Cosmologia Física

Ismael Tereno (FCUL, IA)

Ciências ULisboa

instituto de astrofísica e ciências do espaço

2026

The Inhomogeneous Universe

Statistical properties of the density contrast field II

Projected two-point functions

The 2-pt functions that we saw until now are defined in the cosmological volume, i.e., in a 3D density contrast field.

We can also define angular two-point functions, which are function of two-dimensional (angular) separations and are obtained by ***projecting the 3D 2-pt functions on the sky*.**

A projected 2-pt function is more directly measured in the sky than the original 3-dimensional one → we can always measure an angular separation, but not a radial separation (which needs redshift information) → in general what we really observe is a map of the projected density.

Angular correlation function

An angular correlation function is a 2D correlation function, i.e., obtained by ***projecting the 3D correlation function on the sky*****.**

A projected quantity may be written in general as a weighted (filtered) integral over the third dimension:

$$F(\theta) = \int d\chi \, g(\chi) \, F(f_K(\chi)\theta, \chi)$$

where,

- the 3D coordinates are $\vec{x}$ = $(f_K(\chi)\, \theta_x\, ,\, f_K(\chi)\theta_y,\, \chi)$, with

  χ is the radial coordinate (comoving)

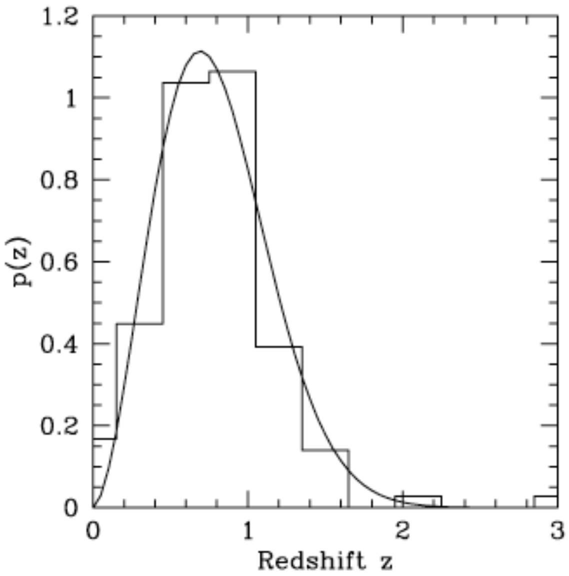
  θ_x is the angular separation (in the x direction) to a reference axis (the line-of-sight)

  $f_K(\chi)\, \theta_x$ is the comoving physical separation corresponding to that angular separation, i.e, the angular separation times the comoving angular diameter distance.

- g(χ) is the weight function used in the projection: for example the **redshift distribution** of the density tracers (galaxies). In this case coordinates χ (redshift z) with more galaxies contribute more to the integral.

$$g(\chi)\, d\chi = p(z)\, dz$$

(A filter (or window or weight function) is needed to account for the various contributions to a given position θ on the sky).



Let us then use this general form to write the projected correlation function:

$$w(\vartheta) = \int d\chi\, g_1(\chi) \int d\chi'\, g_2(\chi') \xi(|\vec{x} - \vec{x}'|)$$

which is function of separation $\vartheta = |\vec{\theta} - \vec{\theta}'|$.

In the projection, each angular separation has contributions from pairs with elements at any radial distance.

We may aproximate it by considering that

- since the 3D correlation function is a decreasing function of separation, **only physically close pairs contribute** (i.e., close in the 3D space and not only in the projected sky) → we consider only pairs with $\chi \sim \chi'$
- the window function has a slow variation in redshift: $g(\chi) \sim g(\chi')$

This is called the Limber approximation.

In the Limber approximation, the two window functions are function of χ and can be written inside the first integral.

Notice that the product of the two window functions is g^2 only in the case that they are not correlated. In general, they are correlated by the correlation function itself → the joint probability $P(g_1, g_2)$ is a conditional probability → there is source clustering and so we should write:

$$w(\vartheta) = \int d\chi\, g^2(\chi)\, [1 + \xi(|\vec{x} - \vec{x}'|)] \int d\chi'\, \xi(|\vec{x} - \vec{x}'|)$$

But this is a second-order effect (order ξ^2). **To first order, the angular correlation function is linear in the 3D correlation function:**

$$w(\vartheta) = \int \int d\chi\, d\chi'\, g^2(\chi)\, \xi(|\vec{x} - \vec{x}'|)$$

The 3d correlation function $\xi = \langle\delta(x)\delta(x')\rangle$ is the Fourier transform of the power spectrum, and so we can write

$$w(\vartheta) = \int \int d\chi\, d\chi'\, g^2(\chi) \int \frac{d^3k}{(2\pi)^3}\, P(\vec{k}, z)\, e^{-i\vec{k}.(\vec{x}-\vec{x}')}$$

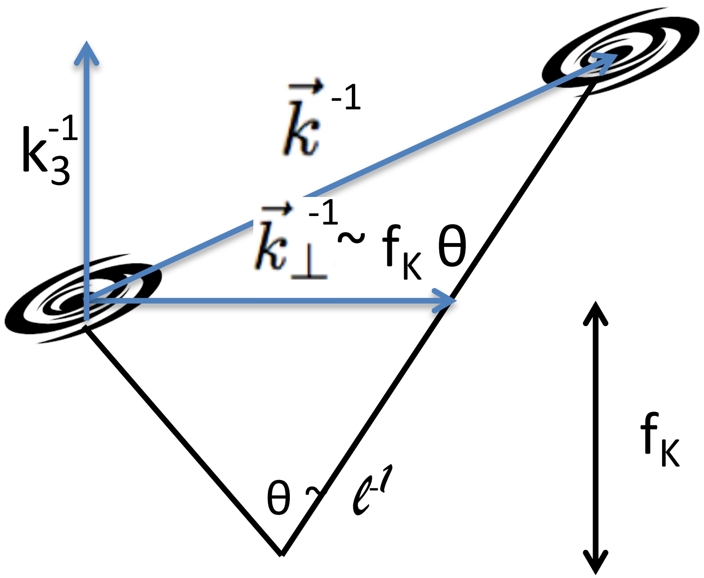
Note the power spectrum evolves in time, and so it also depends explicitly on the redshift z (which is related to χ).

The 3D vector scale can be decomposed in a 2D transversal and a 1D longitudinal component,

$$\vec{k} = (\vec{k}_\perp, k_3)$$

and we can write,

$$w(\vartheta) = \int d\chi\, g^2(\chi) \int d\chi'\, e^{-ik_3(\chi-\chi')} \int \frac{d^3k}{(2\pi)^3} P(\vec{k}, z)\, e^{-i\vec{k}_\perp.(\vec{\theta}-\vec{\theta}')f_K(\chi)}$$



In this expression, there remains no dependence on χ' → the integral over dχ' (or over d(χ-χ') which is the same) is a Dirac delta function $2\pi\, \delta_D(k_3)$ → $k_3 = 0$, i.e:

$$w(\vartheta) = \int d\chi\, g^2(\chi) \int \frac{d^2k_\perp}{(2\pi)^2} P(k_\perp, z)\, e^{-i\vec{k}_\perp . \vec{\vartheta} f_K(\chi)}$$

This is the result, also called the Limber equation - the relation between the angular 2-pt correlation function and the power spectrum.

It shows that only scales in the plane contribute to the angular 2-pt function.

Note on notation - the standard notation is:

ξ(r) - 2-pt correlation function
w(ϑ) - 2-pt angular correlation function
P(k) - power spectrum
C(l) - angular power spectrum

Angular power spectrum: flat sky

The angular power spectrum is the transform of the angular correlation function in the harmonic space.

For **flat-sky (valid for small fields)**, plane-waves $e^{i\vec{\ell}.\vec{\theta}}$
are an orthonormal basis of functions that can be used to make the Fourier transform.

This introduces the 2D angular scale 'l', the reciprocal of the real-space angular separation θ.

The relation between the Fourier angular scale and the real-space angular separation is:

$$\theta = 2\pi/l$$

→ the scale l=100 corresponds to a separation of 3.6 deg
→ the scale l=1000 corresponds to a separation of 21.6 arcmin

Now, the **Fourier transform of the 2-pt angular correlation function** is:

$$C(\ell) = \int d^2\vartheta \, e^{i\vec{\ell}.\vec{\vartheta}} \, w(\vartheta)$$

Inserting in the Limber equation, we find the relation between the angular power spectrum and the power spectrum:

$$C(\ell) = \int d\chi \, g^2(\chi) \int \frac{d^2 k_\perp}{(2\pi)^2} P(k_\perp, z) \int d^2(\vartheta) \, e^{-i\vec{k}_\perp.\vec{\vartheta} f_K(\chi)} \, e^{i\vec{\ell}.\vec{\vartheta}}$$

The last integral is a Dirac delta: $(2\pi)^2 \, \delta_D(\vec{\ell} - k_\perp f_K(\chi))$

This means that 'l' only depends on the transversal components of k, and not on the full 3D k vector,

and allows us to make the dk integration setting

k_transverse = $l \,/\, f_K(\chi)$.

The result is:

$$C(\ell) = \int d\chi\, g^2(\chi)\, P\left(\frac{\ell}{f_K(\chi)}, z\right)$$

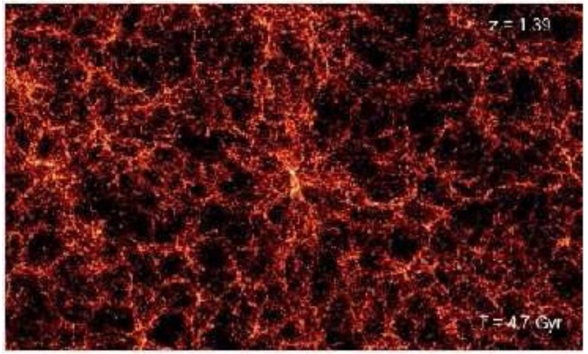
This shows that the amplitude of C for a given angular scale l, is a weighted sum of the amplitudes of P at scales $l/f_K(\chi)$

i.e., at different redshifts, the scales k that contribute to the same angular scale l are different.

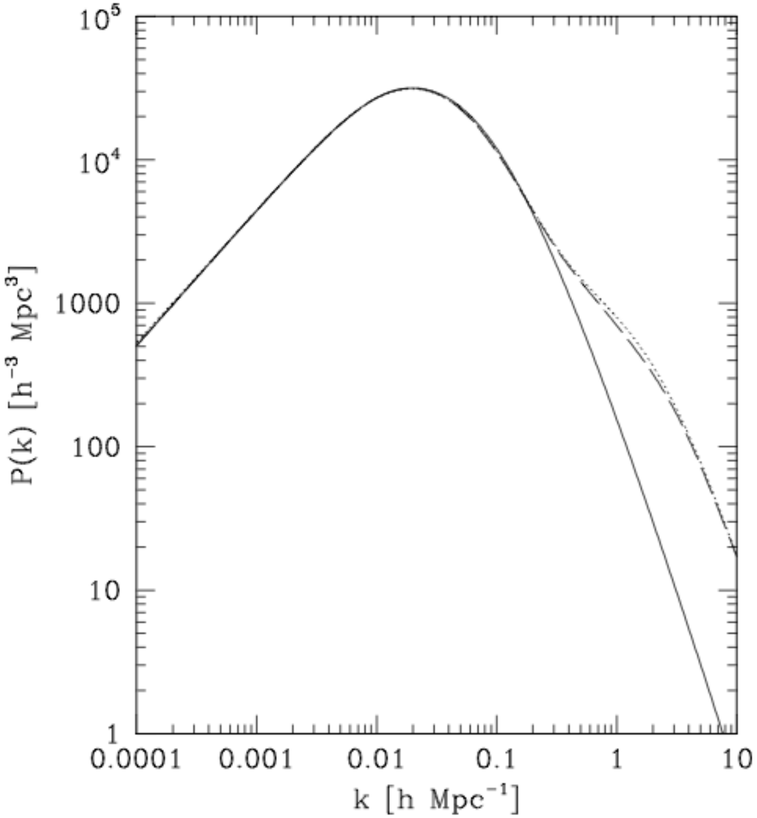
Due to statistical isotropy, the correlation functions only depend on the separation modulus → C(l) is only function of the modulus of ‘l’, as P(k) was function of the modulus of ‘k’.

Decomposing a map in plane waves: the dark matter density contrast

(Note that obviously we still need to study structure formation to find the power spectra, we are just looking at the relations between the various power spectra and correlation functions)



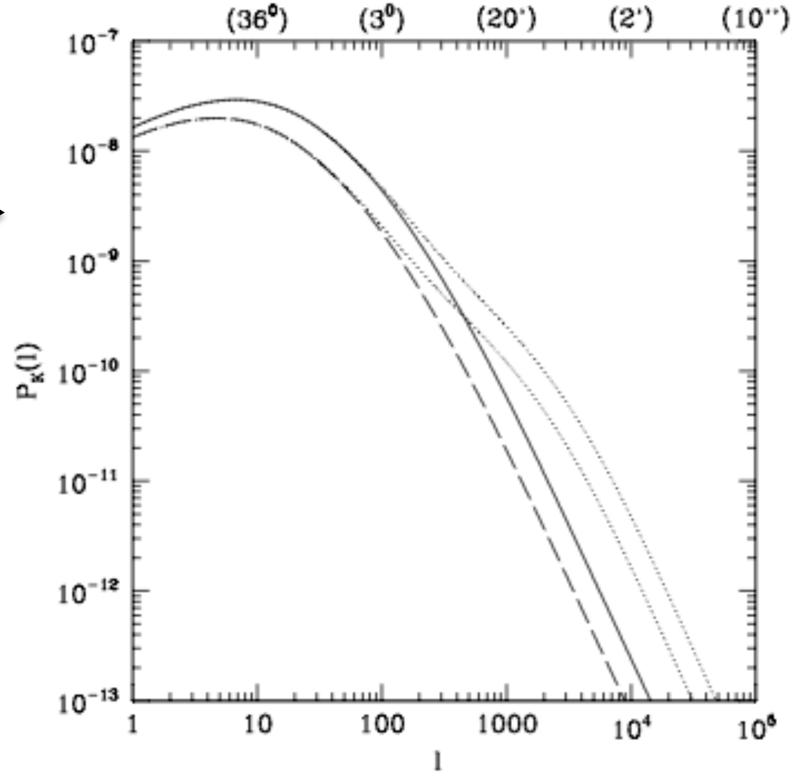
3D matter power spectrum $P_{\delta\delta}(k)$



k

$k = l / f_K$

2D angular matter power spectrum $C_{\delta\delta}(l)$



θ

l

Angular power spectrum: spherical sky

In the spherical full-sky, the flat-sky approximation is not valid for large scales → plane waves are no longer an orthonormal basis.

A better basis are the spherical harmonics Y_{lm}

> Since we are in 2D there are 2 indexes to these functions, just like for Fourier modes $l=(l_x,l_y)$. For spherical harmonics the indexes are called (l,m) and are associated with spherical coordinates θ and φ.

The spherical harmonics form an orthonormal set of functions on the spherical surface:

$$\int d\hat{\mathbf{n}} Y_\ell^{m*}(\hat{\mathbf{n}}) Y_{\ell'}^{m'}(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\sum_{\ell m} Y_\ell^{m*}(\hat{\mathbf{n}}) Y_\ell^{m}(\hat{\mathbf{n}}') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

$$Y_\ell^{m*} = (-1)^m Y_\ell^{-m}$$

The spherical harmonics are defined from the associated Legendre polynomials P_{lm}

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}$$

which in turn are defined from the ordinary Legendre polynomials P_l

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

which are the solutions of Legendre's differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0$$

and can be written as,

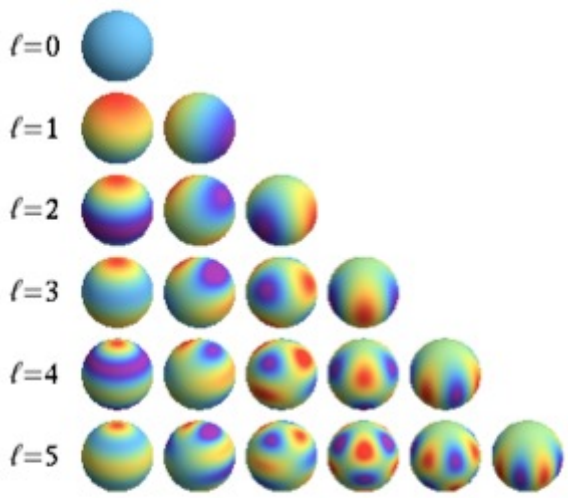
$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

Contrary to cartesian coordinates (where the range of l_x and l_y are independent), in spherical coordinates the range of l and m are not independent: **for each 'l', 'm' runs from -l to l.** → there are 2l+1 values of 'm' for each 'l' → summing over 'm', for a fixed 'l' gives the closure relation:

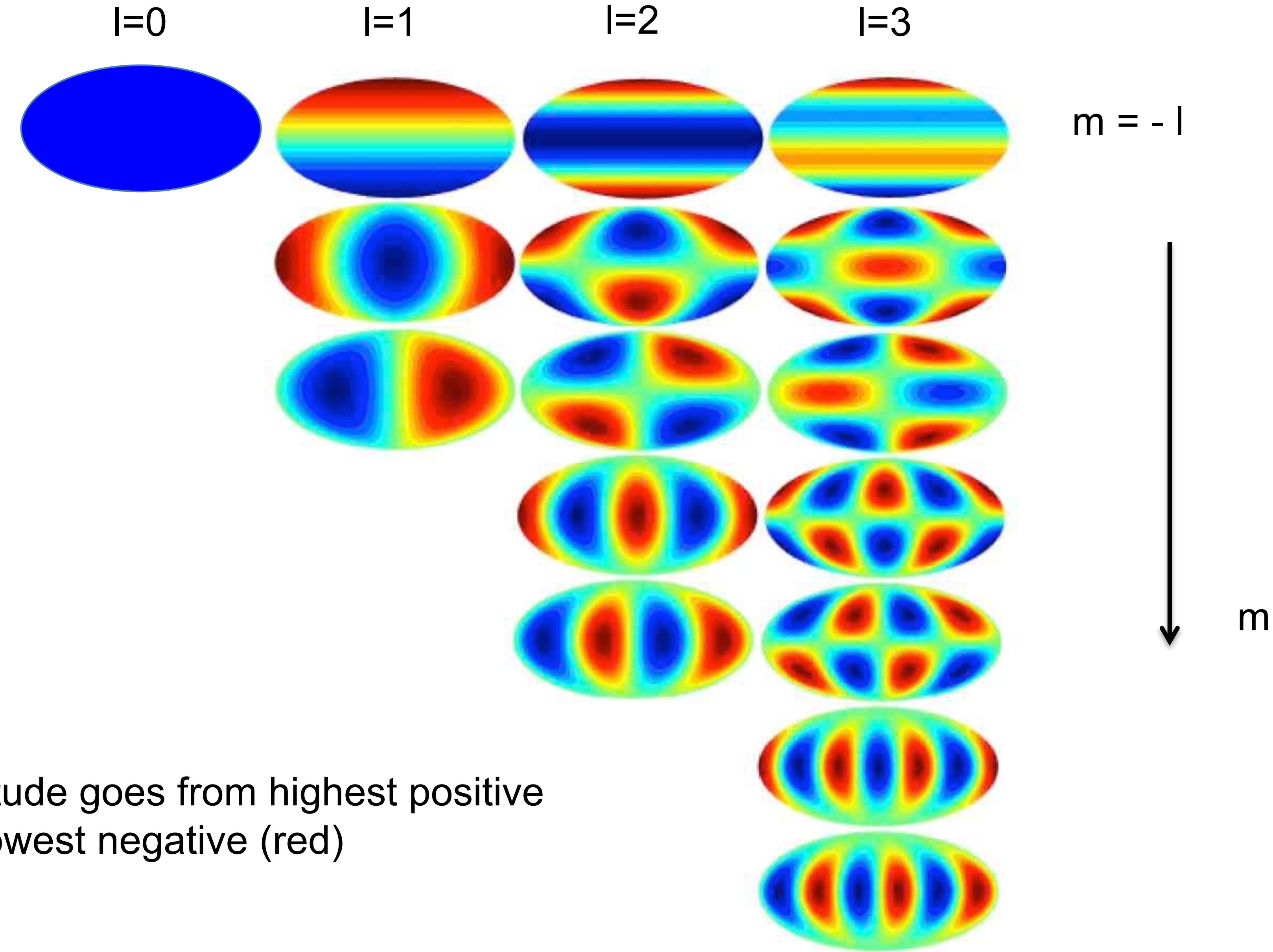
$$\sum_m |Y_{\ell m}(\theta,\phi)|^2 = \frac{2\ell+1}{4\pi}$$

The first spherical harmonics are:

$P_0^0(x) =$	1	$Y_{00} =$	$\sqrt{\frac{1}{4\pi}}$
$P_1^1(x) =$	$-(1-x^2)^{1/2}$	$Y_{11} =$	$-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$
$P_1^0(x) =$	x	$Y_{10} =$	$\sqrt{\frac{3}{4\pi}}\cos\theta$
$P_2^2(x) =$	$3(1-x^2)$	$Y_{22} =$	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\phi}$
$P_2^1(x) =$	$-3(1-x^2)^{1/2}x$	$Y_{21} =$	$-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}$
$P_2^0(x) =$	$\frac{1}{2}(3x^2-1)$	$Y_{20} =$	$\sqrt{\frac{5}{4\pi}}(\frac{3}{2}\cos^2\theta-\frac{1}{2})$



The first spherical harmonics look like this:



The amplitude goes from highest positive (blue) to lowest negative (red)

We see that the (2l+1) 'm' configurations of spherical harmonics for a given 'l' have a similar pattern → they divide the surface of a sphere in (2l) regions of equal area.

l = 0 is constant → monopole
l = 1 is a gradient between 2 poles (the maximum and a minimum) → dipole
(the different basis configurations show the gradient along latitude or along longitude)
l = 2 → quadrupole
l = 3 → octopole

Notice that *the relation between the spherical harmonics angular scale and the real-space angular separation is not unique.*

As an approximation, we may consider that the 2l regions of equal area that divide the surface of the sphere are placed along the meridians. In that case, the width of each region at the equator is

$$\theta = 2\pi/(2l)$$

and so a good indicator is $\theta \sim \pi/l$ (different from the flat sky case)

→ scale l=2 corresponds to a separation of 90 deg (the quadrupole)
→ scale l=100 corresponds to a separation of 1.8 deg
→ scale l=220 corresponds to a separation of 49 arcmin (CMB first peak)
→ scale l=2500 corresponds to a separation of 4.2 arcmin (Planck last data point)

Now, the **spherical harmonic transform of the delta field** is:

$$\delta(\theta,\varphi) = \sum a_{\ell m} Y_{\ell m}(\theta, \phi)$$

The **multipole coefficients** a_{lm} are the equivalent to δ_k in Fourier space (to be precise, this notation a_{lm} is usually reserved for the transform of the CMB temperature contrast δT)

The **correlation function of the transform of the delta field** is $<a_{lm}\, a_{l'm'}>$.
As we saw for the 3D case, the derivation can be made by inserting the inverse transform, which makes appear the correlation function in real space, and various spatial integrals that will result in Dirac deltas and the power spectrum.

The result is:

$$< a_{\ell m} a_{\ell' m'} >= \delta_D(\ell - \ell')\, \delta_D(m - m')\, C_{\ell m}$$

where, once again, the Dirac deltas show the independence of the power spectrum scales.

The correlation function is isotropic → it depends only on the angular separation (l ←→ θ), and not on the direction (m ←→φ). We can thus integrate over m, and get:

$$\sum_m < a_{\ell m} a_{\ell m} > \; = \frac{2\ell+1}{4\pi} \; C_\ell$$

This defines the **isotropic angular power spectrum**, as an average over all directions

$$C_\ell = \frac{4\pi}{2\ell+1} \sum_m < a_{\ell m} a_{\ell m} >$$

This has an impact on observations → the power spectrum on large scales (low multipoles l) corresponds to an average over a small number of independent functions → the large scales are measured with much less precision than small scales →there is a fundamental limit of statistical uncertainty on large scales (called the **cosmic variance**).

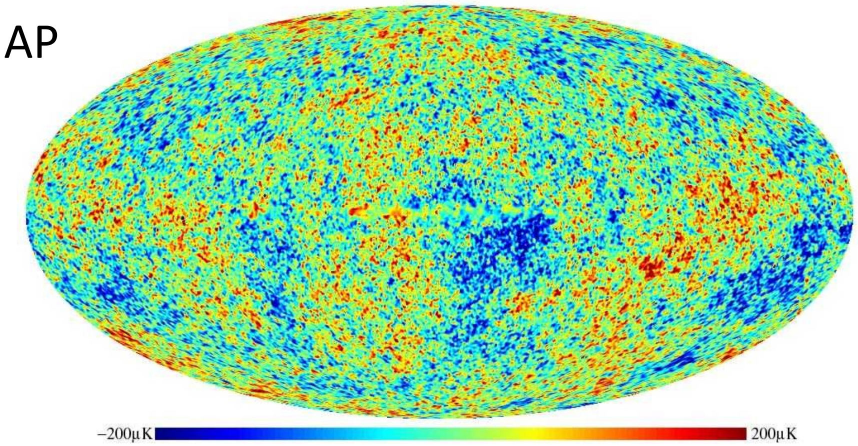
Finally, we can also write the **correlation function in real space**, as function of the isotropic angular power spectrum,

$$<\delta_i \, \delta_j> \; = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell P_\ell(\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j)$$

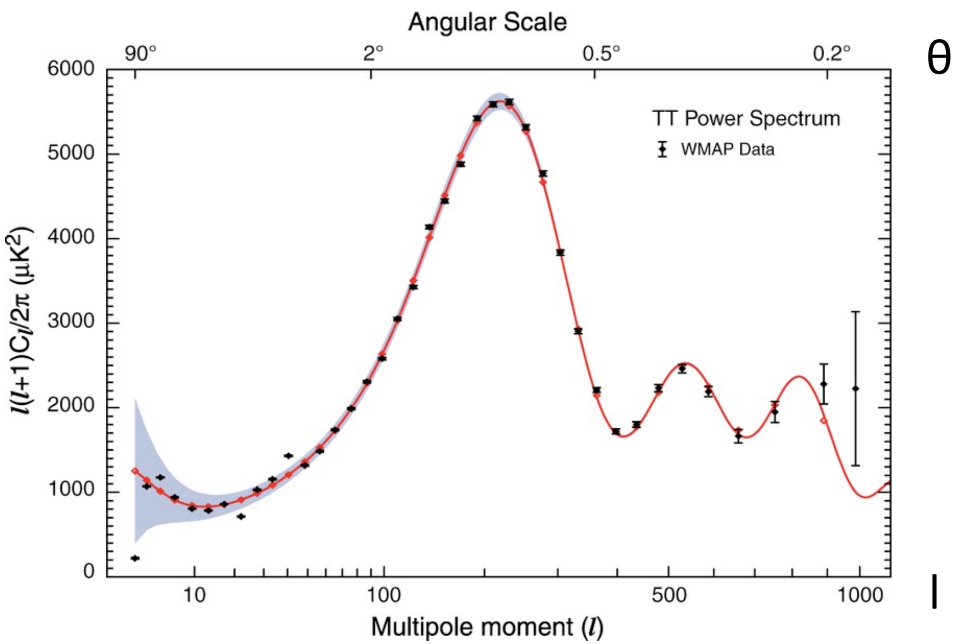
Decomposing a map in spherical harmonics: the CMB temperature contrast

The observed map is one realization
(i.e., one specific m for each l)
of the theoretical C_l computed from the cosmological model, which is
$<a_{lm}\, a_{lm}>$ (any m, all are equivalent)

WMAP



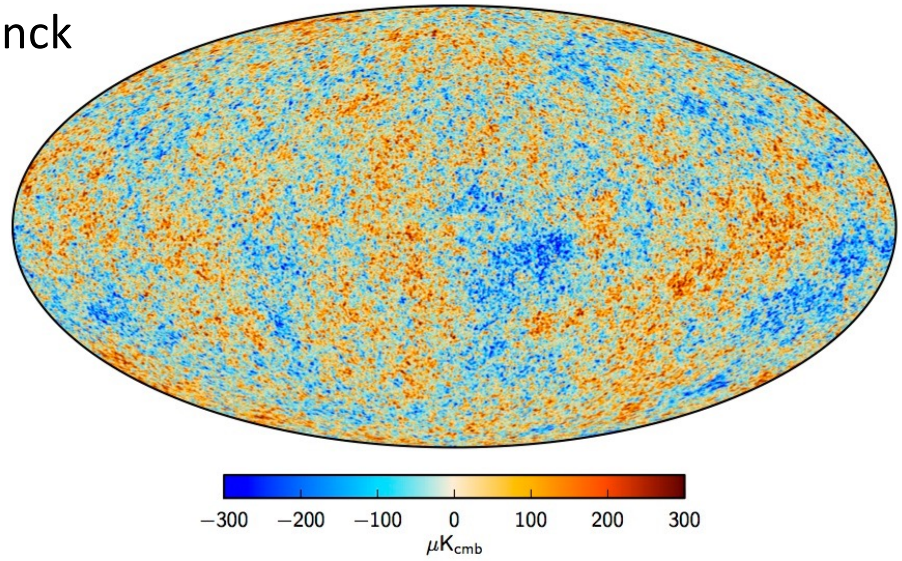
2D angular temperature dimensionless power spectrum $l(l+1)\, C_{TT}(l)$



Cosmological perturbations are functions defined as perturbations around a mean value → its own mean value is zero → the monopole is zero for density contrast fields.

So in cosmology, the monopole is not used and the 'l' range is 1,2,…∞

Planck

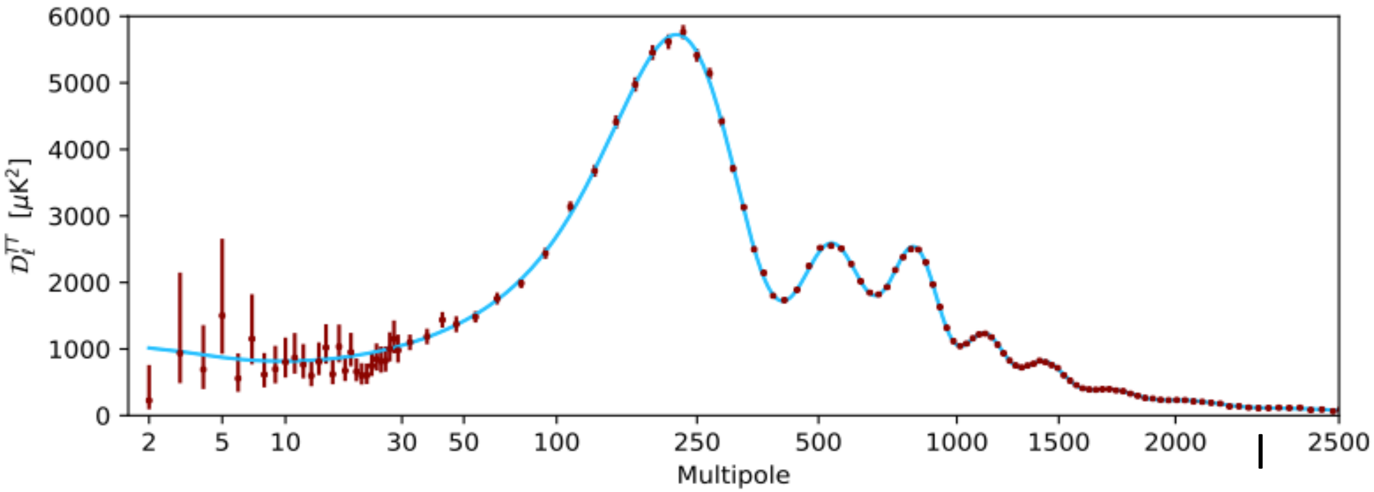


The Planck map was obtained to higher order of the spherical harmonics than the WMAP one.

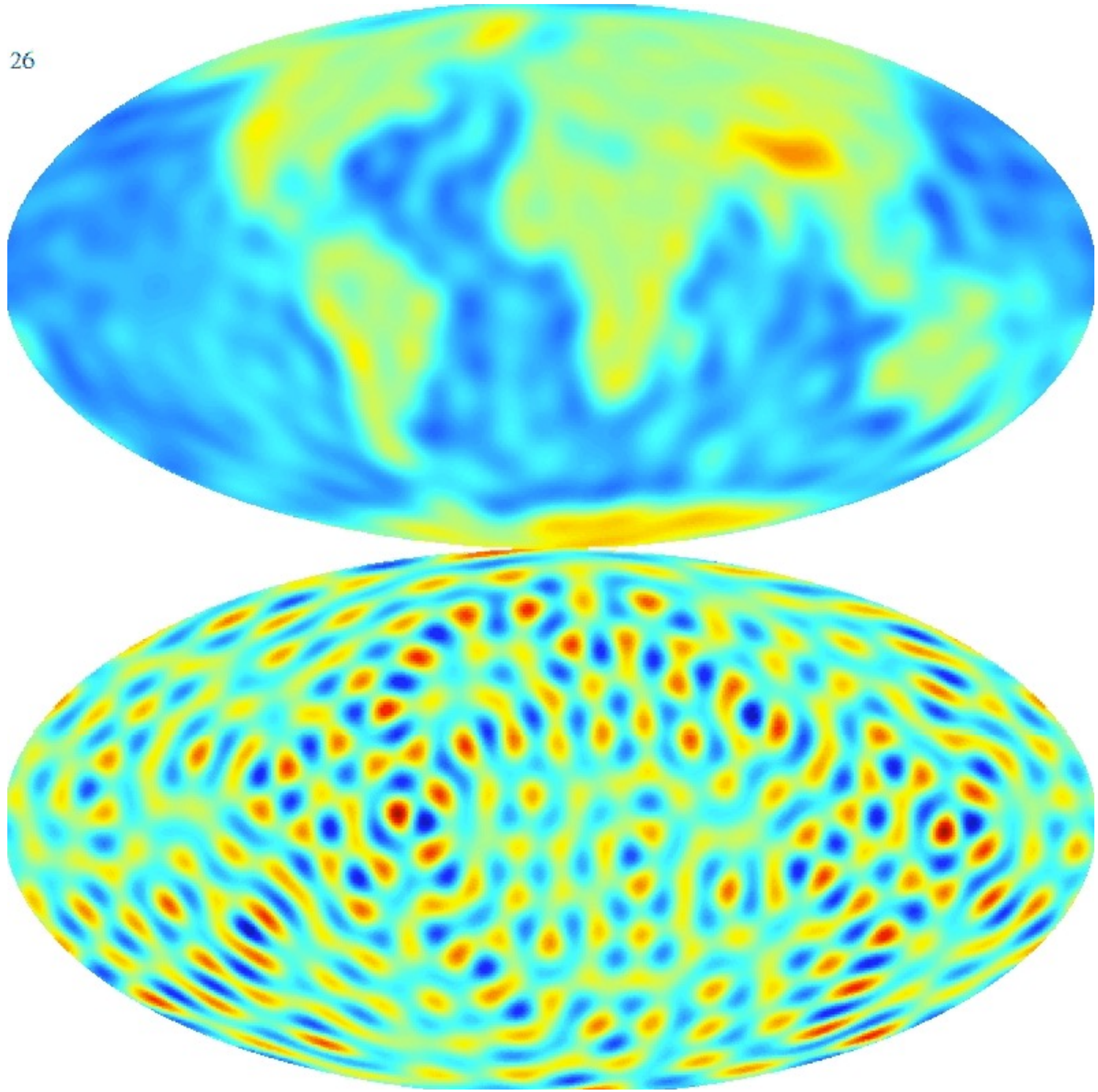
This is noticeable in the map (better resolution and better defined small-scale features)

and in the power spectrum (function measured to higher 'l')

2D angular temperature dimensionless power spectrum l(l+1) C_{TT}(l)



Decomposing a map in spherical harmonics: the Earth



The land distribution is not an isotropic field → the "theoretical" power spectrum is a specific realization, it is not averaged over all the m functions.

Higher-order statistics

Is the density contrast really a Gaussian random field?

- **Primordial non-Gaussianities**

Perhaps not: certain models of inflation can produce non-Gaussian features from the original Gaussian quantum fluctuations

- **Secondary non-Gaussianities**

Definitely not: late-time evolution and other late-time effects produce mode coupling and the cosmological random fields are no longer Gaussian today

The dark matter density field becomes non-Gaussian in the recent universe due to non-linear evolution → δ may only be Gaussian in the linear regime, i.e., while its value is small.

Higher-order moments (eg: order 3 and 4) are in reality non-zero and contain additional cosmological information.

The **histogram of the pixel values in a map** shows the distribution.

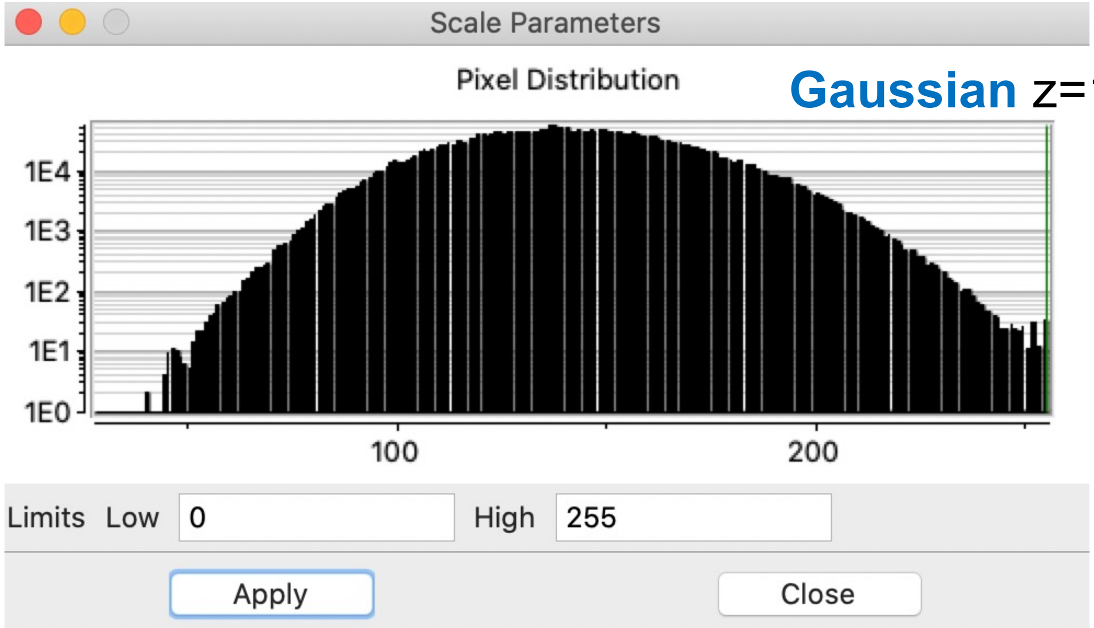
It is a **one-point distribution**

Structure formation:

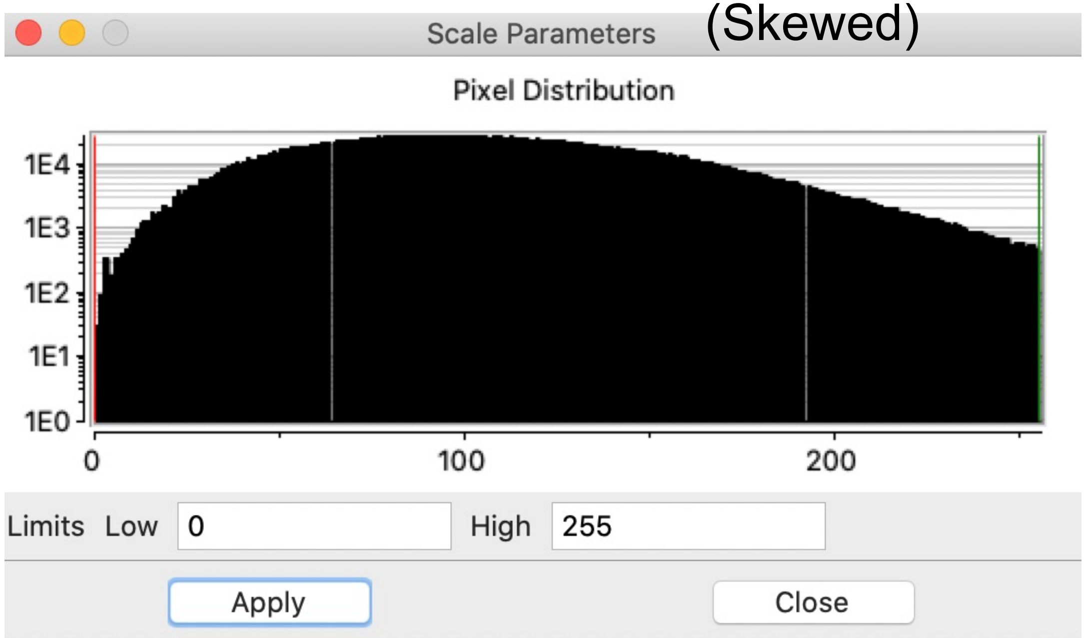
Matter clusters in certain positions, and so **most of the volume has lower density** contrast than in the high-z Gaussian case → peak moves to the left.

But in the positions **where matter clusters, points have high density contrast** → distribution gets a tail

Gaussian z=18

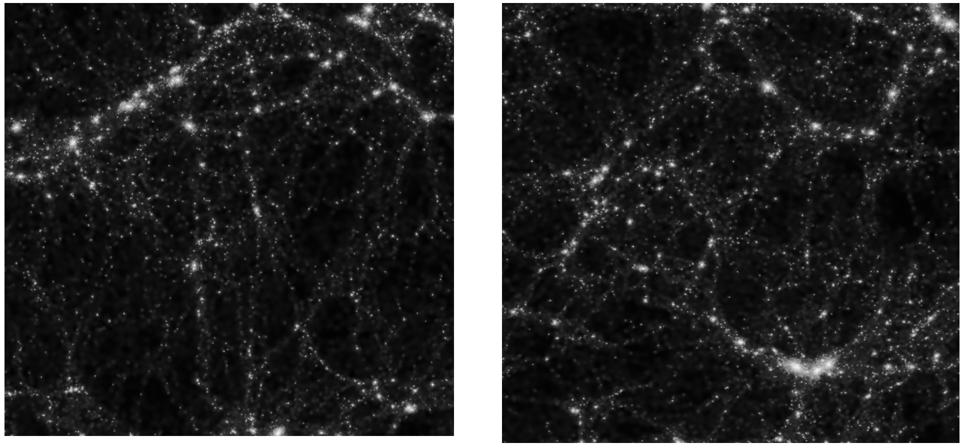


Non-Gaussian z=0
(Skewed)

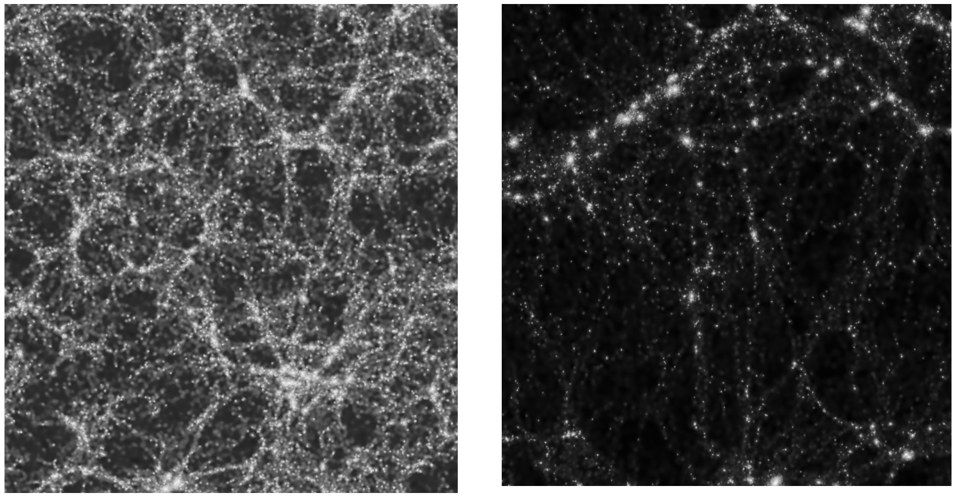


Examples of density contrast fields

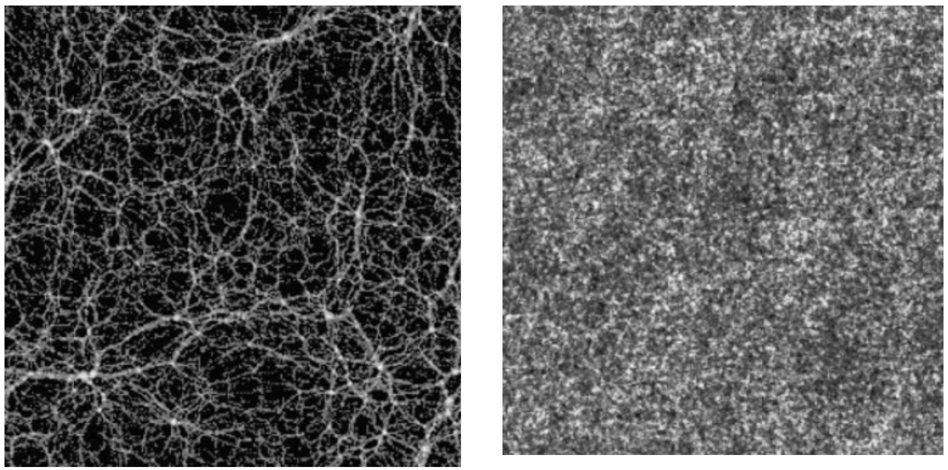
Comparing left and right panels



Same cosmological model (identical statistical moments, P(k), etc)
Same distribution (Gaussian)
Different realizations
→
The maps are statistically equivalent, although not identical



Different cosmological models (different statistical moments, P(k), etc)
Same distribution (Gaussian)
Different realizations
→
Fundamentally non-equivalent



Same cosmological model (identical statistical moments, P(k), etc)
Gaussian distribution (left) and non-Gaussian with identical Gaussian part (right)
Different realizations
→
Non-equivalent from NG effects

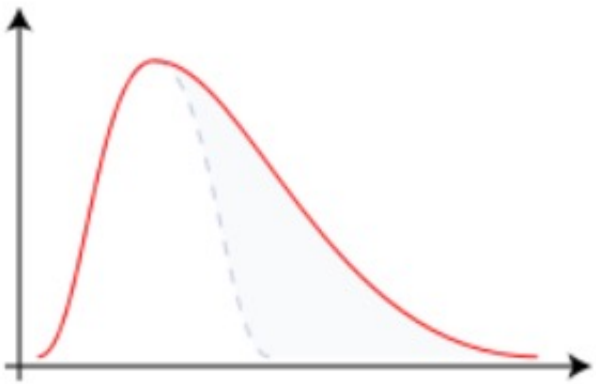
If the distribution is not Gaussian, the covariance matrix (and consequently the 2-pt correlation function and power spectra) do not contain the whole cosmological information

→ we need to consider **higher-order moments.**

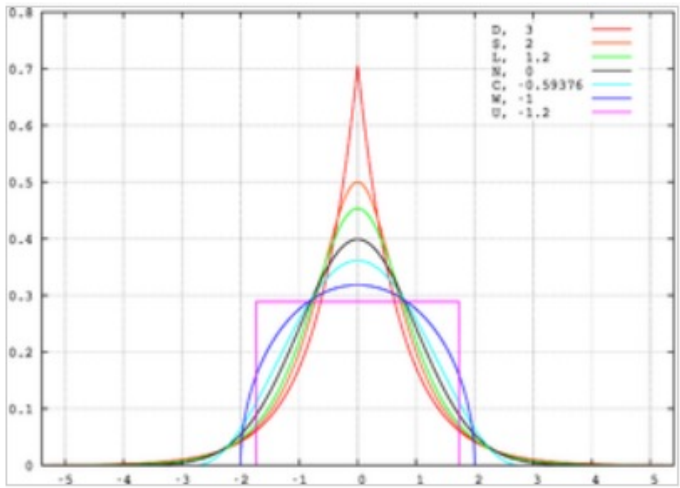
For example:

If the δ distribution is not symmetric → there is a non-zero skewness

If the δ distribution is cuspy → there is a non-zero kurtosis



$$S = \frac{\langle \delta^3 \rangle}{\langle \delta^2 \rangle^{3/2}}$$

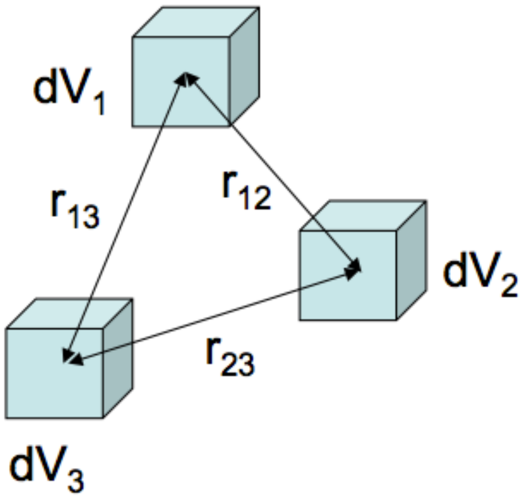


$$K = \frac{\langle \delta^4 \rangle}{\langle \delta^2 \rangle^2} - 3$$

Higher-order moments are computed from n-point correlation functions:

$<\delta_1 \delta_2 \delta_2>$ $<\delta_1 \delta_2 \delta_3 \delta_4>$

The joint probability (and so the clustering properties) of having galaxies in locations 1,2,3 depends on the full conditional probability between the triplet, and also on all combinations of conditional probabilities between pairs:



$$dP_{123} = n^3 \left(1+\xi(r_{12})+ \xi(r_{13})+ \xi(r_{23})+\boxed{\zeta(r_{12}, r_{13}, r_{23})}\right) dV_1\, dV_2\, dV_3$$

An **n-point correlation function** can be written as a sum of terms involving lower-order correlations,
plus an irreducible (also called connected) term

→ this is the **Isserlis theorem** of probability theory (1918).

$$E[\,X_1 X_2 \cdots X_n\,] = \sum_{p\in P_n^2} \prod_{\{i,j\}\in p} E[\,X_i X_j\,] = \sum_{p\in P_n^2} \prod_{\{i,j\}\in p} \mathrm{Cov}(\,X_i, X_j\,)$$

This also implies that

for **variables of zero mean** → the **reducible part** of an **odd n-point correlation function** is zero

$$E[\,X_1 X_2 \cdots X_{2n-1}\,] = 0$$

Wick's theorem (1950) - Note there is a version of Isserlis' theorem used in particle physics that allows to reduce the operators in creation/annihilation processes into sums of products of pairs, which is the basis of the description of the process in terms of Feynman diagrams.

3-pt function

Using "Wick's" theorem, the 3-pt correlation function ζ_{123} - zeta - may be decomposed as

$$< \delta_1 \, \delta_2 \, \delta_3 > = < \delta_1 > < \delta_2 \, \delta_3 > + < \delta_2 > < \delta_1 \, \delta_3 > + < \delta_3 > < \delta_1 \, \delta_2 > + < \delta_1 \, \delta_2 \, \delta_3 >_c$$

This shows that for variables with zero mean → **the 3-pt function is just the connected term**.

In the case of a Gaussian distribution the connected term is zero, and **the 3-pt function is zero** → note that this does not imply that the joint probability becomes just the product of the 3 individual probabilities (with zero correlation) since the conditional probability also depends on the 2-pt correlations.

We can also define the harmonic transformation of the 3-pt function, which is called the bispectrum:

$$\langle \tilde{\delta}(\vec{k})\tilde{\delta}(\vec{q})\tilde{\delta}(\vec{p})\rangle = (2\pi)^3 B(k,q,p)\delta_D(\vec{k}+\vec{q}+\vec{p})$$

4-pt function

In this case the joint probability is

$$dP_{1234} = n^4(1+\xi_{12}+\xi_{13}+\xi_{14}+\xi_{23}+\xi_{24}+\xi_{34}+\zeta_{123}+\zeta_{124}+\zeta_{134}+\zeta_{234}+\mu_{1234})\,dV_1 dV_2 dV_3 dV_4$$

Using "Wick's" theorem, the 4-pt correlation function, μ_{1234}, may be written as

$$<\delta_1\,\delta_2\,\delta_3\,\delta_4> = <\delta_1\,\delta_2><\delta_3\,\delta_4> + <\delta_1\,\delta_3><\delta_2\,\delta_4> + <\delta_1\,\delta_4><\delta_2\,\delta_3> + <\delta_1\,\delta_2\,\delta_3\,\delta_4>_c$$

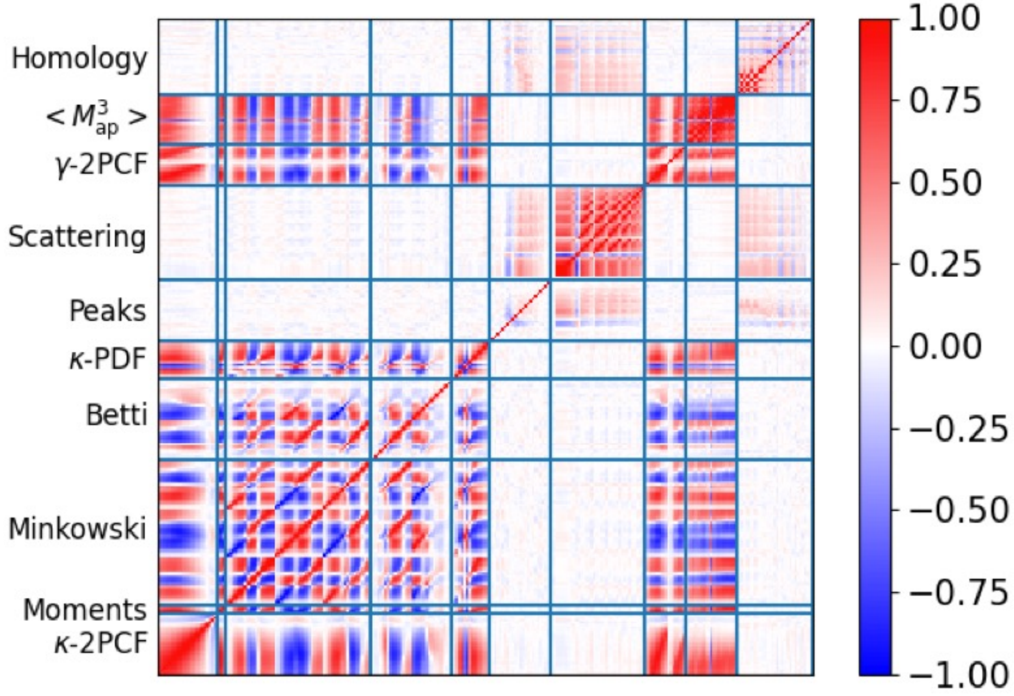
note the **number of terms in the sum** is $\frac{n!}{2^{n/2}\,(n/2)!}$ (in this case, n=4 → n_terms=3)

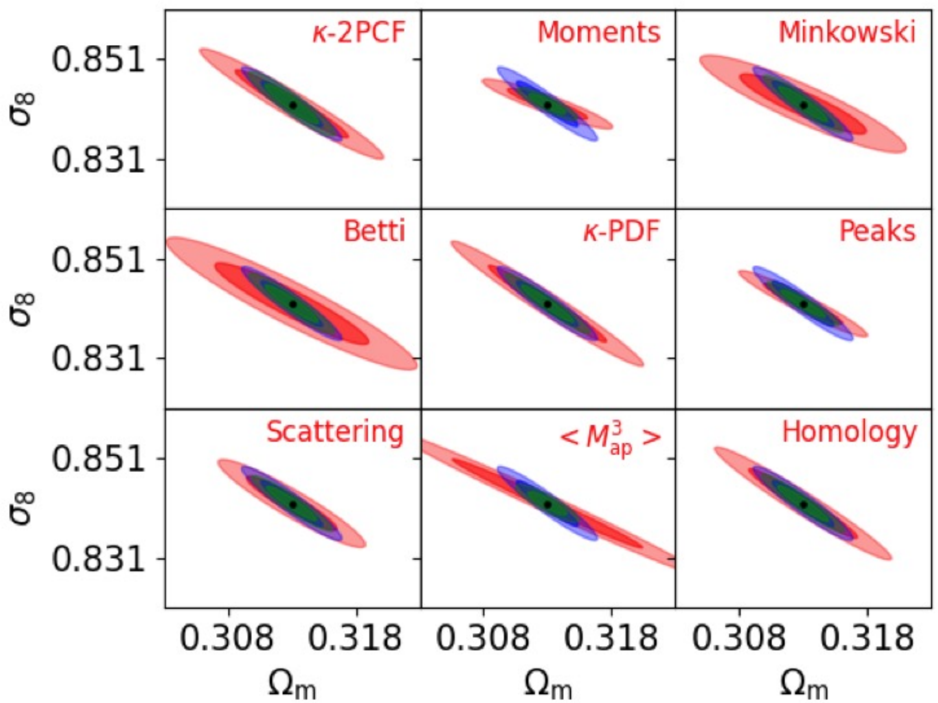
In the case of a Gaussian distribution the connected term is zero**, but the 4-pt function is not zero** → however the "4-point feature" is zero (notice the definition of kurtosis).

The harmonic transformation of the 4-pt function is called the trispectrum: T(k,p,q,s)

There are **different ways of defining higher-order statistics**.

Some are based on n-point correlation functions (like the skewness and kurtosis), others are based on different properties of the field (e.g. number of peaks, area of connected regions - topological features - etc).





Each one combines the underlying cosmological information in a different way → they depend on the cosmological parameters in different, sometimes complementary, ways.

Cosmic variance

Besides being important cosmological functions with valuable information needed to characterize cosmological maps and models**, higher-order statistics are also needed to compute the uncertainty of 2-pt functions.**

The power spectrum measured from a map is one realization of the theoretical power spectrum predicted from the cosmological model.

For example, for a given multipole l, the **measured power spectrum** amplitude may be:

$C_l = \langle a_{l4}\, a_{l4} \rangle$ (or any other value of m) (and other values of m for other multipoles).

Other parts of the map may correspond to other realizations (each sub-map is independent).

The maximum number of independent measurements of C_l from a map is 2l+1

On the other hand, the **theoretical power spectrum,** that we want to estimate from measurements in a map, is C_l with any value of m.

The best way to **estimate** the theoretical power spectrum from a map is to take the average of all possible measurements:

$$\widehat{C}_\ell \equiv \frac{1}{2\ell+1}\sum_m |a_{\ell m}|^2$$

This estimator is unbiased, meaning that if many measurements were made (N → ∞) its average would give exactly the theoretical power spectrum:

$$\langle \widehat{C}_\ell \rangle = C_\ell \pm \sigma / N$$

This is the same as when estimating the mean of a distribution by computing the average of N measurements (the larger is N, the more precise is the estimate).

However, the maximum number of independent measurements that can be made of each multipole is limited: it is given by 2l+1,

so the measured value will estimate the theoretical value with some **minimum uncertainty**. This is called the **cosmic variance**.

(The total uncertainty is in general larger than this, since other measurement errors need to be added to this minimal one).

The uncertainty of the estimator (i.e. the cosmic variance) is defined as the covariance (dispersion) of the estimator. This can be computed theoretically, which is more rigorous than just measuring the dispersion between various measurements (which depends on the specific sample measured)**.**

The expression is:

$$\sigma^2_{C\ell} = \left\langle (\hat{C}_\ell - C_\ell)(\hat{C}_{\ell'} - C_{\ell'}) \right\rangle$$

Note that in general this expression is written as a covariance, i.e., considering l and l'. However, since the multipoles are independent the covariance matrix is diagonal → **only the variances are non-zero → l = l'**

$$\sigma^2_{C\ell} = \left\langle (\hat{C}_\ell - C_\ell)^2 \right\rangle$$

$$\langle (\hat{C}_\ell - \langle C_\ell \rangle)^2 \rangle = \langle \hat{C}_\ell^2 - 2\hat{C}_\ell \langle C_\ell \rangle + \langle C_\ell \rangle^2 \rangle$$

(where $<C_l> = C_l$ is the theoretical value)

$$= \langle \hat{C}_\ell^2 \rangle - 2C_\ell \langle \hat{C}_\ell \rangle + C_\ell^2$$

$$= \langle \hat{C}_\ell^2 \rangle - 2C_\ell^2 + C_\ell^2 = \langle \hat{C}_\ell^2 \rangle - C_\ell^2$$

To evaluate the cosmic variance we need then to compute $\langle \hat{C}_\ell^2 \rangle$ as function of C_l . Naturally, this is:

$$\langle \hat{C}_\ell^2 \rangle = \frac{1}{(2\ell+1)^2} \langle \sum_m (a_{\ell m} a_{\ell m}^*) \sum_{m'} (a_{\ell m'} a_{\ell m'}^*) \rangle$$

$$= \frac{1}{(2\ell+1)^2} \sum_{m,m'} \langle a_{\ell m} a_{\ell m}^* a_{\ell m'} a_{\ell m'}^* \rangle$$

Notice the variances are power spectra squared, i.e., **4-pt functions**

Wick's theorem allows us to write a four-point function in terms of lower order functions. In particular for **Gaussian fields of zero mean**, the 1-pt and 3-pt functions are zero, and we can write:

$$\sum_{m,m'} \langle a_{\ell m} a_{\ell m} a_{\ell m'} a_{\ell m'} \rangle = \sum_{m,m'} \Big[\langle a_{\ell m} a_{\ell m} \rangle \langle a_{\ell m'} a_{\ell m'} \rangle + 2 \langle a_{\ell m} a_{\ell m'} \rangle \langle a_{\ell m} a_{\ell m'} \rangle + \langle a_{\ell m} a_{\ell m} a_{\ell m'} a_{\ell m'} \rangle_c$$

Now, using the result

$$\sum_{m,m'} \langle a_{\ell m} a_{\ell m} \rangle = (2\ell+1)\, C_\ell$$

it is just a question of counting all the terms contributing to the various sums, to find the result:

$$\langle \hat{C}_\ell^2 \rangle = \frac{C_\ell^2}{(2\ell+1)^2} \left[(2\ell+1)^2 + 2(2\ell+1) \right]$$

And so the cosmic variance is:

$$\langle \hat{C}_\ell^2 \rangle - C_\ell^2 = C_\ell^2 \left(1 + \frac{2}{2\ell+1} - 1 \right)$$

i.e.,

$$\sigma^2_{C\ell} = \frac{2}{2\ell+1} C_\ell^2$$

This result shows that this ultimate limit of cosmological observations depends on the amplitude of the angular power spectrum and on the scale l.

Since each scale has (2l+1) independent 'measures' contributing to it → large scales have less independent measures in the full sky than smaller ones → **cosmic variance dominates on large scales**, **we only have 1 universe to observe.**

Thinking of the **ergodic hypothesis**, independent regions of the sky are different realizations → could correspond to different universes (with different parameter values) → creating an intrinsic variance on the measurements → (this is the reason for this limit to be called the cosmic variance).

Also note that since cosmic variance depends on the cosmological parameters, it is not taken into account in Fisher matrix analyses.

The calculation is valid for a **full sky survey**. If the survey covers a smaller area, by a factor **f_sky = Area_survey / Area_fullsky**, there are less independent measures contributing to each scale, and the cosmic variance scales accordingly:

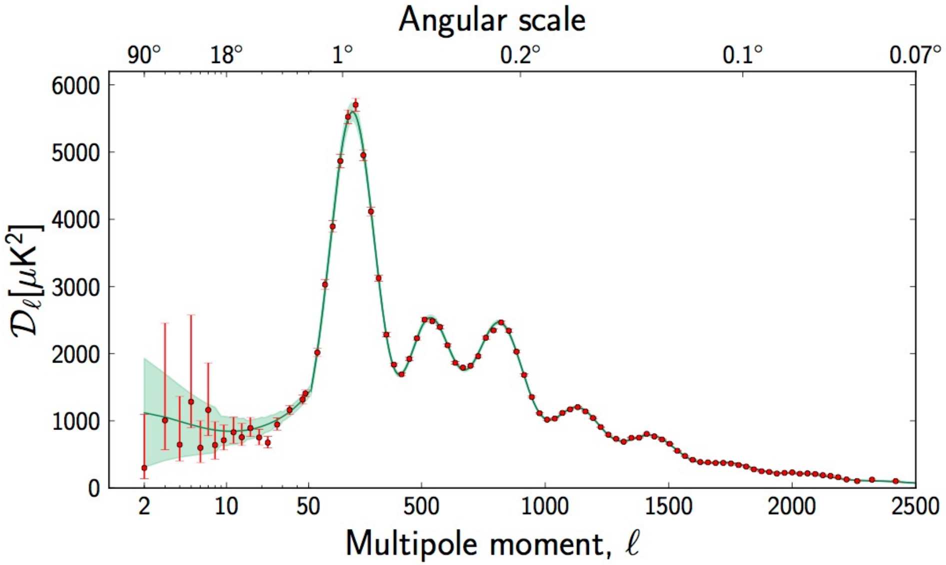
$$\sigma^2_{C\ell} = \frac{1}{\mathrm{f_{sky}}} \frac{2}{2\ell+1} C_\ell^2$$

If we want to limit the cosmic variance in a future survey, we should build a wider survey rather than a deeper one (i.e., increase the survey area).

Moreover, for a fixed f_sky it is useful to observe various separated fields, to average the result over the various fields.

Note that for the largest possible angular scale (l=1), the minimum uncertainty achievable (in the ideal case of a full sky survey and no experimental noise) is a fractional uncertainty of:

$$\sigma_l / C_l = (2/3)^{0.5} = 81\%$$



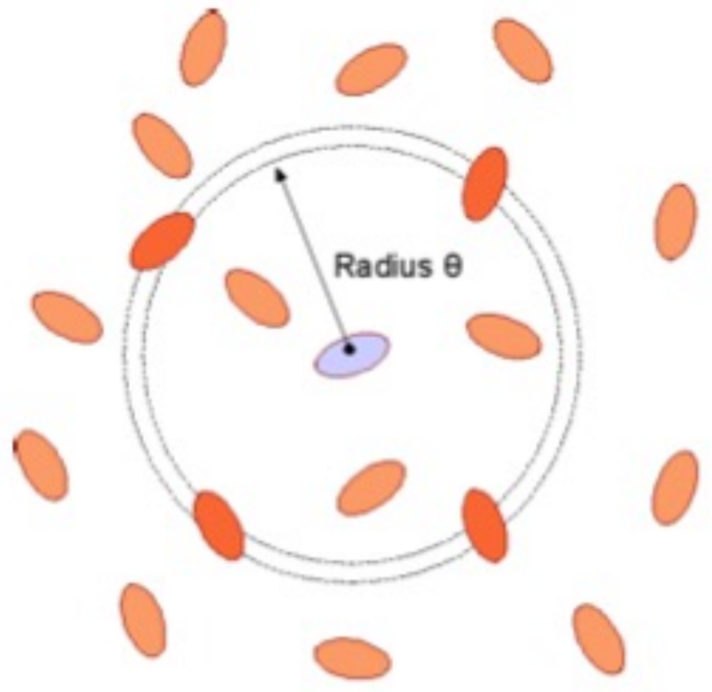
This is the large uncertainty seen in CMB plots, and is a fundamental limitation of cosmological data.

Shot noise

Measurements of discrete galaxies positions can also be used to estimate the power spectrum of the underlying continuous δ field.

Consider N galaxies (particles) of mass m=1 in a volume V, corresponding to a mean density

$$\bar{\rho} = \frac{M}{V} = \frac{N}{V}$$



Assume there is no galaxy bias, i.e., galaxy positions trace perfectly the mass distribution

The density ρ at a location takes values
0 (at a point x with no particle)
or 1 (at a point x with a particle).

With this set up, the density contrast may be written using the Dirac delta function (which will be convenient later on).
Note this is just a sophisticated way of writing 0 or 1.

Note that the integral of the Dirac delta is 1 (over the full infinity range), or zero (if the sum range does not contain the peak).

$$\delta(x) = \frac{\rho(x) - \bar{\rho}}{\bar{\rho}} = \frac{\Sigma_{i=1}^{N} \delta_D(x - x_i)}{N/V} - 1$$

Now, in order to compute the power spectrum, we need first to Fourier transform δ(x):

$$\begin{aligned}
\tilde{\delta}(\vec{\kappa}) &= \int \mathrm{d}^3 x \, \exp(-\mathrm{i}\vec{\kappa}\vec{x})\delta(\vec{x}) \\
&= \int \mathrm{d}^3 x \, \exp(-\mathrm{i}\vec{\kappa}\vec{x}) \left[\frac{V}{N}\sum_{i=1}^{N} \delta_{\mathrm{D}}(\vec{x} - \vec{x}_i) - 1\right] \\
&= \frac{V}{N}\sum_{i} \exp(-\mathrm{i}\vec{k}\vec{x}_i) - (2\pi)^3\delta_{\mathrm{D}}(\kappa)
\end{aligned}$$

where the integral over the Dirac delta sets $x = x_i$ in the plane wave and compute the correlation function in Fourier space $< \delta(k)\delta^*(k') >$

$$\langle\tilde{\delta}(\vec{\kappa})\tilde{\delta}^*(\vec{\kappa}')\rangle = \frac{V^2}{N^2}\sum_{i,j}\langle\exp(-\mathrm{i}\vec{\kappa}\vec{x}_i)\exp(\mathrm{i}\vec{\kappa}'\vec{x}_j)\rangle + (2\pi)^6\delta_\mathrm{D}(\vec{\kappa})\delta_\mathrm{D}(\vec{\kappa}') - (2\pi)^3\delta_\mathrm{D}(\vec{\kappa})\frac{V}{N}\sum_i\langle\exp(-\mathrm{i}\vec{\kappa}\vec{x}_i)\rangle - (2\pi)^3\delta_\mathrm{D}(\vec{\kappa})\frac{V}{N}\sum_j\langle\exp(\mathrm{i}\vec{\kappa}'\vec{x}_j)\rangle$$

To evaluate the 1st term - we may separate the terms i=j from i≠j :

$$\frac{V^2}{N^2}\sum_{i,j}\langle\exp(-\mathrm{i}\vec{\kappa}\vec{x}_i)\exp(\mathrm{i}\vec{\kappa}'\vec{x}_j)\rangle = \frac{V^2}{N^2}\sum_{i=j}\langle\exp(-\mathrm{i}\vec{\kappa}\vec{x}_i)\exp(\mathrm{i}\vec{\kappa}'\vec{x}_i)\rangle + \frac{V^2}{N^2}\sum_{i\neq j}\langle\exp(-\mathrm{i}\vec{\kappa}\vec{x}_i)\exp(\mathrm{i}\vec{\kappa}'\vec{x}_j)\rangle$$

Note: ***What is the sum of a ‘bracketed’ quantity?***

The ensemble average of a random variable ‘x’ is the sum over all its realizations (all elements in a sample).

If we do not have a sample but know the probability function of ‘x’ we could generate a sample and average.

Or, more precisely (and without recurring to numerical methods), we need to **sum over ‘x’ multiplied by its probability → it is a weighted sum**.

In general an ensemble average of a function f is then

<f> = integral (dx f(x) p(x))

or, in 2 dimensions:

$$< f >= \int \int dx_1 \, dx_2 \, p(x_1, x_2) \, f(x)$$

So in order to proceed with the derivation and compute the ensemble averages in this first term, we need first to write the probabilities.

In the case i=j, we need to compute $\langle \exp(-ikx_i)\ \exp(ik'x_j) \rangle$
It is a 1-dimensional problem, the ensemble average is an integral over x_i

What is the probability of having a particle in x_i?

It is just $P(x_i) = 1/V$

So now we can proceed and get:

$$\begin{aligned}\frac{V^2}{N^2}\sum_{i=j}\langle \exp(-\mathrm{i}\vec{\kappa}\vec{x}_i)\exp(\mathrm{i}\vec{\kappa}'\vec{x}_i)\rangle &= \frac{V^2}{N^2}\sum_{i=j}\int \mathrm{d}^3 x_i\,\frac{1}{V}\exp(-\mathrm{i}(\vec{\kappa}-\vec{\kappa}')\vec{x}_i) \\ &= \frac{V^2}{N^2}\frac{N}{V}(2\pi)^3\delta_{\mathrm{D}}(\vec{\kappa}-\vec{\kappa}')\end{aligned}$$

(where the integral gives a Dirac delta and the sum is over the N cases i=j)

In the case i≠j, we need to consider the joint probability of having two particles, one in x_i and another in x_j.

$$\frac{V^2}{N^2} \sum_{i \neq j} \langle \exp(-\mathrm{i}\vec{\kappa}\vec{x}_i) \exp(\mathrm{i}\vec{\kappa}'\vec{x}_j) \rangle$$

This is **the probability of x_i times the conditional probability of x_j given x_i.**

If they are independent this is just $P(xi,xj) = P(xi)\, P(xj) = (1/V)^2$

But if there is a correlation, **the probability of finding a particle in x_j depends on having or not a particle in x_i.**

If they are (positively) correlated the joint probability is larger than $(1/V)^2$:

$$P(xi,xj) = P(xi)\, P(xj|xi) = (\, 1+\xi\, (|xi-xj|)\,) \,/\, (V^2)$$

This is, of course, the definition of correlation function.

So **the ensemble average introduces in a natural way the correlation function of the continuous field in the derivation**.

$$
\begin{aligned}
&= \frac{V^2}{N^2}\sum_{i\neq j}\int \mathrm{d}^3 x_i \mathrm{d}^3 x_j \, \frac{1}{V^2}[1+\xi(|\vec{x}_i-\vec{x}_j|)]\exp(-\mathrm{i}\vec{\kappa}\vec{x}_i)\exp(\mathrm{i}\vec{\kappa}'\vec{x}_j) \\
&= \frac{N-1}{N}\int \mathrm{d}^3 x_i \mathrm{d}^3 z \exp(-\mathrm{i}\vec{\kappa}'\vec{z})\exp(-\mathrm{i}(\vec{\kappa}-\vec{\kappa}')\vec{x}_i)[1+\xi(|\vec{z}|)] \\
&= +(2\pi)^6\delta_{\mathrm{D}}(\vec{\kappa})\delta_{\mathrm{D}}(\vec{\kappa}') + \frac{\text{[redacted]}}{\text{[redacted]}}(2\pi)^3\delta_{\mathrm{D}}(\vec{\kappa}-\vec{\kappa}')\int \mathrm{d}^3 z \exp(-\mathrm{i}\vec{\kappa}'\vec{z})\xi(z)
\end{aligned}
$$

The sum has N(N-1) cases and (1+ξ) separates in 2 terms:

- an integral over the plane waves → giving 2 delta functions

- and the Fourier Transform of the correlation function (where z=|xi-xj|).

Going back to the expression for $< \delta(k)\delta^*(k') >$

The 2nd term has nothing to compute, $(2\pi)^6\delta_D(\vec{\kappa})\delta_D(\vec{\kappa}')$

and **the 3rd and 4th terms**

are similar to the i=j part of the 1st term:

$$
\begin{aligned}
(2\pi)^3\delta_D(\vec{\kappa})\frac{V}{N}\sum_i\langle\exp(-i\vec{\kappa}\vec{x}_i)\rangle &= (2\pi)^3\delta_D(\vec{\kappa})\frac{V}{N}\sum_j\langle\exp(i\vec{\kappa}'\vec{x}_j)\rangle \\
&= (2\pi)^3\delta_D(\vec{\kappa})\frac{V}{N}N\int d^3\,x_i\,\frac{1}{V}\exp(-i\vec{\kappa}\vec{x}_i) \\
&= (2\pi)^6\delta_D(\vec{\kappa})\delta_D(\vec{\kappa}')
\end{aligned}
$$

Putting **all terms together**:

The first term of the i≠j term and the 2nd, 3rd and 4th terms are all double Dirac deltas, and all cancel each other.

The result is then the i=j term, plus the second term of the i≠j term :

$$\begin{aligned}\langle\tilde{\delta}(\vec{\kappa})\tilde{\delta}^*(\vec{\kappa}')\rangle &= (2\pi)^3\delta_{\rm D}(\vec{\kappa}-\vec{\kappa}')\left[\frac{V}{N}+\int {\rm d}^3 z\exp(-{\rm i}\vec{\kappa}'\vec{z})\xi(z)\right] \\ &= (2\pi)^3\delta_{\rm D}(\vec{\kappa}-\vec{\kappa}')\left[\frac{V}{N}+P(|\vec{\kappa}|)\right]\end{aligned}$$

We derived that the correlation function in the Fourier space is the power spectrum plus a constant term (V/N).

(Instead of being just the power spectrum, as we had seen before)

This is a general property of any power spectrum estimated from a discrete spatial distribution.

Why is now the result $P(k)+V/N$ instead of $P(k)$?

The extra contribution comes from the i=j term of the derivation → it is a term of auto-correlation and not a term of covariance → it has no cosmological information related to a scale, because a scale needs a separation → it is a monopole term.

In our derivation, starting from measurements in the real space, it would be very easy to avoid ending up with this term → we just needed to discard auto-correlations in the estimator → consider only pairs of galaxies where the 2 galaxies are different.

But when we estimate directly the power spectrum from a discrete map, in a more indirect way, the result will always implicitly include this monopole → this term cannot be avoided:

$$\hat{P}(k) = P(k) + \frac{V}{N}$$

Notice that, since a scale k is a linear combination of all separations r within the window function, the i=j monopole affects the estimated amplitudes of P(k) for all scales → **it is an overall constant shift in amplitude (i.e., a white noise).**

It affects more the measurements on small scales, where the power spectrum has lower amplitude.

However, the fact that the monopole amplitude is given by V/N tells us that its amplitude will decrease in future surveys → larger V and larger N (with V being limited while N can tend to ∞)

So, **the galaxy power spectrum estimator is not biased**:

$$\left\langle \hat{P}(k) \right\rangle = P(k) + \left\langle \frac{V}{N} \right\rangle = P(k)$$

The monopole adds uncertainty to the estimated power spectrum, but does not bias the measurement. It does not need to be subtracted, it is part of the noise and contributes to the error bars. The monopole term is known as the shot noise (also called discreteness noise).

If we want to limit the shot noise in a future survey, we should build a deeper survey rather than a wider one (i.e., increase the density of galaxies n = N/V).