Proceedings

An Introduction on Relativistic Perturbation Theory

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Abstract: In this essay we discuss the theory of cosmological perturbations in a *soft* way, deriving the Einstein Field Equations for a perturbed space-time and finally touching a bit in Structure Formation.

Keywords: General Relativity; Perturbation Theory; Evolution of Structure; Physical Cosmology

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1. Introduction

In the end, there will be a list of abbreviations that will appear throughout the paper.

Our current understanding of the evolution of the universe is based upon the FLRW cosmological model which can describe the universe at times as early as 10^{-43} seconds! But in 1946 Lifshitz [3] found that the gravitational potential cannot grow within linear perturbation theory and he concluded that galaxies would not have formed by gravitational instability and for this reason we needed to introduce Relativistic Perturbation Theory.

So in this essay we are going to go over what is Relativistic Perturbation Theory (RPT) and how it is constructed starting from our knowledge of General Relativity and Perturbation Theory in hopes to understand the Evolution of Structure of the Big Bang Model. In other words RTP attempts to *explain* how the universe forms astronomic structures such as stars, quasars, galaxies and clusters from perturbations during inflation.

To setup the stage that we are going to be using when analysing RPT we must consider a first important note: RPT applies to an universe that is predominantly homogeneous which is considered to be a good approximation on the largest scales, but on smaller scales we need different techniques.

2. Perturbations of the Metric

We shall begin in the same fashion as we aboard common GR problems: by constructing the Metric Tensor that we are going to work with and, in this case, we want to define our perturbed metric tensor! So we first recall that for a spatially homogeneous and isotropic universe with the FLRW metric we have

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2(\eta)(-\mathrm{d}\eta^2 + \bar{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j) \tag{2.1}$$

where $a(\eta)$ is the scale factor (units are chosen so that the speed of light is unity), $\bar{g}_{ij}$ is the unperturbed metric tensor for a three-space. However, as we said previously the FLRW model is still an incomplete model. So we start our journey we define the perturbed metric tensor as

$$\boxed{g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}} \tag{2.2}$$

where we have the full space-time metric into a background $\bar{g}_{\mu\nu}$, which is a known solution of the Einstein equations, plus some perturbations $\delta g_{\mu\nu}$ which are assumed to be small. It's important to notice that the index $i, j, \cdots$ refer to the 3D spatial components while $\mu, \nu, \cdots$ for 4D space-time components. Unfortunately we do not have a spatially homogeneous universe so we have to parameterize the perturbations to the homogeneous background metric with

$$g_{00} \equiv -a^2(1+2A), \quad g_{0i} \equiv -a^2 B_i, \quad g_{ij} \equiv a^2\left(\bar{g}_{ij} + 2h_{ij}\right) \tag{2.3}$$

where A, B_i and h_{ij} are functions of space and time where the function $A\left(\eta, x^i\right)$ is called the lapse function, and $B_i\left(\eta, x^i\right)$ the shift vector and so the perturbed metric, can be written in a general way as

$$\boxed{\mathrm{d}s^2 = a^2(\eta)\left[-(1+2A)\mathrm{d}\eta^2 - 2B_i\,\mathrm{d}x^i\,\mathrm{d}\eta + (\delta_{ij}+h_{ij})\,\mathrm{d}x^i\,\mathrm{d}x^j\right]} \tag{2.4}$$

which would give us an metric tensor defined as

$$g_{\mu\nu} = a^2(\eta)\begin{pmatrix} 1+2A & -2B_1 & -2B_2 & -2B_3 \\ -2B_1 & (1+h_{11}) & h_{12} & h_{13} \\ -2B_2 & h_{12} & (1+h_{22}) & h_{23} \\ -2B_3 & h_{13} & h_{23} & (1+h_{33}) \end{pmatrix} \tag{2.5}$$

This is to set a general idea for what we want to achieve during this essay and to start build up what will come next. In the metric presented above the variables presented will be explained in the next chapter and further on why we were able to build the metric in such way and what considerations were made to get that result.

3. Scalar-Vector-Tensor decomposition

In order to proceed to have a proper treatment of cosmological perturbation theory it is very helpful to separate the space-time into two different space-times or manifolds: the perturbed space-time, that is close to a simple, symmetric, space-time, the background space-time, that we already know.

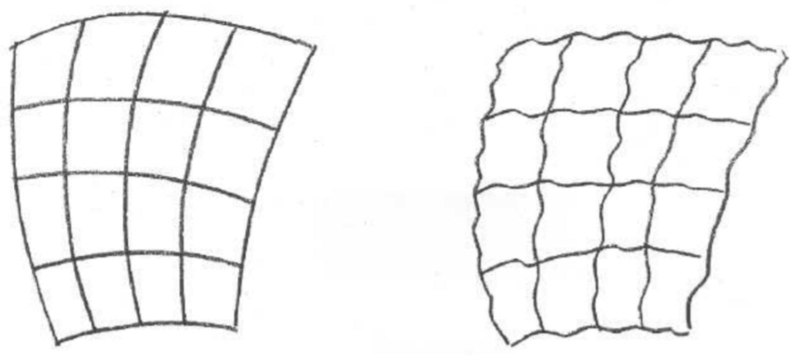

Figure 1. The background space-time and the perturbed space-time.

In order to do this we need to preform a Scalar-Tensor-Vector split which is based upon the idea of decomposing a vector into longitudinal and transverse parts. The longitudinal/transverse decomposition allows us to write a vector field in terms of a scalar (the longitudinal or irrational part) and a part that cannot be obtained from a scalar (the transverse or rotational part). This way we can write h_{ij} function presented in equation (2.4) as

$$h_{ij} = \underbrace{2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E}_{\text{scalar}} + \underbrace{2\partial_{(i}\hat{E}_{j)}}_{\text{vector}} + \underbrace{2\hat{E}_{ij}}_{\text{tensor}} \tag{3.1}$$

where we define, as mentioned above, from the gradients of a scalar and a transverse vector

$$\begin{aligned} \partial_{\langle i}\partial_{j\rangle}E &\equiv \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)E \\ \partial_{(i}\hat{E}_{j)} &\equiv \tfrac{1}{2}\left(\partial_i\hat{E}_j + \partial_j\hat{E}_i\right) \end{aligned} \tag{3.2}$$

The decomposition performed above splits the metric perturbations into 4 + 4 + 2 scalar, vector, tensor degrees of freedom making a total of **10 degrees of freedom** which has

- Scalars: A, B, C, E
- Vectors: $\hat{B}_i, \hat{E}_i$
- Tensors $\hat{E}_{ij}$

An interesting note is that as a property of the GR field equations due to the invariance of the form of physical laws under arbitrary coordinate transformations, out of the 10 degrees of freedom only 6 of which are physical and 4 are not. This implies that we need to distinguish the physical perturbations from the fictitious ones.

4. The gauge issue in cosmology

> The term gauge refers to any specific mathematical formalism to regulate redundant degrees of freedom in the Lagrangian. The transformations between possible gauges, called gauge transformations.

In GR perturbation theory, a gauge transformation implies a coordinate transformation between such coordinate systems in the perturbed space-time. So when we split the metric into the background (unperturbed space-time) and the perturbations around it and by choosing a coordinate system, we explicitly changing the correspondence of the physical Universe to the background homogeneous and isotropic Universe. This way we have the metric perturbations to make a gauge transformation but considering that due to the invariance of the form of physical laws and so would be guage-invariant but the metric perturbations transform non-trivially (or gauge transform).

> To clarify let us imagine a 4-scalar s. The full quantity $s = \bar{s} + \delta s$ lives on the perturbed spacetime. However, we cannot assign a unique background quantity $\bar{s}$ to a point in the perturbed spacetime, because in different gauges this point is associated with different points in the background, with different values of $\bar{s}$. Therefore there is also no unique perturbation δs, but the perturbation is gauge-dependent. The perturbation δs is obtained from a subtraction between two spacetimes, and we consider it as living on the background spacetime. It changes in the gauge transformation.

This explanation is made in more detail in [9] on chapter 4, showing the key steps for the demonstration of what was introduced in the citation. And so we arrive at the gauge issue in cosmology on how we distinguish the physical perturbations from the fictitious ones and decide on which gauge to choose to describe our metric since the perturbations are gauge dependent.

How can we deal with the gauge issue in Cosmology?

1. Define gauge invariant perturbations and solve the corresponding gauge invariant equations. (A1)
2. Fix a gauge choice and keep track of all perturbations and check how quantities transform. (A2)

One thing to note is that performing a gauge-invariant calculations may be technically more difficult, but has the advantage of treating only physical quantities. As a simple example let us consider the coordinate transformation:

$$x \to x, \quad \text{and} \quad t = \int a d\eta \to \bar{t} = t + f(x,t) \tag{4.1}$$

And where for convenience we define the following proprieties energy density: $\varepsilon(x,\eta) = \varepsilon_0(\eta)$ and pressure: $p(\varepsilon)$ in an unperturbed homogeneous Universe:

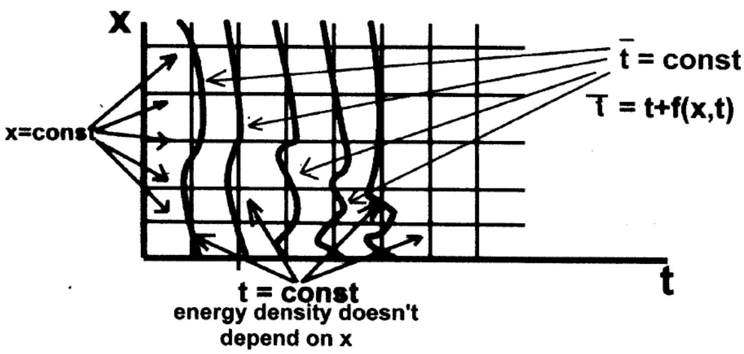

Figure 2. Comparison of the coordinate transformation referenced in the equations

Looking at the figure above (Figure 2) the Universe is homogeneous, but in the new coordinate system it looks like inhomogeneous:

$$\varepsilon(t) = \varepsilon(\bar{t} - f(x,t)) \approx \varepsilon(\bar{t}) - \frac{\partial \varepsilon}{\partial t} f \implies \delta\varepsilon(x,t) = -\frac{\partial \varepsilon}{\partial t} f \tag{4.2}$$

These inhomogeneities/perturbations are not real.

4.1. Gauge Transformations

To solve this problem we have to preform gauge transformations so let us consider the following coordinate transformation

$$X^\mu \mapsto \tilde{X}^\mu \equiv X^\mu + \xi^\mu(\eta, \boldsymbol{x}), \quad \text{where} \quad \xi^0 \equiv T, \quad \xi^i \equiv L^i = \partial^i L + \hat{L}^i \tag{4.3}$$

where we split the spatial shift L^i into a scalar, L, and a divergenceless vector, $\hat{L}^i$. From here we want to obtain a metric for which we can treat perturbations and order to do that we have to choose a gauge transformation that solve the issues that we have discussed previously so starting from the generic form of the metric

$$\mathrm{d}s^2 = g_{\mu\nu}(X)\mathrm{d}X^\mu \mathrm{d}X^\nu = \tilde{g}_{\alpha\beta}(\tilde{X})\mathrm{d}\tilde{X}^\alpha \mathrm{d}\tilde{X}^\beta \tag{4.4}$$

note that the choice of different notation in the second equality is for the new guage defined in (4.3)

$$\mathrm{d}\tilde{X}^\alpha = \frac{\partial \tilde{X}^\alpha}{\partial X^\mu}\mathrm{d}X^\mu \implies g_{\mu\nu}(X) = \frac{\partial \tilde{X}^\alpha}{\partial X^\mu}\frac{\partial \tilde{X}^\beta}{\partial X^\nu}\tilde{g}_{\alpha\beta}(\tilde{X}) \tag{4.5}$$

this simple manipulation allows us to relate the metric in the old coordinates, $g_{\mu\nu}$, to the metric in the new coordinates, $\tilde{g}_{\alpha\beta}$ and with the SVT-decomposition we rewrite the metric perturbations in (2.4).

Without going into too much detail as is beyond the purpose of this essay we can write the gauge transformation rules for the individual components of 4-scalar, 4-vector and type (1,1) 4-tensor perturbations as

$$\begin{aligned}
\widetilde{\delta s} &= \delta s - \bar{s}'\xi^0 \\
\widetilde{\delta w} &= \delta w^0 + \xi^0_{,0}\bar{w}^0 - \bar{w}^0_{,0}\xi^0 \\
\widetilde{\delta w}^i &= \delta w^i + \xi^i, \bar{w}^0 \\
\widetilde{\delta A}^0_0 &= \delta A^0_0 - \bar{A}^0_{0,0}\xi^0 \\
\widetilde{\delta A}^0_i &= \delta A^0_i + \frac{1}{3}\xi^0_{,i}\bar{A}^k_k - \xi^0_{,i}\bar{A}^0_0 \\
\widetilde{\delta A}^i_0 &= \delta A^i_0 + \xi^i_{,0}\bar{A}^0_0 - \frac{1}{3}\xi^i_{,0}\bar{A}^k_k \\
\widetilde{\delta A}^i_j &= \delta A^i_j - \frac{1}{3}\delta^i_j \bar{A}^k_{k,0}\xi^0 \\
\widetilde{\delta A}^k_k &= \delta A^k_k - \bar{A}^k_{k,0}\xi^0 \\
\widetilde{\delta A}^i - \frac{1}{2}\delta^i \widetilde{\delta A}^k_k &= \delta A^i - \frac{1}{2}\delta^i \cdot \delta A^k_n
\end{aligned} \tag{4.6}$$

and the gauge transformation laws

$$\tilde{A} = A - \xi^0_{,0} - \frac{a'}{a}\xi^0 \tag{4.7}$$

$$\tilde{B}_i = B_i + \xi^i_{,0} - \xi^0_{,i} \tag{4.8}$$

$$\tilde{D} = D + \frac{1}{3}\xi^k_{,k} + \frac{a'}{a}\xi^0 \tag{4.9}$$

$$\tilde{E}_{ij} = E_{ij} - \frac{1}{2}\left(\xi^i_{,j} + \xi^j_{,i}\right) + \frac{1}{3}\delta_{ij}\xi^k_{,k} \tag{4.10}$$

which will come in handy when we actually start to use the Gauge Transformations further on the essay and these calculations with a bit more guidelines can be found on reference [9].

4.2. Gauge invariant perturbations

We will now discuss on how to construct the gauge invariant perturbations. With the equations of motion for the scalar and matter perturbations as well as the background quantities one can derive the gauge invariant equations of motion, i.e. equations of motion. And how do we obtain them? by defining special combinations of the SVT perturbations that do not change under coordinate transformations. So using for example the Bardeen potentials

$$\begin{aligned}\Phi &\equiv A + \mathcal{H}\left(B - E'\right) + \left(B - E'\right)' \\ \Psi &\equiv D + \frac{1}{3}\nabla^2 E - \mathcal{H}\left(B - E'\right) = \psi - \mathcal{H}\left(B - E'\right)\end{aligned} \tag{4.11}$$

where we have $\mathcal{H} \equiv a'/a$ and we define derivative as $(\ ' \ ) = \dfrac{\partial}{\partial \eta}$ for convenience and note that the metric perturbation δg_{ij} is not invariant under this change of coordinates so for this reason the functions ϕ, ψ, B and E are introduced to rewrite the perturbation by then substituting in the metric (2.4). These two Bardeen potentials can be considered as the 'real' spacetime perturbations since they cannot be removed by a gauge transformation!

4.3. Gauge fixing choices

As we said before we can fix the gauge and keep track of all perturbations both from the metric and matter. In order to do this we have several choices (these are not only options, just a sample):

⇒ **Newtonian gauge**

The Newtonian guage consists on considering in the metric presentend in equation (2.4) with $B = E = 0$ applying these conditions to the metric and to relate to the Bardeen's potentials we define $A \equiv \Psi, C \equiv -\Phi$ giving

$$\mathrm{d}s^2 = a^2(\eta)\left[(1+2\Phi)\mathrm{d}\eta^2 - (1-2\Psi)\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\right] \tag{4.12}$$

this gauge has an unique proprety of being fixed completly! Which makes it rather simple and easy to work with and will be used and explained in further chapters and Ψ plays the role of the gravitational potential.

⇒ **Spatially-flat gauge**

This gauge is very good to derive the inflationary perturbations and has the conditions $C = E = 0$. This guage leaves the geometric part of the metric unchanged which is really useful for inflationary perturbations since in the very beginning of the universe (according to the big bang model) we had only perturbations in the inflaton field and only after appears geometric perturbations. Note that Ψ and Φ done n

⇒ **Uniform density gauge**

In this gauge we choose the time-slicing in a way that the total density perturbation is set to zero i.e. $\delta\rho = 0$.

⇒ **Comoving gauge**

We chose coordinates in a way that the total momentum density vanishes $q_i = (\bar{\rho} + \bar{P})v_i = 0$ and $q_i = B_i = 0$

5. General form of the equations for cosmological perturbations

We can now start to work towards the Field Equations for Scalar Perturbations. For this chapter, due to its simplicity and usefulness, we will use Newtonian Gauge (often also referenced as the *conformal-Newtonian gauge* or the *longitudinal gauge*). For this derivation we shall assume scalar perturbations only (since they are the ones responsible for the structure of the universe). So our focus will be to obtain the Einstein Field Equations:

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}} \tag{5.1}$$

5.1. Spacetime Curvature: Einstein Tensor

We will start this derivation by obtaining the Einstein Tensor (corresponding to the left side of equation (5.1)). For this purpose we have to obtain the Ricci Tensor, so we must first start by revisiting the line element mentioned previously with the gauge transformation laws shown in equation (4.10) for scalar fields are

$$\begin{aligned}
\tilde{A} &= A - \xi^{0\prime} - \frac{a'}{a}\xi^0 \\
\tilde{B} &= B + \xi' + \xi^0 \\
\tilde{D} &= D - \frac{1}{3}\nabla^2\xi + \frac{a'}{a}\xi^0 \\
\tilde{E} &= E + \xi
\end{aligned} \tag{5.2}$$

and since we are going to use *conformal-Newtonian gauge* to obtain $B = E = 0$ we have to set

$$\begin{aligned}
\xi &= -E \\
\xi^0 &= -B + E'
\end{aligned} \tag{5.3}$$

and so we finally obtain the line element for this *Newtonian gauge* in scalar perturbations as

$$ds^2 = a(\eta)^2 \left[-(1+2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j\right] \tag{5.4}$$

where the metric is simply given by

$$g_{\mu\nu} = a^2 \begin{bmatrix} -1-2\Phi & 0 \\ 0 & (1-2\Psi)\delta_{ij} \end{bmatrix} \tag{5.5}$$

Now with with this information, just as we've seen in GR, we have to find the connection coefficients of the metric which relate with the Ricci tensor, without perturbations, as

$$R_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu,\alpha} - \Gamma^{\alpha}_{\alpha\mu,\nu} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{\nu\mu} - \Gamma^{\alpha}_{\nu\beta}\Gamma^{\beta}_{\alpha\mu} \tag{5.6}$$

but we want to separate the *background* and the perturbations as we've done previously so we consider

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \delta\Gamma^{\alpha}_{\beta\gamma} \tag{5.7}$$

and as we know said coefficients only require the metric to obtain and are given by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right) \tag{5.8}$$

which means that the only thing that we have to find is $g^{\mu\nu}$ and proceed with the derivatives of the metric, so

$$g^{\mu\nu} = a^{-2} \begin{bmatrix} -1+2\Phi & 0 \\ 0 & (1+2\Psi)\delta_{ij} \end{bmatrix} \tag{5.9}$$

since we separated the *background* and the perturbations we can treat both of the connections separately making the calculations strait forward. Firstly we obtain the connections for the *background*

$$\Gamma^0_{00} = \frac{a'}{a} + \Phi' \tag{5.10}$$

$$\Gamma^0_{0k} = \Phi_{,k} \tag{5.11}$$

$$\Gamma^i_{00} = \Phi_{,i} \tag{5.12}$$

$$\Gamma^i_{0j} = \frac{a'}{a}\delta^i_j - \Psi'\delta^i_j \tag{5.13}$$

$$\Gamma^0_{ij} = \frac{a'}{a}\delta_{ij} - \left[2\frac{a'}{a}(\Phi+\Psi) + \Psi'\right]\delta_{ij} \tag{5.14}$$

$$\Gamma^\alpha_{0\alpha} == 4\frac{a'}{a} + \Phi' - 3\Psi' \tag{5.15}$$

$$\Gamma^i_{kl} = -\left(\Psi_{,l}\delta^i_k + \Psi_{,k}\delta^i_l\right) + \Psi_{,i}\delta_{kl} \tag{5.16}$$

$$\Gamma^\alpha_{i\alpha} = \Phi_{,i} - 3\Psi_{,i} \tag{5.17}$$

$$\tag{5.18}$$

an important note is that in $\Gamma^\alpha_{0\alpha}$ and $\Gamma^\alpha_{i\alpha}$ we neglected all terms of higher order of the small quantities Ψ and Φ. In the same fashion we can do the connection coefficients for the perturbations using equation (5.8)

$$\delta\Gamma^0_{00} = \Phi' \tag{5.19}$$

$$\delta\Gamma^0_{0k} = \Phi_{,k} \tag{5.20}$$

$$\delta\Gamma^0_{ij} = -\left[2\mathcal{H}(\Phi+\Psi) + \Psi'\right]\delta_{ij} \tag{5.21}$$

$$\delta\Gamma^i_{00} = \Phi_{,i} \tag{5.22}$$

$$\delta\Gamma^i_{0j} = -\Psi'\delta^i_j \tag{5.23}$$

$$\delta\Gamma^i_{kl} = -\left(\Psi_{,l}\delta^i_k + \Psi_{,k}\delta^i_l\right) + \Psi_{,i}\delta_{kl} \tag{5.24}$$

Now that we have the connection coefficients for the *background* and for the perturbations to arrive at the Ricci Tensor teased in equation (5.6) by only expanding the Ricci Tensor using equation (5.7) as

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta\Gamma^\alpha_{\nu\mu,\alpha} - \delta\Gamma^\alpha_{\alpha\mu,\nu} + \bar{\Gamma}^\alpha_{\alpha\beta}\delta\Gamma^\beta_{\nu\mu} + \bar{\Gamma}^\beta_{\nu\mu}\delta\Gamma^\alpha_{\alpha\beta} - \bar{\Gamma}^\alpha_{\nu\beta}\delta\Gamma^\beta_{\alpha\mu} - \bar{\Gamma}^\beta_{\alpha\mu}\delta\Gamma^\alpha_{\nu\beta} \tag{5.25}$$

Now substituting for the set of connection coefficients for both *background* and the perturbations we have

$$\begin{aligned} &R_{00} = -3\mathcal{H}' + 3\Psi'' + \nabla^2\Phi + 3\mathcal{H}\left(\Phi' + \Psi'\right) \\ &R_{0i} = 2\left(\Psi' + \mathcal{H}\Phi\right)_{,i} \\ &R_{ij} = \left(\mathcal{H}' + 2\mathcal{H}^2\right)\delta_{ij} + \left[-\Psi'' + \nabla^2\Psi - \mathcal{H}\left(\Phi' + 5\Psi'\right) - \left(2\mathcal{H}' + 4\mathcal{H}^2\right)(\Phi+\Psi)\right]\delta_{ij} + (\Psi-\Phi)_{,ij} \end{aligned} \tag{5.26}$$

With this information we can start by computing the Ricci scalar (or the *curvature scalar*) which is given by the sum of

$$R^\mu_\nu = g^{\mu\alpha}R_{\alpha\nu} \tag{5.27}$$

for that we reason we have to raise of the index of $R_{\mu\nu}$ obtained before

$$R^\mu_\nu = g^{\mu\alpha}R_{\alpha\nu} = \left(\bar{g}^{\mu\alpha} + \delta g^{\mu\alpha}\right)\left(\bar{R}_{\alpha\nu} + \delta R_{\alpha\nu}\right) = \bar{R}^\mu_\nu + \delta g^{\mu\alpha}\bar{R}_{\alpha\nu} + \bar{g}^{\mu\alpha}\delta R_{\alpha\nu} \tag{5.28}$$

and so computing with all the information already know we obtain

$$
\begin{aligned}
R^0_0 =& 3a^{-2}\mathcal{H}' + a^{-2}\left[-3\Psi'' - \nabla^2\Phi - 3\mathcal{H}\left(\Phi' + \Psi'\right) - 6\mathcal{H}'\Phi\right] \\
R^0_i =& -2a^{-2}\left(\Psi' + \mathcal{H}\Phi\right)_{,i} \\
R^i_0 =& -R^0_i = 2a^{-2}\left(\Psi' + \mathcal{H}\Phi\right)_{,i} \\
R^i_j =& a^{-2}\left(\mathcal{H}' + 2\mathcal{H}^2\right)\delta^i_j + a^{-2}\left[-\Psi'' + \nabla^2\Psi - \mathcal{H}\left(\Phi' + 5\Psi'\right) - \left(2\mathcal{H}' + 4\mathcal{H}^2\right)\Phi\right]\delta_{ij} + a^{-2}(\Psi - \Phi)_{,ij}
\end{aligned} \tag{5.29}
$$

with this information we can now do the summation o obtain the Ricci scalar

$$
R = R^0_0 + R^i_i = 6a^{-2}\left(\mathcal{H}' + \mathcal{H}^2\right) + a^{-2}\left[-6\Psi'' + 2\nabla^2(2\Psi - \Phi) - 6\mathcal{H}\left(\Phi' + 3\Psi'\right) - 12\left(\mathcal{H}' + \mathcal{H}^2\right)\Phi\right] \tag{5.30}
$$

all things considered we can finally obtain the Einstein tensor, presented in component form, given by

$$
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{5.31}
$$

and so we derive that

$$
G_{ij} = a^{-2}\left(-2\mathcal{H}' - \mathcal{H}^2\right)\delta_{ij} + a^{-2}\left[2\Psi'' + \nabla^2(\Phi - \Psi) + \mathcal{H}\left(2\Phi' + 4\Psi'\right) + \left(4\mathcal{H}' + 2\mathcal{H}^2\right)\Phi\right]\delta_{ij} + a^{-2}(\Psi - \Phi)_{,ij} \tag{5.32}
$$

And with this we derived the 1st part of the Einstein equations! We just now need to apply what we just derived to the perturbed Einstein Equations which reads as follows:

$$
\delta G^\mu_\nu = 8\pi G \delta T^\mu_\nu \tag{5.33}
$$

And obtain the set of values corresponding with the perturbed Einstein Tensor. And so we have

$$
\begin{aligned}
\delta G^0_0 =& a^{-2}\left[-2\nabla^2\Psi + 6\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right)\right] = -8\pi G\delta\rho^N \\
\delta G^0_i =& -2a^{-2}\left(\Psi' + \mathcal{H}\Phi\right)_{,i} = -8\pi G(\bar{\rho} + \bar{p})v^N_{,i} \\
\delta G^i_0 =& 2a^{-2}\left(\Psi' + \mathcal{H}\Phi\right)_{,i} = 8\pi G(\bar{\rho} + \bar{p})v^N_{,i} \\
\delta G^i_j =& a^{-2}\left[2\Psi'' + \nabla^2(\Phi - \Psi) + \mathcal{H}\left(2\Phi' + 4\Psi'\right) + \left(4\mathcal{H}' + 2\mathcal{H}^2\right)\Phi\right]\delta^i_j + a^{-2}(\Psi - \Phi)_{,ij}
\end{aligned} \tag{5.34}
$$

And we can move onto obtaining the Perturbations of the Stress-Energy Tensor to finally obtain the Field equations!

5.2. Perturbations of the Stress-Energy Tensor

We start now the second part of deriving the Einstein field equations, by deriving the perturbed Stress-Energy Tensor but note that we will be considering the background energy tensor is necessarily of the perfect fluid form but this consideration arrives from the fact that the "imperfections" caused by the perturbations can only show up in the energy tensor if there is inhomogeneity or anisotropy but gravity only cares about the energy tensor so we have

$$
\bar{T}^{\mu\nu} = (\bar{\rho} + \bar{p})\bar{u}^\mu\bar{u}^\nu + \bar{p}\bar{g}^{\mu\nu} \tag{5.35}
$$

$$
\bar{T}^\mu_\nu = (\bar{\rho} + \bar{p})\bar{u}^\mu\bar{u}_\nu + \bar{p}\delta^\mu_\nu \tag{5.36}
$$

where u^ν is the 4-velocity vector field of the fluid and the energy tensor of the perturbed universe is

$$
T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu \tag{5.37}
$$

To make a bit of a paralelism with the content mentioned in the beginning of the essay, the energy tensor perturbation has 10 degrees of freedom, of which we were able to divide into a 4+4+2 formulation. Likewise the perturbation can also be divided into perfect fluid + non-perfect, with $5+5$ degrees of freedom! As an important note, the perfect fluid degrees of freedom in δT^{μ}_{ν} are those which keep T^{μ}_{ν} in the perfect fluid form

$$T^{\mu}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + p\delta^{\mu}_{\nu} \tag{5.38}$$

Thus they can be taken as the density perturbation, pressure perturbation, and velocity perturbation

$$\rho = \bar{\rho} + \delta\rho, \quad p = \bar{p} + \delta p, \quad \text{and} \quad u^i = \bar{u}^i + \delta u^i = \delta u^i \equiv \frac{1}{a} v_i \tag{5.39}$$

For convenience we shall define the velocity perturbation as

$$v_i \equiv a u^i \tag{5.40}$$

which in 1st order is equal to the coordinate velocity so

$$\frac{dx^i}{d\eta} = \frac{u^i}{u^0} = \frac{u^i}{\bar{u}^0} = a u^i = v_i \tag{5.41}$$

We also define the relative energy density perturbation

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}} \tag{5.42}$$

So we now want to relate the vector field of the perturbations in terms of the velocity perturbation (or in other words we want to express u^{μ} and u_{ν} in terms of v_i) thus

$$u^{\mu} = \bar{u}^{\mu} + \delta u^{\mu} \equiv \left(a^{-1} + \delta u^0, a^{-1}v_1, a^{-1}v_2, a^{-1}v_3\right) \tag{5.43}$$

$$u_{\nu} = \bar{u}_{\nu} + \delta u_{\nu} \equiv (-a + \delta u_0, \delta u_1, \delta u_2, \delta u_3) \tag{5.44}$$

Noting that $u_{\nu} = g_{\mu\nu}u^{\nu}$ and $u_{\mu}u^{\mu} = -1$. Now just to have a more general expression we will go back to the most general form of pertrubed metric, similar to what we've seen in equation (2.5) at the beginning of the essay but considering perturbations of only 1st order giving such that

$$g_{\mu\nu} = a^2 \begin{bmatrix} -1-2A & -B_i \\ -B_i & (1-2D)\delta_{ij} + 2E_{ij} \end{bmatrix} \tag{5.45}$$

Perturbations in the metric will make the momentum distribution of noninteracting particles anisotropic so we need to take this anisotropic pressure into account. We can now go back to combining equations

$$u_0 = g_{0\mu}u^{\mu} = a^2(-1-2A)\left(a^{-1} + \delta u^0\right) - \delta^{ij}a^2 B_i a^{-1} v_j \tag{5.46}$$

$$= -a - a^2\delta u^0 - 2aA \tag{5.47}$$

from which follows

$$\delta u_0 = -a^2\delta u^0 - 2aA \tag{5.48}$$

$$\delta u_i = u_i = g_{i\mu}u^{\mu} = -aB_i + a v_i \tag{5.49}$$

substituting now in equation (5.38) we obtain that

$$T^{\mu}_{\nu} = \bar{T}^{\mu}_{\nu} + \delta T^{\mu}_{\nu} = \begin{bmatrix} -\bar{\rho} & 0 \\ 0 & \bar{p}\delta^i_j \end{bmatrix} + \begin{bmatrix} -\delta\rho & (\bar{\rho}+\bar{p})(v_i - B_i) \\ -(\bar{\rho}+\bar{p})v_i & \delta p \delta^i_j \end{bmatrix} \tag{5.50}$$

And this is the first part for the perturbed energy tensor we just have to replace by the definition of δT^i_j given by

$$\delta T^i_j = \delta p \delta^i_j + \Sigma_{ij} \equiv \bar{p}\left(\frac{\delta p}{\bar{p}} + \Pi_{ij}\right) \tag{5.51}$$

where we defined

$$\begin{aligned} \Pi_{ij} &\equiv \Sigma_{ij}/\bar{p} \\ \Sigma_{ij} &\equiv \delta T^i_j - \tfrac{1}{3}\delta^i_j \delta T^k_k \end{aligned} \tag{5.52}$$

but now going back to the Newtonian gauge as we are considering scalar perturbations only, so that $v_i = -v_{,i}$ and $B_i = -B_{,i}$ and as we explained in a previous chapter we will now proceed to do the SVT separation from equation (3.2) and we start by defining the quantity

$$\Sigma_{ij} \equiv \delta T^i_j - \frac{1}{3}\delta^i_j \delta T^k_k \tag{5.53}$$

and preform the separation of Π_{ij} into SVT as :

$$\Pi_{ij} = \Pi^S_{ij} + \Pi^V_{ij} + \Pi^T_{ij} \tag{5.54}$$

where $S \rightarrow$ Scalar, $V \rightarrow$ Vector and $T \rightarrow$ Tensor and since that perfect fluid perturbations $(\Pi_{ij} = 0)$ do not have a tensor perturbation component we have

$$\Pi^S_{ij} = \left(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\Pi \tag{5.55}$$

$$\Pi^V_{ij} = -\frac{1}{2}\left(\Pi_{i,j} + \Pi_{j,i}\right) \tag{5.56}$$

Now, with the information that we just got we can proceed onto finding the metric from the Newtonian gauge we come towards the Conformal-Newtonian Gauge in which we just have to revist what was taught in equations (4.6) we just have to apply them to our Stress-Enery Tensor! And we obtain

$$\widetilde{\delta T}^0_0 = -\tilde{\delta}\rho = \delta T^0_0 - \bar{T}^0_{0,0}\xi^0 = -\delta\rho + \bar{\rho}'\xi^0 \tag{5.57}$$

$$\widetilde{\delta T}^i_0 = -(\bar{\rho} + \bar{p})\tilde{v}_i = \delta T^i_0 + \xi^i_{,0}\left(\bar{T}^0_0 - \frac{1}{3}\bar{T}^k_k\right) \tag{5.58}$$

$$\frac{1}{3}\widetilde{\delta T}^k_k = \widetilde{\delta p} = \frac{1}{3}\left(\delta T^k_k - \bar{T}^k_{k,0}\xi^0\right) = \delta p - \bar{p}'\xi^0 \tag{5.59}$$

$$\widetilde{\delta T}^i_j - \frac{1}{3}\delta^i_j\widetilde{\delta T}^k_k = \bar{p}\tilde{\Pi}_{ij} = \delta T^i_j - \frac{1}{3}\delta^i_j\delta T^k_k = \bar{p}\Pi_{ij} \tag{5.60}$$

For scalar perturbations, $v_i = -v_{,i}$ and $\xi^i = -\xi_{,i}$, so that we have

$$\begin{aligned} \tilde{v} &= v + \xi' \\ \tilde{\Pi} &= \Pi \end{aligned} \tag{5.61}$$

We get to the conformal-Newtonian gauge by $\xi^0 = -B + E'$ and $\xi = -E$. Thus

$$\delta\rho^N = \delta\rho + \bar{\rho}'\left(B - E'\right) = \delta\rho - 3\mathcal{H}(1+w)\bar{\rho}\left(B - E'\right) \tag{5.62}$$

$$\delta p^N = \delta p + \bar{p}'\left(B - E'\right) = \delta p - 3\mathcal{H}(1+w)c_s^2\bar{\rho}\left(B - E'\right) \tag{5.63}$$

$$v^N = v - E' \tag{5.64}$$

$$\Pi^N = \Pi \tag{5.65}$$

Considering scalar perturbations only, so that $v_i = -v_{,i}$ and $B_i = -B_{,i}$ and use the conformal-Newtonian gauge we then finaly obtain the energy tensor perturbation in the form

$$\delta T^{\mu}_{\nu} = \begin{bmatrix} -\delta\rho^N & -(\bar{\rho}+\bar{p})v^N_{,i} \\ (\bar{\rho}+\bar{p})v^N_{,i} & \delta p^N \delta^i_j + \bar{p}\left(\Pi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\Pi\right) \end{bmatrix} \tag{5.66}$$

Which is the last remaining stone to obtain the Einstein field equations! So by combining the equation Einstein Perturbed equations shown in equation (5.33) and the results obtained on the following set of equations with δT^{μ}_{ν} that we just obtained in equation (5.66) we get the following relations

$$3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) - \nabla^2\Psi = -4\pi G a^2 \delta\rho^N \tag{5.67}$$
$$\left(\Psi' + \mathcal{H}\Phi\right)_{,i} = 4\pi G a^2 (\bar{\rho}+\bar{p}) v^N_{,i} \tag{5.68}$$
$$\Psi'' + \mathcal{H}\left(\Phi' + 2\Psi'\right) + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \frac{1}{3}\nabla^2(\Phi - \Psi) = 4\pi G a^2 \delta p^N \tag{5.69}$$
$$\left(\partial_i\partial_j - \frac{1}{3}\delta^i_j\nabla^2\right)(\Psi - \Phi) = 8\pi G a^2 \bar{p}\left(\partial_i\partial_j - \frac{1}{3}\delta^i_j\nabla^2\right)\Pi \tag{5.70}$$

Now the equation (8.3) can be simplified to

$$\Psi' + \mathcal{H}\Phi = 4\pi G a^2 (\bar{\rho}+\bar{p}) v^N \tag{5.71}$$

and combining with the equation (8.2) we obtain

$$\nabla^2\Psi = 4\pi G a^2 \bar{\rho}\left[\delta^N + 3\mathcal{H}(1+w)v^N\right] \tag{5.72}$$

where we have defined

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}} \quad \text{and} \quad w \equiv \frac{\bar{p}}{\bar{\rho}} \tag{5.73}$$

from the equation (8.5) one can obtain that

$$\Psi - \Phi = 8\pi G a^2 \bar{p}\Pi \tag{5.74}$$

and so the equations (5.72) and (5.74) we form our constraint equations and our evolution equations

$$\begin{aligned} \nabla^2\Psi &= \frac{3}{2}\mathcal{H}^2\left[\delta^N + 3\mathcal{H}(1+w)v^N\right] \\ \Psi - \Phi &= 3\mathcal{H}^2 w\Pi \\ \Psi' + \mathcal{H}\Phi &= \frac{3}{2}\mathcal{H}^2(1+w)v^N \\ \Psi'' + \mathcal{H}\left(\Phi' + 2\Psi'\right) + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \frac{1}{3}\nabla^2(\Phi - \Psi) &= \frac{3}{2}\mathcal{H}^2\delta p^N/\bar{\rho} \end{aligned} \tag{5.75}$$

where we used from the Friedman equation the following equality's to simplify the result

$$\begin{aligned} \mathcal{H}^2 &= \frac{8\pi G}{3}\bar{\rho}a^2 \\ \mathcal{H}' &= -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p})a^2 \end{aligned} \tag{5.76}$$

And the set of equations (5.75) form the Einstein field equations for scalar perturbations!

6. Curvature Perturbation

We start now the journey towards understanding in what capacity Relativistic Perturbation Theory allows us to explain how the universe forms astronomic structures as we preferences this essay. And our next stepping stone is to take the way we derived RPT on the previous section with a comoving gauge to obtain the curvature perturbation. And we start, as we did last time, by defining the ξ^0 and ξ' that gives us a comoving gauge

$$\begin{aligned} \xi' &= -v \\ \xi^0 &= v - B \end{aligned} \tag{6.1}$$

but note that this does not fully specify the coordinate system in the perturbed space-time, since only ξ' is specified, not ξ. Thus we remain free to do time-independent transformations

$$\tilde{x^i} = x^i - \xi(\vec{x})_{,i} \tag{6.2}$$

while staying in the comoving gauge. With these two equations we can now formulate our gauge transformation laws similar to what we've done before (*The derivation can be found on [9] starting on chapter 16.2*)

$$A^C = A - (v - B)' - \mathcal{H}(v - B) \tag{6.3}$$

$$B^C = B - v + (v - B) = 0 \tag{6.4}$$

$$D^C = -\frac{1}{3}\nabla^2\xi + \mathcal{H}(v - B) \tag{6.5}$$

$$E^C = E + \xi \tag{6.6}$$

$$\psi^C \equiv -\mathcal{R} = \psi + \mathcal{H}(v - B) \tag{6.7}$$

and in this gauge we have defined the Curvature Perturbation in equation (6.7) we can formulate our field equations similar to what we've done before but now we will skip forward the steps needed to arrive at them since the calculations are similar as we've explained earlier and we obtain

$$\nabla^2\Psi = \frac{3}{2}\mathcal{H}^2\delta^C \tag{6.8}$$

$$\Psi - \Phi = 3\mathcal{H}^2 w\Pi \tag{6.9}$$

$$\Psi' + \mathcal{H}\Phi = \frac{3}{2}\mathcal{H}^2(1+w)v^N \tag{6.10}$$

$$\Psi'' + \left(2 + 3c_s^2\right)\mathcal{H}\Psi' + \mathcal{H}\Phi' + 3\left(c_s^2 - w\right)\mathcal{H}^2\Phi + \frac{1}{3}\nabla^2(\Phi - \Psi) = \frac{3}{2}\mathcal{H}^2\frac{\delta p^C}{\bar{\rho}} \tag{6.11}$$

So now we are ready to rewrite (6.7) using (6.5) in relation to relate with the Bardeen Potentials

$$\mathcal{R} = -\Psi - \frac{2}{3(1+w)}\left(\mathcal{H}^{-1}\Psi' + \Phi\right) \tag{6.12}$$

and derivating we obtain a relation similar to

$$-4\pi G a^2(\bar{\rho} + \bar{P})\mathcal{R}' = 4\pi G a^2\mathcal{H}\delta P_{\text{nad}} + \mathcal{H}\frac{\bar{P}'}{\bar{\rho}'}\nabla^2\Psi \tag{6.13}$$

where we have defined the non-adiabatic pressure perturbation

$$\delta P_{\text{nad}} \equiv \delta P - \frac{\bar{P}'}{\bar{\rho}'}\delta\rho \tag{6.14}$$

the right-hand side of equation (6.13) scales as

$$\mathcal{H}k^2\Phi \sim \mathcal{H}k^2\mathcal{R} \implies \frac{d\ln\mathcal{R}}{d\ln a} \sim \left(\frac{k}{\mathcal{H}}\right)^2 \tag{6.15}$$

We can see that on super-Hubble scales, $k \ll \mathcal{H}$, $\mathcal{R}$ doesn't evolve unless non-adiabatic pressure is significant! This is crucial for relating late-time observable, such as the distributions of galaxies, to the initial conditions from inflation!

7. Structure Formation

So finaly, at the end of this long essay we talk about the evolution of structure and we solve the perturbation equations for this particular epoch, and find that the solutions are characterized by quantities that remain constant for the whole epoch. The evolution of the large scale structure and the CMB can be specified during the radiation-dominated + epoch, sufficiently early that all scales k of interest are outside the horizon so we limit this time period to start after BBN. The comoving horizon at the BBN epoch is

$$\mathcal{H}^{-1} \approx \left(\frac{T}{100\text{keV}}\right)^{-1} \cdot 1\text{kpc} \tag{7.1}$$

so all cosmological scales are still well outside horizon then. In this essay we derived the field equations and we could theoretical solve but the has complex interactions between the different species which would complicate things. This interactions are summarised in figure (3).

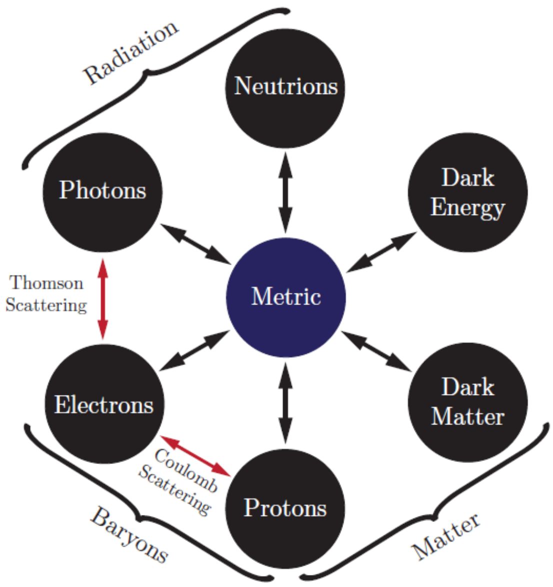

Figure 3. Interactions between the different forms of matter in the universe which can be found on [1]

For adiabatic perturbations, we have $c_s^2 \approx w$ and as we've seen in last chapter from equation (6.11) we get

$$\Psi'' + 3(1+w)\mathcal{H}\Psi' + wk^2\Psi = 0 \tag{7.2}$$

which, in the Super-horizon Limit, k $\ll \mathcal{H}$, becomes

$$\Psi'' + 3(1+w)\mathcal{H}\Psi' = 0 \implies \Psi = \text{const} \tag{7.3}$$

which is independent of f the equation of state w. In the begging on the essay we mentioned that Ψ represented the gravitational potential so we can say that in particular, the gravitational potential is frozen outside the horizon during both the radiation and matter eras. It is easy to see that the Perturbed Poisson equation would be

$$\nabla^2 \delta\Psi = 4\pi G a^2 \bar{\rho}\delta \tag{7.4}$$

and so one can conclude that on superhorizon scales, the density perturbations are simply proportional to the curvature perturbation set up by in ation. The curvature perturbation $\mathcal{R}$ the gravitational doesn't stay constant when the equation of state changes therefore in the superhorizon limit,

$$\mathcal{R} = -\frac{5+3w}{3+3w}\Psi \tag{7.5}$$

which is fantastic since this fact provides an important connection between the source term for the evolution of fluctuations (Ψ) and the primordial initial conditions set up by inflation ($\mathcal{R}$)! So in the transition from a radiation dominated (RD) universe to matter dominated (MD) universe we have that the curvature perturbation must be constant, yet $w = \frac{1}{3}$ and $w = 0$ must have the same ($\mathcal{R}$) so

$$\mathcal{R} = -\frac{3}{2}\Psi_{\mathrm{RD}} = -\frac{5}{3}\Psi_{\mathrm{MD}} \Rightarrow \Psi_{\mathrm{MD}} = \frac{9}{10}\Psi_{\mathrm{RD}} \tag{7.6}$$

which means the gravitational potential decreases by a factor of 9/10 in the phase transition!

8. Summary

In this essay we learned a bit more about the formulation of cosmological perturbation theory and we derived the evolution equations for scalar perturbations in Newtonian gauge with the metric

$$ds^2 = a(\eta)^2\left[-(1+2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j\right] \tag{8.1}$$

we derived the Einstein Field Equations

$$3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) - \nabla^2\Psi = -4\pi G a^2 \delta\rho^N \tag{8.2}$$

$$\left(\Psi' + \mathcal{H}\Phi\right)_{,i} = 4\pi G a^2(\bar{\rho}+\bar{p})v^N_{,i} \tag{8.3}$$

$$\Psi'' + \mathcal{H}\left(\Phi' + 2\Psi'\right) + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \frac{1}{3}\nabla^2(\Phi - \Psi) = 4\pi G a^2 \delta p^N \tag{8.4}$$

$$\left(\partial_i\partial_j - \frac{1}{3}\delta^i_j\nabla^2\right)(\Psi - \Phi) = 8\pi G a^2 \bar{p}\left(\partial_i\partial_j - \frac{1}{3}\delta^i_j\nabla^2\right)\Pi \tag{8.5}$$

Defined the Curvature Perturbation

$$\mathcal{R} = -\Psi - \frac{2}{3(1+w)}\left(\mathcal{H}^{-1}\Psi' + \Phi\right) \tag{8.6}$$

And finished with a bit of physical explanation for the evolution of structure.

During writing of this essay the two main references were [9] and [1] even though the majority of the equations presented were obtained by me in the beginning of this work. With the lack of time necessary to do all the calculations i started following more the guidelines shown in both these references to have more consistency and better explanations throughout the essay.

Abbreviations

The following abbreviations are used in this manuscript:

GR	General Relativity
RPT	Relativistic Perturbation Theory
FLRW	Friedmann–Lemaître–Robertson–Walker
SVT	Scalar-Vector-Tensor
CMB	cosmic microwave background
BBN	Big Bang nucleosynthesis

References

1. Daniel Baumann In *Cosmology - Part III Mathematical Tripos* ; pp. 82–100
2. J. Bardeen In *Gauge-invariant cosmological perturbations*
3. E. Lifshitz In *Republication of: On the gravitational stability of the expanding universe*
4. V.F. Mukhanov, H.A. Feldman, R.H. Brandenberger In *Theory of cosmological perturbations* ; pp. 210–220
5. Edmund Bertschinger In *Cosmological Perturbation Theory and Structure Formation*
6. A. Riotto In *Inflation and the Theory of Cosmological Perturbations*
7. Prof. Dr. Lucio Mayer and Prof. Dr. Jaiyul Yoo In *Theoretical Astrophysics and Cosmology (ETH)* https://www.ics.uzh.ch/ jyoo/class/2019/lectures/ch3.pdf
8. Francesca Lepori In *Relativistic Cosmology from the linear to the non-linear regime*
9. Hannu Kurki-Suonio In *Cosmological Perturbation Theory, part 1*