WIMPs: Weakly interactive Massive Particles

Let's assume decoupling of the Form (annihilation)

$$X + \bar{X} \rightleftharpoons l + \bar{l}$$

l and $\bar{l}$ are massless an tightly coupled particles to the Fluid. Let's assume the following conditions

- $n_l = n_l^{eq}$
- $n_{\bar{l}} = n_{\bar{l}}^{eq}$
- there is no initial assymetry of X and $\bar{X}$:
  $n_X = n_{\bar{X}}$, $n_X^{eq} = n_{\bar{X}}^{eq}$

The Boltzmann equation reads:

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\langle\sigma v\rangle\left[n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} n_3 n_4\right] \rightarrow$$

$$\rightarrow \frac{1}{a^3}\frac{d}{dt}(n_X a^3) = -\langle\sigma v\rangle\left[n_X n_{\bar{X}} - \left(\frac{n_X n_{\bar{X}}}{\cancel{n_l n_{\bar{l}}}}\right)_{eq} \cancel{n_l}\, \cancel{n_{\bar{l}}}\right]$$

n_l^{eq} $n_{\bar{l}}^{eq}$

$$= -\langle\sigma v\rangle\left[n_X n_{\bar{X}} - n_X^{eq} n_{\bar{X}}^{eq}\right] =$$

$$= -\langle\sigma v\rangle\left[n_X^2 - n_X^{eq\,2}\right]$$

We can convert this equation in a Number of particles equation using $N_X = n_X/s$, $N_X^{eq} = n_X^{eq}/s$

$$\frac{1}{a^3}\frac{d}{dt}\left[N_X \underbrace{s a^3}_{S'}\right] = -\langle\sigma v\rangle\left[N_X^2 s^2 - N_X^{eq^2} s^2\right] \Leftrightarrow$$

$$\frac{\cancel{s a^3}}{\cancel{a^3}}\frac{dN_X}{dt} = -\langle\sigma v\rangle s^{\cancel{2}}\left[N_X^2 - N_X^{eq^2}\right] \Leftrightarrow$$

$$\frac{dN_X}{dt} = -\langle\sigma v\rangle s\left[N_X^2 - \left(N_X^{eq}\right)^2\right]$$

Lets make a change of varible $\underbrace{x = \frac{M_X}{T}}$. So the left-hand-side gives:

$$\underbrace{\frac{dN_X}{dt}} = \frac{dN_X}{dx}\frac{dx}{dt} = \frac{dN_X}{dx}\underbrace{\frac{d}{dt}\left(\frac{M_X}{T}\right)} =$$

$$= \frac{dN_X}{dx}\frac{d}{dt}\left(M_X T^{-1}\right) = \frac{dN_X}{dx}\left(M_X (-1) T^{-2}\frac{dT}{dt}\right)$$

(1) $$= -\frac{dN}{dx}\frac{1}{T}\overset{x}{\left(\frac{M_X}{T}\right)}\left(\frac{dT}{dt}\right) = -x\frac{dN}{dx}\frac{1}{T}\frac{dT}{dt}$$

But using entropy conservation we know that

$$T = A\, g_{*s}^{-1/3} a^{-1}$$

lets start with the case whor $g_{*s} \simeq$ constant (away from mom thresholds)

$$\frac{dT}{dt} = A\, g_{*s}^{-1/3} \frac{d a^{-1}}{dt} = A\, g_{*s}^{-1/3} (-1)\, a^{-2} \dot{a}$$

$$= -A g_{*s}^{-1/3} \frac{\dot{a}}{a} a^{-1} = -A g_{*s}^{-1/3} H a^{-1}$$

So going back to (1):

$$\frac{dN_x}{dt} = -x \frac{dN_x}{dx} \frac{1}{T} \left(-A g_{*s}^{-1/3}\right) H a^{-1} =$$

$$= -x \frac{dN_x}{dx} \frac{1}{\left(\cancel{A g_{*s}^{-1/3}}\, \cancel{a^{-1}}\right)} \left(-\cancel{A g_{*s}^{-1/3}}\, \cancel{a^{-1}}\right) H =$$

$$= x H \frac{dN_x}{dx}$$

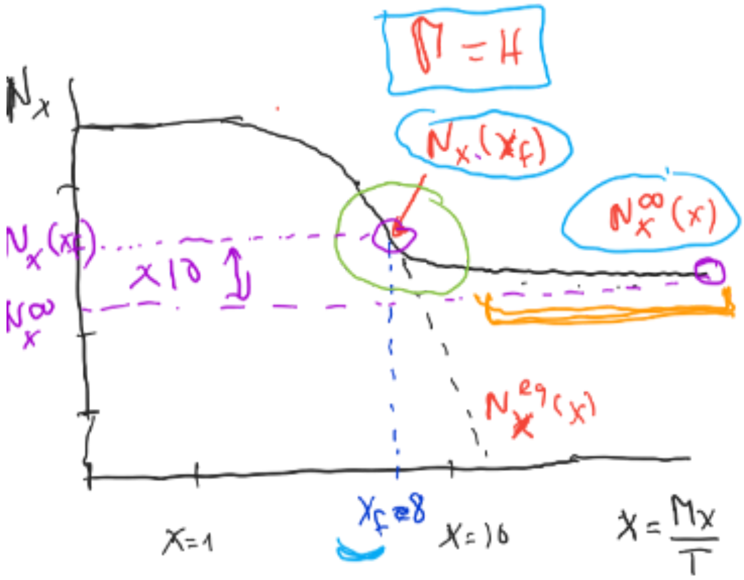
This allows us to go back to Boltzmann equation:

$$\frac{dN_x}{dt} = x H \frac{dN_x}{dx} = -\langle \sigma v \rangle s \left[N_x^2 - \left(N_x^{eq}\right)^2\right]$$

$$\boxed{\frac{dN_x}{dx} \simeq -\frac{\langle \sigma v \rangle s}{x\, H} \left[N_x^2 - N_x^{eq\,2}\right]}$$

Ricatti equation

Ricatti Equation



$$\boxed{\frac{dN}{dx} = -\frac{\langle \sigma v \rangle s}{x H}\left[N_X^2 - \left(N_X^{eq}\right)^2\right]}$$

$$\Downarrow$$

$$\frac{dN_X}{dx} = -\frac{\lambda}{x^2}\left[N_X^2 - \underbrace{\left(\cancel{N_X^{eq}}\right)^2}_{\to 0}\right]$$

$N_X^{eq} \ll N_X$

$x \gg x_f$

Lets assume the $X, \bar{X}$ decouple during the epoch of radiation ~~do~~domination, so:

$$H = \frac{\pi}{3}\left(\frac{g_*}{10}\right)^{1/2}\frac{T^2}{M_{Pl}} = \quad \left\{ \text{but } x = \frac{M_X}{T} \Rightarrow T = \frac{M_X}{x} \right.$$

$$= \frac{\pi}{3}\left(\frac{g_*}{10}\right)^{1/2}\left(\frac{M_X}{x}\right)^2\frac{1}{M_{Pl}} =$$

$$= H(M_X)\,\frac{1}{x^2} \quad \text{where } H(M_X) = \frac{\pi}{3}\left(\frac{g_*}{10}\right)^{1/2}\frac{M_X^2}{M_{Pl}}$$

Now a similar expression can be obtain for the specific entropy:

$$s = \frac{2\pi^2}{45} g_{*s} T^3 = \frac{2\pi^2}{45} g_{*s}\left(\frac{M_X}{x}\right)^3$$

So:

$$\frac{dN_x}{dx} = -\frac{\langle\sigma v\rangle}{x}\,\frac{\frac{2\pi}{45}\,g_{*s}\left(\frac{M_x}{x}\right)}{H(M_x)\,\frac{1}{x^2}}\left[N_x^2-\left(N_x^{eq}\right)^2\right]$$

$$= -\overbrace{\frac{2\pi^2}{45}\,\frac{g_{*s}}{H(M_x)}\,\langle\sigma v\rangle\, M_x^3}^{\lambda}\,\frac{1}{x^2}\left[N_x^2-\left(N_x^{eq}\right)^2\right] =$$

$$= -\frac{\lambda}{x^2}\left[N_x^2-\left(N_x^{eq}\right)^2\right]$$

where $\lambda = \frac{2\pi^2}{45}\,\frac{g_{*s}}{H(M_x)}\,\langle\sigma v\rangle M_x^3 =$

$$= \frac{2\pi^{\cancel{2}}}{45}\,\frac{g_{*s}\,M_x^{\cancel{3}}\,\langle\sigma v\rangle}{\cancel{\pi}/3\left(\frac{g_*}{10}\right)^{1/2} M_x^{\cancel{2}}/M_{Pl}} =$$

$$= \frac{2\pi}{15}\left(\frac{10\, g_{*s}^2}{g_*}\right)^{1/2}\langle\sigma v\rangle\, M_x\, M_{Pl}$$

Away from mass thresholds λ is approximatly constant at high $x = M_x/T$.

At late times, ie $T \ll M \Rightarrow x \gg x_F$, so the Riccatti equation simplifies as:

$$\frac{dN_x}{dx} = -\frac{\lambda}{x^2}\left[N_x^2 - \overset{(N_x^{eq})^2}{0}\right] = -\frac{\lambda}{x^2}N_x^2$$

So integrating with respect to x:

$$dN_x = \frac{dN_x}{dx}\, dx =$$

$$= -\frac{\lambda}{x^2} N_x^2\, dx \Leftrightarrow$$

$$\Leftrightarrow \int_{N_x(x_f)}^{N_x(\infty)} \frac{dN_x}{N_x^2} = \int_{x_f}^{\infty} -\lambda \frac{dx}{x^2} \Rightarrow$$

assuming λ is constant

$$\Leftrightarrow \left[\frac{N_x^{-1}}{\cancel{-1}}\right]_{N_x(x_f)}^{\infty} = -\lambda \left[\frac{x^{-1}}{\cancel{-1}}\right]_{x_f}^{\infty} \Leftrightarrow$$

$$\frac{1}{N_x^{\infty}} - \frac{1}{N_x(x_f)} = -\lambda \left[\cancel{\frac{1}{\infty}}^{\,0} - \frac{1}{x_f}\right] \Leftrightarrow$$

$$\boxed{\frac{1}{N_x^{\infty}} - \cancel{\frac{1}{N_x(x_f)}}^{\,\sim 0} = +\frac{\lambda}{x_f}}$$

is typically 10 times smaller than $N_x(x_f)$

So one can neglect the term

$$\frac{1}{N_x(x_f)} \ll \frac{1}{N_x^{\infty}}$$

So:

$$\frac{1}{N_x^{\infty}} \simeq \frac{\lambda}{x_f} \Leftrightarrow \boxed{N_x^{\infty} \simeq \frac{x_f}{\lambda}}$$

This result gives a good way to estima-

-te the freeze-out (relic) abundances of $X, \bar{X}$