial_\tau \left[\bar{\phi}(\tau) + \frac{f(\tau, \boldsymbol{x})}{a(\tau)} \right] + \partial_i \left[\bar{\phi}(\tau) + \frac{f(\tau, \boldsymbol{x})}{a(\tau)} \right] \partial_i \left[\bar{\phi}(\tau) + \frac{f(\tau, \boldsymbol{x})}{a(\tau)} \right] \\
&= \left[\bar{\phi}' + \frac{f'a - fa'}{a^2} \right] \left[\bar{\phi}' + \frac{f'a - fa'}{a^2} \right] + \frac{1}{a^2} \left(\frac{\partial f}{\partial x} \right)^2 + \frac{1}{a^2} \left(\frac{\partial f}{\partial y} \right)^2 + \frac{1}{a^2} \left(\frac{\partial f}{\partial z} \right)^2 \\
&= \left[\bar{\phi}' + \frac{f'}{a} - f\frac{a'}{a^2} \right] \left[\bar{\phi}' + \frac{f'}{a} - f\frac{a'}{a^2} \right] + \frac{1}{a^2} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 + \left(\frac{\partial f}{\partial z} \right)^2 \right] \\
&= \bar{\phi}'^2 + \frac{2f'\bar{\phi}'}{a} - 2\frac{fa'}{a^2}\bar{\phi}' - 2f'f\frac{a'}{a^3} + f^2\frac{a'^2}{a^4} + \frac{1}{a^2}(\nabla f)^2
\end{aligned} \tag{47}$$

Expanding the potential energy density to the second order we get

$$V(\phi) = V(\bar{\phi}) + V_{,\phi}\frac{f}{a} + V_{,\phi\phi}\left(\frac{f}{a}\right)^2 \tag{48}$$

To get the linearised equation of motion for the field f we select the second and third term in (47) and the second term in (48). The determinant of a diagonal matrix is simply the product between the diagonal elements and therefore $g = -a^8$. Now we compute the action $S^{(1)}$

$$\begin{aligned}
S^{(1)} &= \int \mathrm{d}\tau \mathrm{d}^3 x \, a^4 \left[\frac{1}{2a^2} \left(\frac{2f'\bar{\phi}'}{a} - 2\frac{fa'}{a^2}\bar{\phi}' \right) - V_{,\phi}\frac{f}{a} \right] \\
&= \int \mathrm{d}\tau \mathrm{d}^3 x \left[a\bar{\phi}'f' - a'\bar{\phi}'f - a^3 V_{,\phi} f \right]
\end{aligned} \tag{49}$$

Notice that $g^{00} = g_{00}^{-1} = a^{-2}$. Integrating the term $a\bar{\phi}' f'$ by parts and dropping the boundary term, we find

$$\begin{aligned} S^{(1)} &= -\int \mathrm{d}\tau \mathrm{d}^3x \left[\partial_\tau \left(a\bar{\phi}'\right) + a'\bar{\phi}' + a^3 V_{,\phi}\right] f \\ &= -\int \mathrm{d}\tau \mathrm{d}^3x \, a \left[\bar{\phi}'' + 2\mathcal{H}\bar{\phi}' + a^2 V_{,\phi}\right] f \end{aligned} \tag{50}$$

Equating $S^{(1)} = 0$ for any f we obtain the Klein-Gordon equation

$$\bar{\phi}'' + 2\mathcal{H}\bar{\phi}' + a^2 V_{,\phi} = 0 \tag{51}$$

Now, to get the equation of motion for the quadratic fluctuations f we select the last three terms of (47) and the last term of (48). Now we compute the action for the second order fluctuations $S^{(2)}$

$$\begin{aligned} S^{(2)} &= \int \mathrm{d}\tau \mathrm{d}^3x \, a^4 \left[\frac{1}{2a^2}\left(f^2 \frac{a'^2}{a^4} + \frac{f'^2}{a^2} - 2f'f\frac{a'}{a^3}\right) - \frac{1}{2a^4}(\nabla f)^2 - V_{,\phi\phi}\frac{f^2}{a^2}\right] \\ &= \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(f')^2 - (\nabla f)^2 - 2\mathcal{H}ff' + \left(\mathcal{H}^2 - a^2 V_{,\phi\phi}\right) f^2\right] \end{aligned} \tag{52}$$

Notice that $g^{ii} = g_{ii}^{-1} = -a^{-2}$. Using $ff' = \frac{1}{2}\left(f^2\right)'$ and integrating that term by parts, we get

$$\begin{aligned} S^{(2)} &= \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(f')^2 - (\nabla f)^2 + \left(\mathcal{H}' + \mathcal{H}^2 - a^2 V_{,\phi\phi}\right) f^2\right] \\ &= \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(f')^2 - (\nabla f)^2 + \left(\frac{a''}{a} - a^2 V_{,\phi\phi}\right) f^2\right] \end{aligned} \tag{53}$$

During slow-roll inflation and recalling (31) (34) we have

$$\frac{V_{,\phi\phi}}{H^2} \approx \frac{3M_{\rm pl}^2 V_{,\phi\phi}}{V} = 3\eta_{\rm v} \ll 1 \tag{54}$$

Since $a' = a^2 H$ with $H \approx const$ (quasi-de Sitter expansion) we also have

$$\frac{a''}{a} \approx 2a'H = 2a^2H^2 \gg a^2 V_{,\phi\phi} \tag{55}$$

For that reason we can neglet the term $a^2 V_{,\phi\phi}$ and the action can be approximated as

$$S^{(2)} \approx \int \mathrm{d}\tau \mathrm{d}^3x \frac{1}{2}\left[(f')^2 - (\nabla f)^2 + \frac{a''}{a} f^2\right] \tag{56}$$

One can identify the Lagrangian density $\mathcal{L}$ in (56)

$$\mathcal{L} = \frac{1}{2}\left[(f')^2 - (\nabla f)^2 + \frac{a''}{a} f^2\right] \tag{57}$$

The Euler-Lagrange equation in GR is

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi\right)}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \tag{58}$$

So applying it for the field f and writing $(\nabla f)^2 = \partial_i f \partial_i f$ and $(f')^2 = \partial_\tau f \partial_\tau f$ we finally obtain the Mukhanov-Sasaki equation in real spacetime

$$\partial_\tau \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\tau f\right)}\right) + \partial_i \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_i f\right)}\right) - \frac{\partial \mathcal{L}}{\partial f} = 0 \Leftrightarrow \frac{1}{2}\left[2\partial_\tau\partial_\tau f - 2\partial_i\partial_i f - 2\frac{a''}{a} f\right] = 0 \Leftrightarrow$$

$$\Leftrightarrow f'' - \nabla^2 f - \frac{a''}{a} f = 0 \tag{59}$$

or, for each Fourier mode $f \longrightarrow f_{\boldsymbol{k}}$, with $\nabla^2 f_{\boldsymbol{k}} = -k^2 f_{\boldsymbol{k}}$, we get the Mukhanov-Sasaki equation in Fourier spacetime

$$f''_{\boldsymbol{k}} + \left(k^2 - \frac{a''}{a}\right) f_{\boldsymbol{k}} = 0 \tag{60}$$

3.6. Quantum Fluctuations in de Sitter Space

We have finally come to the highlight of this work: the full computation of the quantum-mechanical fluctuations (assuming a pure de Sitter space) generated during inflation and their relation to cosmological perturbations.

3.6.1. Canonical Quantisation

First of all, we return to the Lagrangian density (57) for the inflaton fluctuation f and define its conjugate momenta

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial f'} = f' \tag{61}$$

To do the treatment of quantum fluctuations, the first thing to do is to promote the classical fields to quantum operators, ie $f(\tau, \boldsymbol{x}) \longrightarrow \hat{f}(\tau, \boldsymbol{x})$ and $\pi(\tau, \boldsymbol{x}) \longrightarrow \hat{\pi}(\tau, \boldsymbol{x})$. Since $\pi(\tau, \boldsymbol{x})$ was defined as the momentum conjugate to $f(\tau, \boldsymbol{x})$, $\hat{\pi}(\tau, \boldsymbol{x})$ is also the momentum conjugate to $\hat{f}(\tau, \boldsymbol{x})$ (but in a quantum fashion) and hence these two operators satisfy the equal time canonical commutation relation (CCR) due to Heisenberg's uncertainty principle

$$\left[\hat{f}(\tau, \boldsymbol{x}), \hat{\pi}(\tau, \boldsymbol{x}')\right] = i\delta\left(\boldsymbol{x} - \boldsymbol{x}'\right) \tag{62}$$

Note that we fixed $\hbar = 1$. The CCR (62) means that modes at different points in space are independent and then the corresponding operators commute. Now that we have the CCR between $\hat{\pi}(\tau, \boldsymbol{x})$ and $\hat{f}(\tau, \boldsymbol{x})$ in real space, let's find out their CCR in Fourier space by doing the Fourier transform of (62)

$$\begin{aligned}
\left[\hat{f}_{\boldsymbol{k}}(\tau), \hat{\pi}_{\boldsymbol{k}'}(\tau)\right] &= \int \frac{d^3x}{(2\pi)^{3/2}} \int \frac{d^3x'}{(2\pi)^{3/2}} \underbrace{\left[\hat{f}(\tau, \boldsymbol{x}), \hat{\pi}(\tau, \boldsymbol{x}')\right]}_{i\delta(\boldsymbol{x}-\boldsymbol{x}')} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}'} \\
&= i\int \frac{d^3x}{(2\pi)^3} e^{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}} \\
&= i\delta\left(\boldsymbol{k} + \boldsymbol{k}'\right)
\end{aligned} \tag{63}$$

i.e the modes commute if they have different wavelenghts. The Fourier components $\hat{f}_{\boldsymbol{k}}(\tau)$ can be decomposed as follows

$$\hat{f}_{\boldsymbol{k}}(\tau) = f_k(\tau)\hat{a}_{\boldsymbol{k}} + f_k^*(\tau) a_{\boldsymbol{k}}^\dagger \tag{64}$$

where $\hat{a}_{\boldsymbol{k}}$ is the annihilation operator, $a_{\boldsymbol{k}}^\dagger$ is the creation operator and $f_k(\tau)$ and its complex conjugate $f_k^*(\tau)$ are two linearly independent solutions of (60) with $\omega_k(\tau) \equiv k^2 - a''/a$. We have switched to the scalar notation k on the subscript because these functions are the same for all Fourier modes with $k = |\boldsymbol{k}|$ since evolution does not depend on direction (ω_k depends only on the modulus k). We now define a new quantity, the Wronskian, to help us simplify some calculations and normalize f_k such that

$$W\left[f_k, f_k^*\right] \equiv -i\left(f_k \partial_\tau f_k^* - \left(\partial_\tau f_k\right) f_k^*\right) \equiv 1 \tag{65}$$

If one puts (64) into (63) then

$$W\left[f_k, f_k^*\right] \times \left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger\right] = \delta\left(\boldsymbol{k} + \boldsymbol{k}'\right) \tag{66}$$

But (65) implies that

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger\right] = \delta\left(\boldsymbol{k} + \boldsymbol{k}'\right) \tag{67}$$

Equation (65) also provides one of the boundary conditions on the solutions of (60). The second boundary condition that fixes the mode functions completely comes from vacuum selection.

3.6.2. Boundary Conditions and Bunch-Davies Vacuum

We must choose a vacuum state for the fluctuations

$$\hat{a}_{\boldsymbol{k}}|0\rangle = 0 \tag{68}$$

in which we can produce excited states by application of $\hat{a}^{\dagger}_{\boldsymbol{k}}$

$$\left|m_{\boldsymbol{k}_1}, n_{\boldsymbol{k}_2}, \cdots\right\rangle = \frac{1}{\sqrt{m!n!\cdots}}\left[\left(a^{\dagger}_{\boldsymbol{k}_1}\right)^m \left(a^{\dagger}_{\boldsymbol{k}_2}\right)^n \cdots\right]|0\rangle \tag{69}$$

and which corresponds to specifying an additional boundary conditions for f_k. The usual choice is Minkowski vacuum of a comoving observer in the far past ($\tau \longrightarrow -\infty$) when all comoving scales were far inside the Hubble horizon ($k \gg aH$). As previously said, inflationary period is considered a quasi-de Sitter period so $H \approx const$ and $a(t) \approx e^{Ht}$ (see (18)). With these considerations in mind we now simplify the term a''/a present in (60)

$$\tau = \int_0^t \frac{dt'}{a(t')} \approx \int_0^t e^{-Ht'}dt' = -\frac{e^{-Ht}}{H} = -\frac{1}{aH} \Leftrightarrow$$

$$\Leftrightarrow a \approx -\frac{1}{\tau H} \Rightarrow \frac{a''}{a} \approx \frac{2}{\tau^2} \tag{70}$$

Therefore in the limit $\tau \longrightarrow -\infty$ all observable modes had time-independent frequencies as we can see below

$$\omega_k^2 = k^2 - \frac{a''}{a} \approx k^2 - \frac{2}{\tau^2} \overset{\tau \longrightarrow -\infty}{\longrightarrow} k^2 \tag{71}$$

This makes (60) become

$$f_k'' + k^2 f_k \approx 0 \tag{72}$$

which is the equation of a simple harmonic oscillator with time-independent frequency. There are two linearly independent solutions which are $f_k \propto^{-ik\tau}$ and $f_k \propto e^{ik\tau}$. The positivity of the normalisation condition (65) selects the minus sign and therefore what makes the vacuum state to be the ground state is a positive-frequency solution. This means that the resolution of (60) has the following initial condition

$$\lim_{\tau \to -\infty} f_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau} \tag{73}$$

We say that (73) defines a preferable set of Fourier functions and a unique physical vacuum - the Bunch-Davies vacuum (see [8] for more information about this topic). If we consider the inflation to be perfect, i.e $\varepsilon = 0$, then we are at slow-roll inflation regime and hence we are able to do this treatment in de Sitter space. This means that inflation persists as the (conformal) time passes and then the term $-2/\tau^2$ has to be taken into account in solving the Mukhanov-Sasaki equation because τ can be big enough (not big in module) to give a significant contribution. Therefore (60) becomes

$$f_k'' + \left(k^2 - \frac{2}{\tau^2}\right) f_k = 0 \tag{74}$$

The exact solution of (74) is

$$f_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right) \tag{75}$$

where α and β are constants fixed by the initial conditions of the problem. The initial condition (73) forces $\beta = 0$ and $\alpha = 1$, which leads to the final solution

$$f_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \tag{76}$$

Now that f_k is completely determined, the evolution of the mode is also determined including its superhorizon dynamics (where we have the conserved curvature perturbation $\mathcal{R}$).

3.6.3. Zero-Point Fluctuations

With $f_k(\tau)$ fixed we can now calculate the quantum statistics of the operator

$$\hat{f}(\tau, \boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\left[f_k(\tau)\hat{a}_{\boldsymbol{k}} + f_k^*(\tau)a_{\boldsymbol{k}}^\dagger\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{77}$$

Starting with the mean value $\langle\hat{f}\rangle$ we obtain

$$\begin{aligned}\langle\hat{f}\rangle \equiv&\langle 0|\hat{f}(\tau,\mathbf{0})|0\rangle\\ =&\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\left\langle 0\left|f_k(\tau)\hat{a}_{\boldsymbol{k}} + f_k^*(\tau)\hat{a}_{\boldsymbol{k}}^\dagger\right|0\right\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\\ =&0\end{aligned} \tag{78}$$

since $\hat{a}$ annihilates $|0\rangle$ when acting on it from the left, and $\hat{a}^\dagger$ annihilates $\langle 0|$ when acting on it from the right. Let's now calculate the mean square $\langle|\hat{f}|^2\rangle$

$$\begin{aligned}\langle|\hat{f}|^2\rangle &\equiv \left\langle 0\left|\hat{f}^\dagger(\tau,\mathbf{0})\hat{f}(\tau,\mathbf{0})\right|0\right\rangle\\ &= \int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\left\langle 0\left|\left(f_k^*(\tau)\hat{a}_{\boldsymbol{k}}^\dagger + f_k(\tau)\hat{a}_{\boldsymbol{k}}\right)\left(f_{k'}(\tau)\hat{a}_{\boldsymbol{k}'} + f_{k'}^*(\tau)\hat{a}_{\boldsymbol{k}'}^\dagger\right)\right|0\right\rangle\\ &= \int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}f_k(\tau)f_{k'}^*(\tau)\left\langle 0\left|\left[\hat{a}_{\boldsymbol{k}},\hat{a}_{\boldsymbol{k}'}^\dagger\right]\right|0\right\rangle\\ &= \int\frac{\mathrm{d}^3k}{(2\pi)^3}|f_k(\tau)|^2\\ &= \int\frac{\mathrm{d}k}{(2\pi)^3}|f_k(\tau)|^2\,4\pi k^2\\ &= \int\mathrm{d}\ln k\frac{k^3}{2\pi^2}|f_k(\tau)|^2\end{aligned} \tag{79}$$

This means that the variance of inflaton fluctuations $\langle|\hat{f}|^2\rangle - \langle\hat{f}\rangle^2$ receive non-zero quantum fluctuations for $k \neq 0$. We now define the quantity in the integral in $\mathrm{d}\ln k$ above as the (dimensionless) power spectrum

$$\Delta_f^2(k,\tau) \equiv \frac{k^3}{2\pi^2}|f_k(\tau)|^2 \tag{80}$$

This means that the square of the classical solution determines the variance of quantum fluctuations. If one remembers from the beginning of this chapter, we defined the inflaton perturbation as $\delta\phi = f/a$ which implies that the power spectrums of $\delta\phi$ and f are related in the following way

$$\Delta_{\delta\phi}^2(k,\tau) = \frac{\Delta_f^2(k,\tau)}{a^2} \tag{81}$$

Let's compute $\Delta^2_{\delta\phi}(k,\tau)$ by parts. We start calculating $|f_k(\tau)|^2$ using the mode function (76)

$$|f_k(\tau)|^2 = f_k(\tau)f_k^*(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1-\frac{i}{k\tau}\right)\frac{e^{+ik\tau}}{\sqrt{2k}}\left(1+\frac{i}{k\tau}\right) = \frac{1}{2k}\left(1+\frac{1}{(k\tau)^2}\right) \tag{82}$$

This makes (80) like

$$\Delta^2_f(k,\tau) = \frac{k^3}{2\pi^2}\frac{1}{2k}\left(1+\frac{1}{(k\tau)^2}\right) = \left(\frac{1}{2\pi}\right)^2\left(k^2+\frac{1}{\tau^2}\right) \tag{83}$$

In (70) we saw that we can express a in terms of the conformal time τ. This relation becomes a equality in a de Sitter space because $H = const$ and since $a' = a^2H$ we get

$$a = -\frac{1}{\tau H} \Rightarrow a' = \frac{1}{\tau^2 H} \Leftrightarrow (aH)^2 = \frac{1}{\tau^2} \tag{84}$$

Substituting (84) into (83) we get

$$\Delta^2_f(k,a) = \left(\frac{1}{2\pi}\right)^2\left(k^2+(aH)^2\right) = \left(\frac{aH}{2\pi}\right)^2\left(\left(\frac{k}{aH}\right)^2+1\right) \tag{85}$$

Therefore (81) becomes

$$\Delta^2_{\delta\phi}(k,a) = \left(\frac{H}{2\pi}\right)^2\left(1+\left(\frac{k}{aH}\right)^2\right) \tag{86}$$

In the superhorizon limit we know that $k \ll aH$ so

$$\Delta^2_{\delta\phi}(k,a) = \left(\frac{H}{2\pi}\right)^2\left(1+\left(\frac{k}{aH}\right)^2\right) \overset{\text{superhorizon}}{\longrightarrow} \left(\frac{H}{2\pi}\right)^2 \tag{87}$$

We will use the approximation that the power spectrum at horizon crossing ($k = aH$) is

$$\Delta^2_{\delta\phi}(k) \approx \left(\frac{H}{2\pi}\right)^2\Bigg|_{k=aH} \tag{88}$$

3.6.4. Quantum-to-Classical Transition

We now briefly mention the Quantum-to-Classical Transition. One can proove that in the superhorizon limit $k \ll aH$ (or $k\tau \longrightarrow 0$) the operators $\hat{f}(\tau,\boldsymbol{x})$ and $\hat{\pi}(\tau,\boldsymbol{x})$ are proportional to each other and hence commute. This is literally the signature of classical modes. So after horizon crossing the inflaton fluctuation $\delta\phi$ can be seen as a classical stochastic field and therefore we can 'promote' the quantum expectation value to a classical ensemble average.

3.7. Primordial Perturbations from Inflation - Curvature Perturbations

As we cam see by Figure (2), at horizon crossing $k = aH$ we pass from the inflationary scalar field fluctuations to the conserved curvature perturbations. Just like their variances (43), the power spectra of $\mathcal{R}$ and $\delta\phi$ are related by a similar expression

$$\Delta^2_{\mathcal{R}} = \left(\frac{H}{\dot{\phi}}\right)^2\Delta^2_{\delta\phi} = \frac{1}{2\varepsilon}\frac{\Delta^2_{\delta\phi}}{M^2_{\text{pl}}} \tag{89}$$

If we put the approximation (88) into (89) at horizon crossing we get

$$\Delta^2_{\mathcal{R}}(k) = \frac{1}{8\pi^2}\frac{1}{\varepsilon}\frac{H^2}{M^2_{\text{pl}}}\Bigg|_{k=aH} \tag{90}$$

Another way to write the power spectra of $\Delta^2_{\mathcal{R}}$ is

$$\Delta^2_{\mathcal{R}}(k) \equiv A_s \left(\frac{k}{k_\star}\right)^{n_s-1} \tag{91}$$

where A_s is the amplitude of the spectrum, $k_\star$ is a reference scale $n_s - 1$ is the scalar spectral index which is used to quantify the deviation from scale-invariance defined by

$$n_s - 1 \equiv \frac{d \ln \Delta^2_{\mathcal{R}}}{d \ln k} \tag{92}$$

The value $n_s = 1$ corresponds to a perfect scale invariance because the power spectra does not depend on the scale ($\Delta_{\mathcal{R}^2} \propto k^0$). One can deduce another expression for this parameter

$$n_s - 1 = -2\varepsilon - \eta \tag{93}$$

This parameter is very useful because it allows us to assess how different was the inflation from the perfect de Sitter limit. According to [1], observations have detected the small deviation from scale-invariance predicted by inflation

$$n_s = 0.9603 \pm 0.0073 \tag{94}$$

and hence the comoving curvature perturbation is not strictly conserved on superhorizon scales.

4. Final words

We reached the end of this 'lecture' having seen the way in which inflation solves the problems of the standard Big Bang cosmology, namely the horizon problem, the various consequences that it produces in quantum terms and beyond, and finally that there are at least one parameter (in fact there are more, see [2] for more information) that can be measured, with which you can get an idea of how inflation occurred. This work arose from my desire to learn more about inflation, including the subject that was given in the course "Universo Primitivo". My main goal was to approach the topic of inflaton's quantum fluctuations in the most pedagogical way possible and that was only possible by also presenting all the basic theory behind it. I could have gone into detail in the part of cosmological perturbation theory, but I felt that it would be enough to use the main results to move forward than to present that topic extensively. My two main references were [1] and [2] but i also checked [3] and [7] sometimes. If one says that this work is a mediocre review on inflation i would be happy. Finally, i would like to dedicate this work to my family, my closest friends and to my girlfriend for you to know why i was absent in the first half of February.

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